Orchestrating Distributive Process: Baron and Ferejohn Meet Tullock*

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Abstract

This paper examines the process of dividing a fixed surplus as a multilateral bargaining game with costly recognition. We establish the existence of equilibrium and explore the roles of institutional rules that govern this process. Specifically, we consider a design problem in which the designer maximizes a general objective function by determining both the voting rule—i.e., the minimum number of votes required to approve a proposal—and the mechanism for proposer recognition, modeled as a biased generalized lottery contest. We demonstrate that any feasible outcome regarding equilibrium efforts and recognition probabilities can be implemented using a rule that incorporates a dictatorial voting rule. In other words, the designer can always maximize her interests by resolving the distributive process through a simple static biased contest.

Keywords: Multilateral Bargaining; Costly Recognition; Contest Design

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1 Introduction

Economic agents often negotiate to divide scarce resources. Different units within a firm split advertising or R&D budgets; academic departments within a school deliberate over the allocation of hiring lines; and electoral candidates bargain over the distribution of campaign funding from a political party. The seminal study by Baron and Ferejohn (1989) provides an ingenious and intuitive framework for the analysis of distributive politics.

A canonical multilateral bargaining model typically involves an agent who proposes a plan to share a fixed surplus, seeking to secure a minimum number of favorable votes from peers for its approval. Conventional wisdom in the literature holds that the proposer enjoys a disproportionately large share thanks to their ability to set the agenda, while the vast majority of studies assume that the proposer is chosen at random. The rent associated with the proposing right would arguably tempt agents to engage in costly influencing activities to acquire this privilege. Such dynamics are indeed commonplace in reality. For instance, an academic department might promote its faculty members' publications as leverage for priority in future hiring, and disputing parties may lobby mediators for more favorable bargaining positions. To the best of our knowledge, Yildirim (2007) is the first to endogenize the recognition of the proposer and integrate the allocation of proposing right into the process of distributive politics. He considers a contest prior to the bargaining, in which agents expend effort to vie for the proposing right—and thereby generate a bargaining game with costly recognition.

This enriched framework broadens the scope for exploring the strategic nature and implications of distributive process. A pivotal question naturally arises: How do the institutional rules that govern the process affect agents' incentives to engage in influencing efforts and the distribution of recognition opportunities? We define the bargaining protocol by a k-majority rule, where k denotes the minimum number of favorable votes (including the proposer) required for approval. Does a more inclusive voting rule—i.e., one with a larger k—stifle competition for recognition and reduce effort? Insights from such analysis would further extend this inquiry to a rule-design problem, wherein a designer, such as a mediator, can fine-tune the relevant institutional rules to promote specific goals. In this paper, we address this challenge.

We examine a bargaining game with costly recognition that generalizes that of

Yildirim (2007)—allowing agents to be heterogeneous in contest technologies, effort cost functions, and patience levels—and establish equilibrium existence in Section 2.2. Despite the intuitiveness of the game, the answer to the question outlined above remains elusive. The game can be viewed as a contest with an endogenously determined prize. To vie for recognition, agents weigh their potential payoffs from winning—i.e., being recognized—against those from losing. This payoff differential effectively functions as the prize spread that motivates their efforts in the contest. The winner offers a subset of peers—namely, agents in his winning coalition—their equilibrium continuation values in the dynamic bargaining process to secure their votes. A loser, on the other hand, receives his equilibrium continuation value in exchange for his approval if he is included in a winning coalition or nothing if he is excluded. The prize spreads—which depend on agents' continuation values—motivate their efforts, while agents' efforts determine their recognition probabilities, the formation of each agent's winning coalition, and, ultimately, their equilibrium continuation values. The endogeneity, together with agents' heterogeneity, complicates the analysis and differentiates the game from traditional bargaining or contest models. We demonstrate that agents' total efforts for recognition can be nonmonotonic with respect to the voting rule k, and dismiss usual comparative statics in general. We elaborate on the game-theoretic subtleties caused by variations in the voting rule k in Section 2.3.

To further illuminate the nature of this game, we take a rule design approach and allow a designer to maximize a general objective function that depends on the profile of agents' equilibrium recognition probabilities and weakly increases with their efforts. The designer sets (i) the voting rule, k, and (ii) the recognition mechanism—that is, the rules that govern the contest for the right to propose. By varying the contest rules, the designer can effectively bias the competition in favor of certain contenders, and thereby tilt the playing field and reshape agents' incentives. For instance, the dean of a school might prioritize one department over others when evaluating their performance; similarly, a mediator may not be entirely neutral.

This analysis differs from the setups for biased contest design in two fundamental dimensions. First, we do not require that the objective function increase with agents' efforts, as in the vast majority of the contest literature, and the designer can either benefit or suffer from each agent's input. Second, adjusting the contest rules changes agents' incentives to supply effort, which, in turn, alters the prize spreads among agents. The endogenously determined prize in this game undermines established contest design results derived from settings with a fixed prize, which we elaborate on in Appendix B.

Despite these challenges, our analysis delivers an unambiguous message: When the designer can set both the voting rule and the recognition mechanism, the general objective function can always be maximized by a *dictatorial voting rule* with $k = 1$, although the specific form of recognition mechanism depends on the particular environmental factors. More importantly, we establish that any feasible outcome of the game regarding agents' efforts and recognition probabilities can be implemented through a rule with a dictatorial voting rule. In this case, a proposal is accepted with the consent of only the proposer, which results in an agent's capturing the entire surplus once recognized and receiving nothing otherwise. The game thus reduces to a standard static contest with a fixed prize. In other words, the designer can achieve any feasible outcome by resolving the distributive process through a contest. To maximize the designer's interests, it is thus without loss of generality to search for the optimal rules within the set of possible rules with $k = 1$.

The universal optimality of the dictatorial voting rule naturally leads to an alternative and potentially more general inquiry: Suppose the designer can flexibly adjust the recognition mechanism for a given k. Does a less inclusive voting rule—i.e., a smaller k—necessarily improve the value of the objective function? We demonstrate that this does not necessarily hold. However, our analysis shows that when agents are relatively impatient—i.e., when their patience levels are capped by an upper bound—any feasible outcome regarding equilibrium efforts and recognition probabilities can be implemented using an alternative rule with a different recognition mechanism and a less inclusive voting rule, i.e., with a smaller k. Consequently, general monotonicity is restored: The objective function weakly decreases with k, and the designer always prefers a less inclusive voting rule.

Link to the Literature Our paper adds to the literature on multilateral bargaining by providing a comprehensive analysis to endogenize the bargaining protocol and the proposer recognition mechanism. An extensive body of studies has been built on the framework established by Baron and Ferejohn (1989)—e.g., Merlo and Wilson (1995, 1998); Banks and Duggan (2000); Eraslan (2002); Eraslan and Merlo (2002, 2017); Diermeier and Fong (2011); Diermeier, Prato, and Vlaicu (2015, 2016); Ali, Bernheim, and Fan (2019); and Evdokimov (2023). The majority of this literature assumes that the proposer is exogenously and randomly selected from the agents.

A small but growing strand of the literature considers the selection of the proposer to be an integral part of the political process, and examines the endogenous formation of bargaining protocols. Yildirim (2007) models the process of selecting proposers as a contest in which agents exert costly effort to gain power, and pioneers the integration of a contest model (generalized Tullock contest) with a multilateral bargaining game to endogenize the recognition mechanism. Yildirim (2010) compares total effort and distributive outcomes between persistent and transitory recognition procedures, and Ali (2015) models the recognition process as an all-pay auction.

Our paper extends the effort to incorporate recognition mechanisms in a holistic distribution process and models the recognition process as an influencing competition. Our work is closely related to Yildirim (2007). Similar to Yildirim, we adopt a generalized Tullock contest, but we introduce heterogeneous production technologies with fewer restrictions, as well as nonlinear effort cost functions. Yildirim conducts comparative statics of the prevailing voting rule for homogeneous agents and shows that a more inclusive voting rule—i.e., a larger minimum number of required votes—always leads to lower total effort. In contrast, we explore optimal rule design in a setting that allows for a general design objective, heterogeneous agents, and multiple design instruments (voting rule and recognition mechanism). Agents' heterogeneity catalyzes complex effects with varying voting rules or contest rules, which prevents standard comparative static analysis and differentiates our game from conventional bargaining or contest models.

Several papers examine the endogenous formation of a bargaining protocol without using a contest approach. Diermeier, Prato, and Vlaicu (2015, 2016) employ a pre-bargaining process to determine proposal power in bargaining over policy. In McKelvey and Riezman (1992, 1993); Muthoo and Shepsle (2014); and Eguia and Shepsle (2015), recognition probability is determined by seniority, which is endogenously voted on at the end of each session. Kim (2019) assumes that current and past proposers are excluded from the pool of eligible candidates when a round of bargaining fails to reach consensus. Jeon and Hwang (2022) assume that an agent's recognition probability and bargaining power depend on the previous bargaining outcome in a dynamic legislative bargaining model, which leads to an oligopolistic outcome as the result of an evolutionary process. Agranov, Cotton, and Tergiman (2020) examine, both theoretically and experimentally, a repeated multilateral bargaining model in which the agenda setter can retain his power with the majoritarian support of other committee members.

Our paper is closely related to Jeon and Hwang (2022) and Ali, Bernheim, and Fan (2019), who demonstrate that power concentration could arise in the equilibrium. The former study attributes the endogenous formation of oligopoly to the influence of past bargaining outcomes. The latter shows that the prevailing information structure could lead to extreme power in terms of distributive outcomes when the voting rule is not unanimous. Neither involves a contest of proposing rights or an endogenously set voting rule, which is the focus of this paper.

Our paper is also naturally linked to the literature on contest design and, particularly, that on optimally biased contests. We develop a technique similar to that of Fu and Wu (2020) and Fu, Wu, and Zhu (2023), who characterize the optimum without explicitly solving for the equilibrium. Our analysis complements these studies by embedding the contest in a multilateral sequential bargaining framework, which generates an endogenous prize spread.

The rest of the paper is structured as follows. Section 2 sets up the model and characterizes the equilibrium. Section 3 allows a designer to set the voting rule and the recognition mechanism and solves the design problem. Section 4 concludes. Appendix A collects the proofs and derivations for examples that are not provided in the main text, and Appendix B provides an example that demonstrates how our results differ from those in the literature on contest design.

2 Multilateral Sequential Bargaining with Costly Recognition

In this section, we first lay out the modeling details, then conduct equilibrium analysis and discuss the nature of this game.

2.1 Model Setup

A set of $n \geq 2$ agents, indexed by $\mathcal{N} := \{1, 2, \ldots, n\}$, decide how to divide a dollar. In each period $t = 0, 1, 2, \ldots$, one agent (proposer) makes a proposal $\boldsymbol{s}_t \in \triangle^{n-1} := \{(s_{1,t}, \ldots, s_{n,t}) : 0 \leq s_{i,t} \leq 1, \sum_{i \in \mathcal{N}} s_{i,t} = 1\}$, where $s_{i,t}$ denotes the share of the dollar each agent $i \in \mathcal{N}$ is to receive under this proposal. Agents simultaneously vote in favor of or against the proposal. We assume a "k-majority" voting rule—with $1 \leq k \leq n$—for this sequential bargaining process: The proposal is approved if at least k agents accept it (including the proposer). Specifically, $k = n$ implies a unanimous rule wherein the proposal can be vetoed by any single dissident; $k = \lfloor n/2 \rfloor + 1$ refers to a simple majority rule; with $k = 1$, the proposer dictates the decision process.

At the beginning of each period t, a contest takes place in which each agent exerts an effort $x_{i,t} \geq 0$ to vie for the proposing right, which incurs a cost $c_i(x_{i,t})$. We assume that $c_i(\cdot)$ is twice differentiable and satisfies $c_i(0) = 0$, $c_i'(\cdot) > 0$, and $c_i''(\cdot) \geq 0$. For a given effort profile $\boldsymbol{x}_t := (x_{1,t}, \ldots, x_{n,t})$, an agent i is recognized as the proposer for period t with a probability

$$p_i(\boldsymbol{x}_t) = \begin{cases} \dfrac{f_i(x_{i,t})}{\sum_{j \in \mathcal{N}} f_j(x_{j,t})}, & \sum_{j \in \mathcal{N}} f_j(x_{j,t}) > 0, \\ \dfrac{1}{n}, & \sum_{j \in \mathcal{N}} f_j(x_{j,t}) = 0, \end{cases} \tag{1}$$

where $f_i(\cdot)$ is called the impact function in the contest literature. The function indicates each agent's technology for converting his effort into output in the competition, which is twice differentiable and satisfies $f_i(\cdot) \geq 0$, $f_i'(\cdot) > 0$, and $f_i''(\cdot) \leq 0$.

Each agent is risk neutral and has a discount factor $\delta_i \in (0, 1)$, which measures the degree of his patience. If a proposal is approved in period τ, an agent i's discounted payoff is

$$\Pi_i := \delta_i^{\tau} s_{i,\tau} - \sum_{t=0}^{\tau} \delta_i^t c_i(x_{i,t}),$$

where $s_{i,\tau}$ is the share he receives under the approved proposal and $c_i(x_{i,t})$ the effort cost incurred in each period $t \in \{0, \ldots, \tau\}$.[1]

The bargaining game with costly recognition can be described as $\langle (f_i(\cdot))_{i \in \mathcal{N}}, (c_i(\cdot))_{i \in \mathcal{N}}, \boldsymbol{\delta}, k \rangle$, where $(f_i(\cdot))_{i \in \mathcal{N}}$ denotes the set of impact functions, $(c_i(\cdot))_{i \in \mathcal{N}}$ the set of effort cost functions, $\boldsymbol{\delta} := (\delta_1, \ldots, \delta_n)$ the set of discounting factors, and k the voting rule. We allow agents to differ in their impact functions, cost functions, and the degrees of patience.

We assume that agents use stationary strategies whereby for each period t, agents' period-t actions are independent of the history (see Theorem 1 for details of the strategies). We adopt the solution concept of the stationary subgame perfect equilibrium

[1]If no agreement is reached, agent i's discounted payoff is $\Pi_i = -\sum_{t=0}^{+\infty} \delta_i^t c_i(x_{i,t})$.

(SSPE) and drop the time subscript t throughout. A strategy profile is an SSPE if it is stationary and constitutes a subgame perfect equilibrium.

2.2 Equilibrium Analysis

Let $\boldsymbol{v} := (v_1, \ldots, v_n)$ be the set of agents' equilibrium expected payoffs and consider stage-undominated voting strategies, such that agents vote as if they were pivotal. Suppose that an agent is not recognized as the proposer. He accepts a proposal if his share exceeds the discounted continuation value—i.e., $s_i \geq \delta_i v_i$—and rejects it otherwise. The proposer, in contrast, needs to select $k-1$ agents to form the least costly winning coalition and offers them their continuation values. His expected vote-buying cost is

$$w_i = \sum_{j \neq i} \psi_{ij} \delta_j v_j,$$

where ψ_{ij} gives the probability of agent i's including another agent j in his winning coalition. For each $j \in \mathcal{N}$, we further define $\mu_j := \sum_{i \neq j} \psi_{ij} p_i$ as agent j's probability of being included in others' winning coalitions before a proposer is recognized.

For each agent $i \in \mathcal{N}$, the expected gross payoff conditional on winning the competition and being the proposer is $1 - w_i$, and that when not being selected is $\frac{\mu_i}{1-p_i}\delta_i v_i$. The payoff differential between winning the competition and losing it, $1 - w_i - \frac{\mu_i}{1-p_i}\delta_i v_i$, is the effective prize spread that motivates his effort. He chooses effort x_i that solves the maximization problem on the right-hand side of the following Bellman equation:

$$v_i = \max_{x_i \geq 0} \left\{ p_i(x_i, \boldsymbol{x}_{-i})(1 - w_i) + [1 - p_i(x_i, \boldsymbol{x}_{-i})] \times \frac{\mu_i}{1 - p_i(x_i, \boldsymbol{x}_{-i})} \delta_i v_i - c_i(x_i) \right\}. \tag{2}$$

The first-order condition ensues:

$$\underbrace{c_i'(x_i)}_{\text{marginal cost of effort}} \geq \underbrace{\frac{f_i'(x_i)}{f_i(x_i)} \times p_i(1-p_i) \times \overbrace{\left(1 - w_i - \frac{\mu_i}{1-p_i}\delta_i v_i\right)}^{\text{effective prize spread}}}_{\text{marginal benefit of effort}}. \tag{3}$$

Our analysis concludes the following.

Theorem 1 *For each game $\langle (f_i(\cdot))_{i\in\mathcal{N}}, (c_i(\cdot))_{i\in\mathcal{N}}, \boldsymbol{\delta}, k\rangle$, there exists an SSPE characterized by $(\boldsymbol{x}, \boldsymbol{v})$ and $\{\psi_{ij}\}_{i\neq j}$. In the equilibrium, each agent $i \in \mathcal{N}$ exerts effort x_i in each period. If selected as the proposer, he forms a winning coalition of $k-1$ agents such that agent j is included with probability ψ_{ij} and offers the agent $\delta_j v_j$. Otherwise, he accepts a proposer's offer if and only if his share is no less than $\delta_i v_i$. The equilibrium is unique when $k = 1$.*

Theorem 1 establishes the equilibrium existence of the game. Two remarks are in order. First, the game can be viewed as a contest with an endogenous prize, since w_i, p_i, μ_i, and v_i all depend on agents' effort profile $\boldsymbol{x} = (x_1, \ldots, x_n)$ and vice versa. These nuances differentiate the model from a standard contest with a fixed prize or a standard multilateral sequential bargaining game, dismissing the regularity typically assumed in conventional frameworks. As a result, the equilibria may not be unique.[2] However, uniqueness is restored when $k = 1$: With a dictatorial voting rule, the game reduces to a standard contest with a fixed prize, since an agent secures the entire surplus for being the proposer while receiving nothing once he loses the competition, which yields a fixed prize spread of 1.

Second, our game differs from the setting of Yildirim (2007) in several respects. Yildirim (2007) assumes linear cost function and homogeneous impact function $f(\cdot)$, with $f(0) = 0$, and weakly decreasing elasticity $xf'(x)/f(x)$. All of these restrictions are relaxed in our setting.

2.3 Role of Voting Rule k

We now discuss the role played by the voting rule k in shaping the equilibrium, which illuminates the strategic nature of the game. More specifically, we explore how the change in k affects agents' effort incentives and the resultant total equilibrium efforts $\sum_{i\in\mathcal{N}} x_i$.

Recall that the game is a contest with an endogenous prize spread $1 - w_i - \frac{\mu_i}{1-p_i}\delta_i v_i$. Imagine a more inclusive voting rule—i.e., increasing k. It generates a direct effect on agents' prize spreads, which we call the (direct) *prize effect*. A larger k changes both an agent's winning value—i.e., $1 - w_i$—and losing value, i.e., $\frac{\mu_i}{1-p_i}\delta_i v_i$. A proposer

[2]Fixing a recognition probability profile—i.e., fixing an effort profile—the literature on multilateral bargaining has noticed that there exist multiple equilibria that differ in $\{\psi_{ij}\}$, but they result in the same profile of $(\mu_1, \ldots, \mu_n)$. An additional layer of equilibrium multiplicity may arise within our context, in the sense that agents' effort profile may differ across equilibria.

has to buy more votes if he wins, and he needs to buy votes from a different set of his peers; each peer would demand a different offer, since their continuation values change. All of these change w_i. Further, a losing candidate may expect a different payoff because he is more likely to be included in some winning coalitions, while the minimum share he would accept also varies with the change in his continuation value. These change $\frac{\mu_i}{1-p_i}\delta_i v_i$ accordingly.

The ultimate effect on $\sum_{i \in \mathcal{N}} x_i$ is ambiguous. First, it is a priori unclear whether all agents expect a smaller prize spread when k increases. Second and more importantly, the changes in prize spreads are *nonuniform* among asymmetric agents, which, in turn, catalyzes the (indirect) *rebalancing effect.*

To see this, consider the simple case with $k = 1$, such that all agents have a prize spread of 1 irrespective of their patience levels, since the proposer can expropriate all surplus. Suppose that k increases to 2. This decreases the effective spreads of all agents, but the decrease is asymmetric. Agents' patience δ_i, for instance, may play a role. Ceteris paribus, the most patient agent is least likely to be included in a winning coalition, since high patience elevates his continuation value and therefore others' costs of buying his vote. His losing value, $\frac{\mu_i}{1-p_i}\delta_i v_i$, tends to rise less than that of others, which in turn causes a smaller decrease in his prize spread than that of others. This agent is thus motivated by the largest prize spread, which could encourage him to step up effort. The tilted playing field in the competition alters agents' incentives indeterminately.

We construct an example to illustrate the subtlety.

Example 1 *Suppose that* $n = 4$, $f_i(x_i) = \eta_i x_i$, $c_i(x_i) = \eta_i x_i$, *with* $\boldsymbol{\eta} = (1, 0.2, 0.2, 0.2)$ *and* $\boldsymbol{\delta} = (0.1, 0.5, 0.5, 0.5)$. *The equilibria under different voting rules are depicted in Table 1.*

	$k = 1$	$k = 2$	$k = 3$	$k = 4$
Winning probability of agent 1	0.2500	0.2322	0.2421	0.2500
Winning probability of agents 2-4	0.2500	0.2559	0.2526	0.2500
Equilibrium effort of agent 1	0.1875	0.1711	0.1656	0.1570
Equilibrium efforts of agents 2-4	0.9375	0.9433	0.8641	0.7849
Total effort	3.0000	3.0011	2.7578	2.5116

Table 1: Equilibrium Outcomes in Example 1.

Table 1 demonstrates that the equilibrium total effort varies nonmonotonically with k and is maximized at $k = 2$. There are two types of agents: 1 impatient agent and 3 patient agents. When $k = 1$, heterogeneity in effort cost and that in impact function perfectly offset each other, and all agents win with equal probability. Each agent needs to pay the vote-buying cost when k increases to 2 and thus the prize effect arises, which tends to reduce the prize spread and the equilibrium effort. The impatient agent is always included in the winning coalition, which increases his losing value and decreases his prize spread more than that of his patient counterparts. As a result, patient agents have a larger prize spread and therefore a stronger prize incentive. The rebalancing effect thus comes into play. In this example, the indirect rebalancing effect dominates the direct prize effect when k increases from 1 to 2 and maximizes the total effort.

This observation sharply contrasts with the result of Yildirim (2007). With symmetric agents, Yildirim shows that total effort strictly decreases with k. Intuitively, with symmetric agents, an increase in k decreases agents' prize spreads and weakens their incentives, while the rebalancing effect is muted due to symmetry. As shown in Table 1, no explicit comparative statics with respect to k can be obtained in general when agents are heterogeneous.

3 Rule Design

To further understand the nature of this game and obtain more general insight, we now take a rule design approach that allows for more freedom in varying the rules that govern the entire distributive process. In particular, a designer sets both the voting rule k and the mechanism of proposer recognition (i.e., contest rules) that determines the probability of each agent's recognition for every given effort profile.

To proceed, we assume that the impact function $f_i(\cdot)$ in (1) takes the form

$$f_i(\cdot) = \alpha_i \cdot h_i(\cdot) + \beta_i, \forall\, i \in \mathcal{N}, \tag{4}$$

where $h_i(\cdot)$ is twice differentiable and satisfies $h_i(0) = 0$, $h_i'(\cdot) > 0$, and $h_i''(\cdot) \leq 0$. The designer imposes the multiplicative weights $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n_+ \setminus \{(0, \ldots, 0)\}$—which scale one's output up or down—and additive headstarts $\boldsymbol{\beta} := (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n_+$. We can view $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ as nominal scoring rules that manipulate agents' relative competitiveness and tilt the balance of the competition. Alternatively, they can be viewed as resources assigned to agents that alter their productivity or influence (see, e.g., Fu and Wu, 2022).

Both multiplicative weights $\boldsymbol{\alpha}$ and additive headstarts $\boldsymbol{\beta}$ are broadly adopted in modeling biased contests: Epstein, Mealem, and Nitzan (2011) and Franke, Kanzow, Leininger, and Schwartz (2014), for instance, consider the former; Konrad (2002); Siegel (2009, 2014); and Kirkegaard (2012) focus on the latter; and Franke, Leininger, and Wasser (2018) and Fu and Wu (2020) allow for both. It is noteworthy that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ play different roles in impacting the contest's outcome: $\boldsymbol{\alpha}$ alter the marginal returns of agents' efforts, while $\boldsymbol{\beta}$ directly add to their effective output regardless of their efforts.

The designer's utility depends on agents' equilibrium efforts and recognition probabilities—i.e., the ex ante distribution of recognition opportunities. More formally, she sets $(\boldsymbol{\alpha}, \boldsymbol{\beta}, k)$ to maximize an objective function $\Lambda(\boldsymbol{x}, \boldsymbol{p})$, where $\boldsymbol{x} := (x_1, \ldots, x_n)$ and $\boldsymbol{p} := (p_1, \ldots, p_n)$ denote the profiles of equilibrium efforts and agents' recognition probabilities, respectively. The objective function accommodates a diverse array of preferences. Consider, for example, $\Lambda(\boldsymbol{x}, \boldsymbol{p}) = \sum_{i \in \mathcal{N}} x_i - \lambda \sum_{i \in \mathcal{N}} \left|p_i - \frac{1}{n}\right|$, with $\lambda \geq 0$. When $\lambda = 0$, this objective boils down to maximizing equilibrium total effort, which is popularly assumed in the contest design literature. The term $\sum_{i \in \mathcal{N}} \left|p_i - \frac{1}{n}\right|$—i.e., the mean absolute deviation of $\boldsymbol{p}$—increases in the dispersion of $\boldsymbol{p}$. When $\lambda > 0$, the function thus depicts a preference for more equitable distribution of recognition opportunities, which compels the designer to set rules to reduce $\sum_{i \in \mathcal{N}} \left|p_i - \frac{1}{n}\right|$.[3]

Our analysis differs from conventional studies on optimally biased contests in two key respects. First, the literature typically assumes that contestants' efforts accrue to the designer's benefit. For instance, Fu and Wu (2020) require that $\Lambda(\boldsymbol{x}, \boldsymbol{p})$ weakly increases with x_i, for each $i \in \mathcal{N}$. In contrast, our framework does not impose these restrictions, and thus allows for the possibility that the designer may incur costs from certain agents' input and may be negatively affected by some agents' efforts more than others.

Second, the endogenously determined prize significantly complicates the role of contest rules in determining equilibrium and invalidates prior findings in the contest literature. Recall that the equilibrium of the game is governed by the first-order

[3]Eraslan and Merlo (2017) examine the distributive implications of voting rules. They show that unanimity may paradoxically lead to a more unequal distributive outcome. It is noteworthy that in our context, the designer's fairness concern refers to her preference for ex ante distribution of bargaining power among agents—i.e., the recognition probability profile—instead of ex post distribution of the surplus.

condition

$$\underbrace{c_i'(x_i)}_{\text{marginal cost of effort}} \geq \underbrace{\frac{f_i'(x_i)}{f_i(x_i)} \times p_i(1-p_i) \times \overbrace{\left(1 - w_i - \frac{\mu_i}{1-p_i}\delta_i v_i\right)}^{\text{effective prize spread}}}_{\text{marginal benefit of effort}}.$$

In contrast, a standard static contest, with a fixed prize with unit value, would require the following in equilibrium:

$$\underbrace{c_i'(x_i)}_{\text{marginal cost of effort}} \geq \underbrace{\frac{f_i'(x_i)}{f_i(x_i)} \times p_i(1-p_i) \times 1}_{\text{marginal benefit of effort}}.$$

A change in $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ alters agents' relative competitiveness in the competition, and the tilted playing field reshapes their effort incentives, which we call a *(direct) rebalancing effect*. In our dynamic bargaining process, the rebalancing effect further causes an *(indirect) prize effect*: Each agent's effective prize spread, $1 - w_i - \frac{\mu_i}{1-p_i}\delta_i v_i$, is endogenously determined, and a change in efforts alters w_i, μ_i, p_i, and v_i. Such indirect effect is absent in a standard static contest. In Appendix B, we show that the results established in prior literature on contest design may lose their bite.

3.1 Main Result

Despite these complexities, our analysis obtains an unambiguous conclusion when k and $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ can be set altogether.

Theorem 2 *Consider an arbitrary rule $(\boldsymbol{\alpha}, \boldsymbol{\beta}, k)$ with $k > 1$ that induces an equilibrium with an outcome $(\boldsymbol{x}, \boldsymbol{p})$, where $\boldsymbol{x} = (x_1, \ldots, x_n)$ is agents' effort profile and $\boldsymbol{p} = (p_1, \ldots, p_n)$ is the associated recognition probability profile. There always exists an alternative rule $(\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}, 1)$ that induces the same outcome $(\boldsymbol{x}, \boldsymbol{p})$ in the unique equilibrium. As a result, the objective function $\Lambda(\boldsymbol{x}, \boldsymbol{p})$ can always be maximized by a set of rules that involve a dictatorial voting rule with $k = 1$.*

With $k = 1$, the bargaining process collapses and reduces the game to a standard static contest with a fixed prize of unit value, since both w_i and $\frac{\mu_i}{1-p_i}\delta_i v_i$ are zero. Theorem 2 thus states that the designer can implement any feasible outcome $(\boldsymbol{x}, \boldsymbol{p})$ by resolving the distributive process through a static contest, although the specific form of the associated recognition mechanism $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ depends on the particular context. The rule design problem for the bargaining game with costly recognition reduces to a standard contest design problem: To maximize objective function $\Lambda(\boldsymbol{x}, \boldsymbol{p})$, it is without loss of generality to search for the optimum within the set of possible rules with $k = 1$.

Notably, by setting $k = 1$, the ambiguous indirect prize effect caused by a change in $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is muted, since the prize spread is now fixed and independent of agents' continuation values. This eliminates the dynamic linkage within the game. Our analysis shows that with a fixed prize, adjusting $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ alone—rebalancing the playing field of the competition without introducing the complications of the prize effect—is sufficient to induce any outcome $(\boldsymbol{x}, \boldsymbol{p})$ that can be achieved by rules that involve $k \neq 1$.

Consider scenarios in which agents' efforts are productive and benefit the designer, such that $\Lambda(\boldsymbol{x}, \boldsymbol{p})$ weakly increases with x_i for every $i \in \mathcal{N}$. For instance, a mediator might value the lobbying efforts of disputing parties, or a school's dean might seek to encourage departments' research efforts in exchange for priority in future resource allocation. Fu and Wu (2020) show that in a standard contest with a fixed prize, such an objective function $\Lambda(\boldsymbol{x}, \boldsymbol{p})$ can be maximized through a set of optimally chosen multiplicative biases, while setting all headstarts β_i to zero. Their result and approach do not apply due to the endogenously determined prize when $k \geq 2$ (see Appendix B for more details). Theorem 2 revives the relevance of Fu and Wu (2020) in our context when efforts are productive. We conclude the following.

Corollary 1 *Suppose that the objective function $\Lambda(\boldsymbol{x}, \boldsymbol{p})$ weakly increases with x_i for every $i \in \mathcal{N}$. When the designer can flexibly choose $(\boldsymbol{\alpha}, \boldsymbol{\beta}, k)$, $\Lambda(\boldsymbol{x}, \boldsymbol{p})$ can always be maximized by a rule $(\boldsymbol{\alpha}, \mathbf{0}, 1)$—i.e., a dictatorial voting rule and zero headstart with $\beta_i = 0$ for every $i \in \mathcal{N}$.*

This claim is straightforward but further illustrates the logic of our analysis. Setting $k = 1$ maximizes the prize spread and provides the strongest prize incentive for agents, which encourages them to strive for recognition. The designer can then adjust $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ accordingly to exploit agents' heterogeneity and optimally balance the playing field. The two sets of instruments thus serve distinct roles. As in Fu and Wu (2020), headstarts are less effective in incentivizing efforts than multiplicative biases, as the latter directly alter an agent's marginal benefit from exerting effort, whereas the former do not. Consequently, the optimum requires $\beta_i = 0$, which ensures that positive effort is necessary for an agent to secure victory for the privilege.

3.2 Extension and Discussion

Theorem 2 establishes the general optimality of a dictatorial voting rule. This observation inspires us to further explore the general effect of varying k: Despite the unavailability of the comparative statics of k when only the voting rule changes, as shown in Example 1, does there exist a monotone relationship between the functional value of $\Lambda(\boldsymbol{x}, \boldsymbol{p})$ and the voting rule k when the designer can adjust the recognition mechanism optimally for every given k? In other words, does a less inclusive voting rule—i.e., a smaller k—improve the value of the objective function when the recognition mechanism can be adjusted accordingly?

The following example demonstrates that this conjecture does not hold in general, even for a simple objective function.

Example 2 *Suppose $n = 7$. Further, $h_i(x_i) = x_i$ and $\delta_i = 0.999$ for all agents $i \in \mathcal{N}$. We construct the following vectors: $\tilde{\boldsymbol{p}} := (0.005, 0.005, 0.005, 0.1, 0.1, 0.1, 0.685)$, $r = 839.9 \times \tilde{p}_7(1 - \tilde{p}_7)$, and $\tilde{\boldsymbol{x}} = (0.0037, 0.0037, 0.0037, 0.0144, 0.0144, 0.0144, 0.0001^{\frac{1}{r}})$. Agents' effort cost functions take the following form:*

$$c_i(x_i) = \begin{cases} x_i, & x_i \leq \tilde{x}_i \text{ and } i \leq 6, \\ x_i^r, & x_i \leq \tilde{x}_i \text{ and } i = 7, \\ \tilde{x}_i + \gamma(x_i - \tilde{x}_i), & x_i > \tilde{x}_i \text{ and } i \leq 6, \\ \tilde{x}_i^r + \gamma(x_i - \tilde{x}_i), & x_i > \tilde{x}_i \text{ and } i = 7, \end{cases}$$

where γ is a sufficiently large constant. Assume an objective function $\Lambda = \sum_{i \in \mathcal{N}} x_i - \lambda \sum_{i \in \mathcal{N}} |p_i - \tilde{p}_i|$ with a sufficiently large λ. The designer can freely set $(\boldsymbol{\alpha}, \boldsymbol{\beta})$. It can be verified that setting k to either 5 or 1 maximizes the objective function, while $k = 4$ is suboptimal, which indicates the nonmonotonicity of the designer's payoff with respect to k.

Example 2 reveals the nuances of this game. The designer values agents' efforts in this context. On the one hand, increasing k requires a proposer to buy more votes, which, ceteris paribus, directly reduces each agent's winning prize and effort incentive. On the other hand, the cost for each vote may also change due to the altered dynamics involved in the bargaining process; this indirectly affects agents' winning prizes and could either increase or decrease them. By Example 2, the cost of an individual agent's vote can be reduced when the voting rule becomes more inclusive. Thus the latter indirect effect may dominate the former direct effect, which results in the nonmonotonicity of the designer's payoff with respect to k.

Despite the complexity, our analysis obtains the following, which extends the limit of our main result.

Theorem 3 *Suppose that $\delta_i \leq \frac{1}{2}$ for each $i \in \mathcal{N}$. Consider an arbitrary outcome of the game $(\boldsymbol{x}, \boldsymbol{p})$ that can be induced by a rule $(\boldsymbol{\alpha}, \boldsymbol{\beta}, k)$ with $k \in \{2, \ldots, n\}$. There always exists an alternative rule $(\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}, k-1)$ that induces the same equilibrium outcome $(\boldsymbol{x}, \boldsymbol{p})$. As a result, if the designer can flexibly adjust $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ for a given k, the objective function Λ weakly decreases in k.*

Theorem 3 confirms that with additional restrictions on δ_i, any feasible outcome $(\boldsymbol{x}, \boldsymbol{p})$ can be implemented using a less inclusive voting rule, provided that the recognition mechanism $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is adjusted accordingly. The designer can further improve the outcome beyond $(\boldsymbol{x}, \boldsymbol{p})$ by fine-tuning the recognition mechanism. As a result, the designer always prefers a less inclusive voting rule when agents are relatively impatient—i.e., when δ_i is bounded from above by $1/2$.

When agents are relatively impatient, the dynamic linkage within the game weakens, which limits the nuanced indirect effects triggered by changes in k and $(\boldsymbol{\alpha}, \boldsymbol{\beta})$. The designer can then leverage biases in the recognition contest $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ to achieve her objective without complications arising from the change in the cost for each agent's vote. This result is even more intuitive when agents' efforts benefit the designer, and she aims to incentivize them. With weaker dynamic linkage, the direct effect of increasing k—i.e., more votes required to approve a proposal—dominates the indirect effect—i.e., the change in the cost of buying each vote. Consequently, she can reduce k to increase each agent's winning prize and expand the prize spread, thereby amplifying effort incentives while being less concerned about its indirect impact on bargaining dynamics.

4 Conclusion

In this paper, we explore a multilateral bargaining game with costly recognition, in which a set of agents divide a fixed amount of resources and the proposer is recognized through a contest. We examine the roles played by the institutional rules that govern the distributive process. We illustrate the complexity of the game and consider a general design problem in which a designer's payoff depends on agents' efforts and their recognition probabilities. The designer can deploy two sets of design instruments: (i) the voting rule that governs how proposals are accepted or rejected and (ii) the recognition mechanism that determines how the proposer is selected based on agents' productive efforts. We demonstrate that any feasible outcome regarding equilibrium efforts and recognition probabilities can be implemented by a contest—i.e., a rule that involves a dictatorial voting rule. As a result, the designer can always maximize her objective function by resolving the distributive process through a simple static contest.

Appendix A: Proofs

Proof of Theorem 1

Proof. We first characterize the SSPE assuming its existence, then prove equilibrium existence.

Equilibrium Characterization Denote by V^{Δ} the k-th lowest continuation value. Let $\mathcal{N}_1 := \{i \in \mathcal{N} : \delta_i v_i < V^{\Delta}\}$, $\mathcal{N}_2 := \{i \in \mathcal{N} : \delta_i v_i = V^{\Delta}\}$, and $\mathcal{N}_3 := \{i \in \mathcal{N} : \delta_i v_i > V^{\Delta}\}$. Evidently, agent i, when becoming the proposer, buys out the votes of the cheapest "winning coalition"—i.e., $\mathcal{N}_1$ and a subset of $\mathcal{N}_2$, from which we can conclude

$$\psi_{ij} \begin{cases} = 1, & j \in \mathcal{N}_1, \\ \in [0,1], & j \in \mathcal{N}_2, \\ 0, & j \in \mathcal{N}_3, \end{cases} \text{ and } \mu_i \begin{cases} = 1 - p_i, & i \in \mathcal{N}_1, \\ \in [0, 1-p_i], & i \in \mathcal{N}_2, \\ = 0, & i \in \mathcal{N}_3. \end{cases} \tag{5}$$

Define

$$V_L := 1 - \sum_{j \in \mathcal{N}_1} \delta_j v_j - \big(k - |\mathcal{N}_1|\big) V^{\Delta}. \tag{6}$$

Agent i's expected cost is then

$$w_i = \begin{cases} 1 - V_L - \delta_i v_i, & i \in \mathcal{N}_1, \\ 1 - V_L - V^\Delta, & \text{otherwise.} \end{cases}$$

The effective prize spread $1 - w_i - \frac{\mu_i}{1-p_i}\delta_i v_i$ in (3) can be expressed as

$$1 - w_i - \frac{\mu_i}{1-p_i}\delta_i v_i = V_L + \frac{1-\mu_i-p_i}{1-p_i}V^\Delta = \begin{cases} V_L, & i \in \mathcal{N}_1, \\ V_L + \frac{1-p_i-\mu_i}{1-p_i}V^\Delta, & i \in \mathcal{N}_2, \\ V_L + V^\Delta, & i \in \mathcal{N}_3. \end{cases} \tag{7}$$

We are ready to lay out the conditions for equilibrium characterization. An SSPE can be characterized by $(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{p}, \boldsymbol{\mu}, V_L, V^\Delta)$. Combining (3) and (7) yields

$$\frac{c_i'(x_i)f_i(x_i)}{f_i'(x_i)} \geq p_i(1-p_i)\left(V_L + \frac{(1-p_i-\mu_i)V^\Delta}{1-p_i}\right). \tag{8}$$

Next, consider the expected payoff v_i. By (2), we have

$$v_i = p_i(1-w_i) + \mu_i\delta_i v_i - c_i(x_i) = \begin{cases} \frac{1}{1-\delta_i}\left(p_i V_L - c_i(x_i)\right), & i \in \mathcal{N}_1, \\ \frac{V^\Delta}{\delta_i}, & i \in \mathcal{N}_2, \\ p_i(V_L + V^\Delta) - c_i(x_i), & i \in \mathcal{N}_3. \end{cases} \tag{9}$$

Combining (2), (5), and (9) yields

$$\mu_i \begin{cases} = 1 - p_i, & i \in \mathcal{N}_1, \\ \in [0, 1-p_i] \text{ solves } \frac{V^\Delta}{\delta_i} = p_i V_L + (\mu_i + p_i)V^\Delta - c_i(x_i), & i \in \mathcal{N}_2, \\ = 0, & i \in \mathcal{N}_3. \end{cases} \tag{10}$$

Each agent chooses exactly $k-1$ agents in his winning coalition—i.e., $\sum_{j \neq i} \psi_{ij} = k - 1,\ \forall i \in \mathcal{N}$. Therefore,

$$\sum_{i \in \mathcal{N}} \mu_i = \sum_{i \in \mathcal{N}} \sum_{j \neq i} \psi_{ji} p_j = \sum_{j \in \mathcal{N}} p_j \sum_{i \neq j} \psi_{ji} = \sum_{j \in \mathcal{N}} (k-1)p_j = k - 1. \tag{11}$$

Last, (6) can be rewritten as

$$V_L + \sum_{i \in \mathcal{N}_1} (\delta_i v_i) + \left(k - |\mathcal{N}_1|\right) V^\Delta = 1. \tag{12}$$

To characterize an SSPE, it suffices to find $(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{p}, \boldsymbol{\mu}, V^\Delta, V_L)$ that satisfies (8)-(12).

Equilibrium Existence Let $Y := \sum_{i \in \mathcal{N}} f_i(x_i)$. By (1), we have $p_i = f_i(x_i)/Y$, which implies that

$$x_i = f_i^{-1}(Y p_i), \text{ for } p_i \in \left[f_i(0)/Y, 1\right], \tag{13}$$

and

$$\sum_{i \in \mathcal{N}} p_i = 1. \tag{14}$$

Substituting (13) into (8) yields

$$\frac{Y c_i'\left(f_i^{-1}(Y p_i)\right)}{f_i'\left(f_i^{-1}(Y p_i)\right)} \geq (1 - p_i)(V_L + V^\Delta) - \mu_i V^\Delta, \text{with equality holding if } p_i > \frac{f_i(0)}{Y}. \tag{15}$$

Rewriting (10) and (11) and substituting (9) into (12) yield

$$\mu_i = \frac{1}{V^\Delta} \text{med} \left\{ 0, V^\Delta(1 - p_i), \frac{V^\Delta}{\delta_i} - p_i(V_L + V^\Delta) + c_i\left(f_i^{-1}(Y p_i)\right) \right\}, \tag{16}$$

$$\sum_{i \in \mathcal{N}} \mu_i = k - 1, \tag{17}$$

and

$$\sum_{i \in \mathcal{N}_1} \frac{\delta_i}{1 - \delta_i} \left[p_i V_L - c_i\left(f_i^{-1}(Y p_i)\right)\right] + \left(k - |\mathcal{N}_1|\right) V^\Delta + V_L = 1, \tag{18}$$

where $\text{med}\{\cdot, \cdot, \cdot\}$ gives the median of the input.

To prove equilibrium existence, it suffices to show that there exists $(\boldsymbol{p}, \boldsymbol{\mu}, Y, V^\Delta, V_L)$ to satisfy conditions (14)-(18). The proof consists of four steps. First, fixing (Y, V^Δ, V_L), we show that there exists a unique $(\boldsymbol{p}, \boldsymbol{\mu})$ to satisfy (15) and (16). Second, fixing (V^Δ, V_L), there exists $Y \geq \sum_{i \in \mathcal{N}} f_i(0)$ to satisfy (14). Third, fixing V_L, there exists V^Δ to satisfy (17). Last, we show that there exists V_L to satisfy (18).

Step I Substituting (16) into (15) yields

$$\frac{Yc_i'\big(f_i^{-1}(Yp_i)\big)}{f_i'\big(f_i^{-1}(Yp_i)\big)} \geq \text{med}\left\{(1-p_i)(V_L+V^\Delta), (1-p_i)V_L, V_L+V^\Delta-\frac{V^\Delta}{\delta_i}-c_i(f_i^{-1}\big(Yp_i)\big)\right\}, \tag{19}$$

with equality holding if $p_i > \frac{f_i(0)}{Y}$.

Let

$$\phi(p_i) := \frac{Yc_i'\big(f_i^{-1}(Yp_i)\big)}{f_i'(f_i^{-1}(Yp_i))} - \text{med}\left\{(1-p_i)(V_L+V^\Delta), (1-p_i)V_L, V_L+V^\Delta-\frac{V^\Delta}{\delta_i}-c_i\big(f_i^{-1}(Yp_i)\big)\right\}.$$

Note that $f_i(\cdot)$ is increasing and concave by assumption. This implies that $\phi(\cdot)$ strictly increases with p_i. Therefore, if $\phi\big(\frac{f_i(0)}{Y}\big) \geq 0$, or equivalently, if

$$\frac{Yc_i'(0)}{f_i'(0)} \geq \text{med}\left\{\left(1-\frac{f_i(0)}{Y}\right)V_L, \left(1-\frac{f_i(0)}{Y}\right)(V_L+V^\Delta), V_L+V^\Delta-\frac{V^\Delta}{\delta_i}\right\}, \tag{20}$$

then $p_i = \frac{f_i(0)}{Y}$. Otherwise, if $\phi\big(\frac{f_i(0)}{Y}\big) < 0$, or equivalently, if

$$\frac{Yc_i'(0)}{f_i'(0)} < \text{med}\left\{\left(1-\frac{f_i(0)}{Y}\right)V_L, \left(1-\frac{f_i(0)}{Y}\right)(V_L+V^\Delta), V_L+V^\Delta-\frac{V^\Delta}{\delta_i}\right\}, \tag{21}$$

then $p_i > \frac{f_i(0)}{Y}$; moreover, p_i is uniquely pinned down by $\phi(p_i) = 0$, or equivalently,

$$\frac{Yc_i'\big(f_i^{-1}(Yp_i)\big)}{f_i'(f_i^{-1}\big(Yp_i)\big)} = \text{med}\left\{(1-p_i)(V_L+V^\Delta), (1-p_i)V_L, V_L+V^\Delta-\frac{V^\Delta}{\delta_i}-c_i\big(f_i^{-1}(Yp_i)\big)\right\}. \tag{22}$$

Further, μ_i can be uniquely solved from (16). Therefore, fixing (Y, V^Δ, V_L), there exists a unique pair (p_i, μ_i) to satisfy (15) and (16), which we denote by $\big(p_i(Y, V^\Delta, V_L), \mu_i(Y, V^\Delta, V_L)\big)$ with slight abuse of notation.

Step II We show that fixing (V^Δ, V_L) and $\big\{p_i(Y, V^\Delta, V_L), \mu_i(Y, V^\Delta, V_L)\big)\}_{i\in\mathcal{N}}$, there exists $Y \geq \sum_{i\in\mathcal{N}} f_i(0)$ to satisfy (14). By definition of $p_i(Y, V^\Delta, V_L)$, we have that

$p_i(Y, V^\Delta, V_L) \geq \frac{f_i(0)}{Y}$, which implies

$$\sum_{i \in \mathcal{N}} p_i \left(\sum\nolimits_{j \in \mathcal{N}} f_j(0), V^\Delta, V_L \right) \geq 1.$$

Next, we claim that

$$\lim_{Y \to +\infty} \sum_{i \in \mathcal{N}} p_i(Y, V^\Delta, V_L) = 0.$$

To see this, first consider the case of $f_i'(0) < +\infty$. Then (20) holds as Y approaches infinity, in which case $p_i = \frac{f_i(0)}{Y}$ and

$$\lim_{Y \to +\infty} p_i(Y, V^\Delta, V_L) = \lim_{Y \to +\infty} \frac{f_i(0)}{Y} = 0.$$

Next, consider the case of $f_i'(0) = +\infty$. Then (21) holds for all Y and $p_i(Y, V^\Delta, V_L)$ solves (22). As Y approaches infinity, the right-hand side of (22) approaches infinity; therefore, the left-hand side must be finite, which indicates that $p_i(Y, V^\Delta, V_L)$ approaches 0.

By the intermediate value theorem, there exists $Y \geq \sum_{i \in \mathcal{N}} f_i(0)$ such that

$$\sum_{i \in \mathcal{N}} p_i(Y, V^\Delta, V_L) = 1.$$

Fixing (V^Δ, V_L), we denote the largest Y that solves the above equation by $Y(V^\Delta, V_L)$ in the subsequent analysis.

Step III Fixing V_L, $Y(V^\Delta, V_L)$, and $\{p_i(Y, V^\Delta, V_L), \mu_i(Y, V^\Delta, V_L)\}_{i \in \mathcal{N}}$, we show that there exists V^Δ such that (17) holds, i.e.,

$$\sum_{i \in \mathcal{N}} \mu_i \left(Y(V^\Delta, V_L), V^\Delta, V_L \right) = k - 1. \tag{23}$$

First, consider the case in which V^Δ approaches 0. For each $i \in \mathcal{N}$, when $p_i = \frac{f_i(0)}{Y}$, we have that

$$\lim_{V^\Delta \searrow 0} \mu_i \left(Y(V^\Delta, V_L), V^\Delta, V_L \right)$$

$$= \lim_{V^\Delta \searrow 0} \frac{1}{V^\Delta} \text{med} \left\{ 0, V^\Delta \left(1 - \frac{f_i(0)}{Y(V^\Delta, V_L)} \right), \frac{V^\Delta}{\delta_i} - \frac{f_i(0)}{Y(V^\Delta, V_L)} (V_L + V^\Delta) \right\} = 0,$$

where the second equality follows from the fact that $\frac{V^\Delta}{\delta_i} - \frac{f_i(0)}{Y(V^\Delta, V_L)}(V_L + V^\Delta) \leq 0 \leq V^\Delta \big(1 - \frac{f_i(0)}{Y(V^\Delta, V_L)}\big)$ as V^Δ approaches 0.

When $p_i > \frac{f_i(0)}{Y}$, by (16), $\mu_i \left(Y(V^\Delta, V_L), V^\Delta, V_L \right) = 0$ for sufficiently small V^Δ. Therefore, we have that

$$\lim_{V^\Delta \searrow 0} \sum_{i \in \mathcal{N}} \mu_i \left(Y(V^\Delta, V_L), V^\Delta, V_L \right) = 0.$$

Next, consider the case in which V^Δ approaches infinity. For each $i \in \mathcal{N}$, we have that

$$\begin{aligned} 0 \leq & V^\Delta \left[1 - p_i \left(Y(V^\Delta, V_L), V^\Delta, V_L \right) \right] \\ \leq & \frac{V^\Delta}{\delta_i} - p_i \left(Y(V^\Delta, V_L), V^\Delta, V_L \right) (V_L + V^\Delta) + c_i \left(f_i^{-1} \left(Y(V^\Delta, V_L) p_i \left(Y(V^\Delta, V_L), V^\Delta, V_L \right) \right) \right); \end{aligned}$$

together with (16), we can obtain that

$$\mu_i \left(Y(V^\Delta, V_L), V^\Delta, V_L \right) = 1 - p_i \left(Y(V^\Delta, V_L), V^\Delta, V_L \right), \text{ as } V^\Delta \to +\infty.$$

Therefore, we have that

$$\lim_{V^\Delta \to +\infty} \sum_{i \in \mathcal{N}} \mu_i \left(Y(V^\Delta, V_L), V^\Delta, V_L \right) = \lim_{V^\Delta \to +\infty} \sum_{i \in \mathcal{N}} \left[1 - p_i \left(Y(V^\Delta, V_L), V^\Delta, V_L \right) \right] = n-1.$$

Note that $\mu_i(Y, V^\Delta, V_L)$ and $Y(V^\Delta, V_L)$ are continuous for all $i \in \mathcal{N}$, and $0 \leq k-1 \leq n-1$. It follows immediately that there exists $V^\Delta \geq 0$ to satisfy (23). In what follows, fixing V_L, let us denote the largest V^Δ that solves (23) by $V^\Delta(V_L)$.

Step IV We show that there exists $V_L \in [0, 1]$ to satisfy (18), i.e.,

$$\sum_{i \in \mathcal{N}_1} \frac{\delta_i}{1 - \delta_i} \left[p_i V_L - c_i \left(f_i^{-1}(Y p_i) \right) \right] + \left(k - |\mathcal{N}_1| \right) V^\Delta + V_L = 1, \tag{24}$$

where $V^\Delta = V^\Delta(V_L)$, $Y = Y(V^\Delta, V_L)$, and $p_i = p_i(Y, V^\Delta, V_L)$ for $i \in \mathcal{N}$, as defined above.

Note that the left-hand side of (24) is always nonnegative; moreover, it is evident that the left-hand side is no less than 1 when $V_L = 1$. To conclude the proof, it suffices to show that $\lim_{V_L \searrow 0} V^\Delta(V_L) = 0$, from which we can conclude that the left-hand side of (24) approaches 0 as $V_L \searrow 0$.

Suppose, to the contrary, that $\limsup_{V_L \searrow 0} V^\Delta(V_L) > 0$. It can then be verified that the following strict inequality holds as $V_L \searrow 0$ and $V^\Delta \to \limsup_{V_L \searrow 0} V^\Delta(V_L)$:

$$V^\Delta(1 - p_i) < \frac{V^\Delta}{\delta_i} - p_i(V_L + V^\Delta) + c_i\left(f_i^{-1}(Yp_i)\right), \ \forall\, i \in \mathcal{N}.$$

Recall that $\mathcal{N}_2$ is nonempty by definition. That is, there exists some agent $j \in \mathcal{N}_2$. By (16), we have that

$$V^\Delta(1 - p_j) \geq \frac{V^\Delta}{\delta_j} - p_j(V_L + V^\Delta) + c_j\left(f_j^{-1}(Yp_j)\right).$$

A contradiction. ■

Derivation for Equilibria in Example 1

Proof. First, consider the case of $k = 1$. The game reduces to a static Tullock contest. Let $Y := \sum_{i \in \mathcal{N}} \eta_i x_i$. The equilibrium conditions can be derived as

$$Y = 1 - p_i,$$

from which we can solve for the equilibrium aggregate effort Y, the equilibrium recognition probabilities $\boldsymbol{p} = (p_1, p_2, p_3, p_4)$, and the equilibrium efforts $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$ as follows:

$$Y = \frac{3}{4},$$

$$p_i = 1 - Y = \frac{1}{4}, \ \forall i \in \mathcal{N},$$

and

$$\boldsymbol{x} = Y\boldsymbol{p} \oslash \boldsymbol{\eta} = \left(\frac{3}{16}, \frac{15}{16}, \frac{15}{16}, \frac{15}{16}\right).$$

Next, consider the case of $k = 2$. The equilibrium conditions in the proof of Theorem 1—i.e., conditions (13), (14), (15), (16), (17), and (18)—for this example can be expressed as follows:

$$Yp_i = \eta_i x_i,$$

$$\sum_{i\in\mathcal{N}} p_i = 1,$$

$$Y = (1-p_i)(V_L + V^{\Delta}) - \mu_i V^{\Delta},$$

$$\mu_i = \frac{1}{\delta_i} - p_i - \frac{p_i V_L - Y p_i}{V^{\Delta}},$$

$$\sum_{i\in\mathcal{N}} \mu_i = 1,$$

$$\frac{1}{9}\left[p_1 V_L - Y p_1\right] + V_L + V^{\Delta} = 1.$$

It can be verified that $\boldsymbol{p} = (0.2322, 0.2559, 0.2559, 0.2559)$, $\boldsymbol{x} = (0.1711, 0.9433, 0.9433, 0.9433)$, $\boldsymbol{\mu} = (0.7678, 0.0774, 0.0774, 0.0774)$, $V_L = 0.9600$, and $V^{\Delta} = 0.0342$ constitute an SSPE of the game. The equilibria for the cases of $k = 3$ and $k = 4$ can be similarly verified. ∎

Proof of Theorem 2 and Corollary 1

Proof. We first prove Theorem 2. By (1), (4), and (8), we have

$$p_i = \frac{\alpha_i h_i(x_i) + \beta_i}{\sum_{j\in\mathcal{N}}\left[\alpha_j h_j(x_j) + \beta_j\right]},$$

and

$$c_i'(x_i)\frac{\alpha_i h_i(x_i) + \beta_i}{\alpha_i h_i'(x_i)} \geq p_i(1-p_i)\left(V_L + \frac{(1-p_i-\mu_i)V^{\Delta}}{1-p_i}\right), \tag{25}$$

with equality holding if $x_i > 0$.

We construct $(\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}})$ as follows. For $x_i = 0$, we set $(\hat{\alpha}_i, \hat{\beta}_i) = (0, p_i)$. For $x_i > 0$, note by (12) that we have that

$$1 = \sum_{i\in\mathcal{N}_1}(\delta_i v_i) + (k - |\mathcal{N}_1|)V^{\Delta} + V_L \geq V^{\Delta} + V_L, \tag{26}$$

where the inequality follows from $v_i \geq 0$ and $|\mathcal{N}_1| \leq k-1$. Combining (25) and (26) yields

$$\frac{c_i'(x_i)h_i(x_i)}{h_i'(x_i)} \leq c_i'(x_i)\frac{\alpha_i h_i(x_i)+\beta_i}{\alpha_i h_i'(x_i)} = p_i(1-p_i)\left(V_L + \frac{(1-p_i-\mu_i)V^{\Delta}}{1-p_i}\right) \leq p_i(1-p_i).$$

Define $\hat{\theta}_i := p_i(1-p_i)h_i'(x_i)/c_i'(x_i) - h_i(x_i)$. The above inequality indicates $\hat{\theta}_i \geq 0$. Set

$$\left(\hat{\alpha}_i, \hat{\beta}_i\right) := \left(\frac{p_i}{h_i(x_i)+\hat{\theta}_i}, \hat{\alpha}_i\hat{\theta}_i\right). \tag{27}$$

It remains to verify that $(\boldsymbol{x}, \boldsymbol{p})$ constitutes the unique equilibrium effort profile and recognition probabilities under $(\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}, 1)$. When $k = 1$, the game degenerates to a standard static contest with prize value of 1. It suffices to show that the equilibrium recognition probability p_i satisfies

$$p_i = \frac{\hat{\alpha}_i h_i(x_i)+\hat{\beta}_i}{\sum_{j\in\mathcal{N}}\left[\hat{\alpha}_j h_j(x_j)+\hat{\beta}_j\right]}, \tag{28}$$

and x_i solves

$$\max_{x_i\geq 0} \frac{\hat{\alpha}_i h_i(x_i)+\hat{\beta}_i}{\sum_{j\in\mathcal{N}}\left[\hat{\alpha}_j h_j(x_j)+\hat{\beta}_j\right]} - c_i(x_i). \tag{29}$$

Note that $p_i = \hat{\alpha}_i h_i(x_i) + \hat{\beta}_i$ for all $i \in \mathcal{N}$ by construction (see, e.g., (27)). Therefore, $\sum_{j\in\mathcal{N}}(\hat{\alpha}_j h_j(x_j) + \hat{\beta}_j) = \sum_{j\in\mathcal{N}} p_j = 1$, which implies (28).

Next, we verify that x_i solves the maximization problem (29). For agent $i \in \mathcal{N}$ with $x_i = 0$, it is evident that choosing $x_i = 0$ dominates $x_i > 0$ under $(\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}, 1)$ because $\hat{\alpha}_i = 0$. For agent $i \in \mathcal{N}$ with $x_i > 0$, by (27), we have that

$$c_i'(x_i)\frac{\hat{\alpha}_i h_i(x_i)+\hat{\beta}_i}{\hat{\alpha}_i h_i'(x_i)} = c_i'(x_i)\frac{h_i(x_i)+\hat{\theta}_i}{h_i'(x_i)} = p_i(1-p_i),$$

which is exactly the first-order condition for the maximization problem (29).

The above analysis shows that the optimum can be achieved by $k = 1$, in which case the game reduces to a standard static contest. By Theorem 2 in Fu and Wu (2020), the optimum can be achieved by choosing multiplicative biases $\boldsymbol{\alpha}$ only and setting headstart $\boldsymbol{\beta}$ to zero if fixing $\boldsymbol{p}$, $\Lambda(\boldsymbol{x}, \boldsymbol{p})$ weakly increases with x_i for all $i \in \mathcal{N}$. ■

Derivation for the Optimal Voting Rule in Example 2

Proof. By Theorem 2, the optimum can be achieved by setting $k = 1$. It remains to show that the optimum can be achieved by setting $k = 5$, but not $k = 4$. Evidently, when λ is sufficiently large, the optimum requires that $\boldsymbol{p} = \tilde{\boldsymbol{p}}$. Moreover, when γ is sufficiently large, each agent i's equilibrium effort x_i cannot exceed $\tilde{x}_i$. Therefore, it suffices to show that fixing $\boldsymbol{p} = \tilde{\boldsymbol{p}}$, the equilibrium effort is $\boldsymbol{x} = \tilde{\boldsymbol{x}}$ at $k = 5$, and the designer cannot induce $\boldsymbol{x} = \tilde{\boldsymbol{x}}$ at $k = 4$.

When $k = 5$, it can be verified that $\mathcal{N}_1 = \{1, 2, 3\}$, $\mathcal{N}_2 = \{4, 5, 6\}$, and $\mathcal{N}_3 = \{7\}$. Moreover, the equilibrium effort is $\tilde{\boldsymbol{x}}$, and the equilibrium winning probability is $\tilde{\boldsymbol{p}}$. In this case, agent 7's effective prize spread is $V_L + V^\Delta = 0.8399$, and his first-order condition holds with equality:

$$r\tilde{x}_7 c_7'(\tilde{x}_7) = (V_L + V^\Delta)\tilde{p}_7(1 - \tilde{p}_7).$$

Next, we show that when $k = 4$, the designer cannot induce $\boldsymbol{p} = \tilde{\boldsymbol{p}}$ and $\boldsymbol{x} = \tilde{\boldsymbol{x}}$ simultaneously. In fact, fixing $\boldsymbol{p} = \tilde{\boldsymbol{p}}$ and $\boldsymbol{x} = \tilde{\boldsymbol{x}}$, by (16)-(18), we have that $V_L = 0.7439$ and $V^\Delta = 0.0669$, with $V_L + V^\Delta < 0.8399$. However, agent 7's first-order condition requires that

$$r\tilde{x}_7^r = \tilde{x}_7 c_7'(\tilde{x}_7) = 0.8399 \times \tilde{p}_7(1 - \tilde{p}_7) \le (V_L + V^\Delta)\tilde{p}_7(1 - \tilde{p}_7).$$

A contradiction. ■

Proof of Theorem 3

Proof.

Plugging (10) into (11), we have that

$$\sum_{i \in \mathcal{N}_2} \left[\left(\frac{1}{\delta_i} - p_i^* \right) V^\Delta - p_i^* V_L + c_i(x_i^*) \right] + \sum_{i \in \mathcal{N}_1} (1 - p_i^*) = (k-1)V^\Delta. \tag{30}$$

Recall by (18), we have that

$$\sum_{i \in \mathcal{N}_1} \frac{\delta_i}{1 - \delta_i} \left[p_i^* V_L - c_i(x_i^*) \right] + \left(k - |\mathcal{N}_1| \right) V^\Delta + V_L = 1. \tag{31}$$

Note that holding fixed $(\boldsymbol{x}^*, \boldsymbol{p}^*)$, we can adjust the contest rule to satisfy the above two equilibrium conditions as the voting rule k varies, which yields a new pair (V^Δ, V_L). To prove the theorem, it remains to verify the following first-order condition under the less inclusive voting rule $k-1$ and the new pair (V^Δ, V_L):

$$\frac{c_i'(x_i^*)h_i(x_i^*)}{h_i'(x_i^*)} \le p_i^*(1-p_i^*)\left[V_L + V^\Delta - \frac{\mu_i}{1-p_i^*}V^\Delta\right], \forall\, i \in \mathcal{N}.$$

Evidently, it suffices to show that the effective prize spread, $V_L + V^\Delta - \frac{\mu_i}{1-p_i^*}V^\Delta$, is non-increasing in k.

We treat k as a continuous variable. Clearly, μ_i, V^Δ, and V_L are all continuous in k. Moreover, for all but finitely many values of k, the sets $\mathcal{N}_1$, $\mathcal{N}_2$, and $\mathcal{N}_3$ remain unchanged in a neighborhood of k, which indicates that μ_i, V^Δ, and V_L are differentiable with respect to k. Therefore, it suffices to show that the derivative of the effective prize spread, $V_L + V^\Delta - \frac{\mu_i}{1-p_i^*}V^\Delta$, with respect to k is nonpositive whenever it is differentiable. Taking the derivatives of (30) and (31) with respect to k yields that

$$\frac{dV_L}{dk} = -\frac{(\mathcal{B}+\mathcal{D})V^\Delta}{\mathcal{A}\mathcal{D}+\mathcal{B}\mathcal{C}} \text{ and } \frac{dV^\Delta}{dk} = -\frac{(\mathcal{C}-\mathcal{A})V^\Delta}{\mathcal{A}\mathcal{D}+\mathcal{B}\mathcal{C}},$$

where

$$\begin{aligned}
\mathcal{A} :=& 1 + \sum_{i\in\mathcal{N}_1} \frac{p_i^*\delta_i}{1-\delta_i} > 0, \\
\mathcal{B} :=& k - |\mathcal{N}_1| > 0, \\
\mathcal{C} :=& \sum_{i\in\mathcal{N}_2} p_i^* > 0, \\
\mathcal{D} :=& \sum_{i\in\mathcal{N}_1}(1-p_i^*) + \sum_{i\in\mathcal{N}_2}\left(\frac{1}{\delta_i} - p_i^*\right) - (k-1) > 0.
\end{aligned}$$

Evidently, $\mathcal{A}\mathcal{D}+\mathcal{B}\mathcal{C} > 0$ and $\mathcal{B}+\mathcal{D} > 0$. Moreover, we have that

$$\mathcal{C}-\mathcal{A} = \sum_{i\in\mathcal{N}_2} p_i^* - 1 - \sum_{i\in\mathcal{N}_1}\frac{p_i^*\delta_i}{1-\delta_i} \le -\sum_{i\in\mathcal{N}_1}\frac{p_i^*\delta_i}{1-\delta_i} \le 0.$$

Therefore, we have that

$$\frac{dV_L}{dk} = -\frac{(\mathcal{B}+\mathcal{D})V^{\Delta}}{\mathcal{A}\mathcal{D}+\mathcal{B}\mathcal{C}} \le 0 \text{ and } \frac{dV^{\Delta}}{dk} = -\frac{(\mathcal{C}-\mathcal{A})V^{\Delta}}{\mathcal{A}\mathcal{D}+\ \mathcal{B}\mathcal{C}} \ge 0. \tag{32}$$

We consider the following three cases.

Case I: $i \in \mathcal{N}_1$. By (10), $\mu_i = 1 - p_i^*$ and the effective prize spread is V_L. We can then conclude from (32) that V_L is non-increasing in k.

Case II: $i \in \mathcal{N}_2$. By (10), the effective prize spread is

$$\begin{aligned}
V_L + V^{\Delta} - \frac{\mu_i}{1-p_i^*}V^{\Delta} =& V_L + V^{\Delta} - \frac{V^{\Delta}}{\delta_i(1-p_i^*)} + \frac{p_i^*}{1-p_i^*}(V_L+V^{\Delta}) - c_i(x_i^*)\\
=& \frac{V_L}{1-p_i^*} - \frac{(1-\delta_i)V^{\Delta}}{\delta_i(1-p_i^*)} - c_i(x_i^*).
\end{aligned}$$

Carrying out the algebra, we can obtain that

$$\frac{d}{dk}\left(V_L + V^{\Delta} - \frac{\mu_i}{1-p_i^*}V^{\Delta}\right) = \frac{1}{1-p_i^*} \times \frac{dV_L}{dk} - \frac{1-\delta_i}{\delta_i(1-p_i^*)} \times \frac{dV^{\Delta}}{dk} \le 0.$$

Case III: $i \in \mathcal{N}_3$. By (10), $\mu_i = 0$ and the effective prize spread is $V_L + V^{\Delta}$. By (32), we have that

$$\frac{d(V_L+V^{\Delta})}{dk} = \frac{(\mathcal{B}+\mathcal{C}+\mathcal{D}-\mathcal{A})V^{\Delta}}{\mathcal{A}\mathcal{D}+\mathcal{B}\mathcal{C}}.$$

It remains to prove

$$\mathcal{B}+\mathcal{C}+\mathcal{D}-\mathcal{A} \ge 0.$$

Carrying out the algebra, we have that

$$\begin{aligned}
\mathcal{B}+\mathcal{C}+\mathcal{D}-\mathcal{A} =& k - |\mathcal{N}_1| + \sum_{i\in\mathcal{N}_2} p_i^* + \sum_{i\in\mathcal{N}_1}(1-p_i^*) + \sum_{i\in\mathcal{N}_2}\left(\frac{1}{\delta_i} - p_i^*\right) - (k-1) - 1 - \sum_{i\in\mathcal{N}_1}\frac{p_i^*\delta_i}{1-\delta_i}\\
=& \sum_{i\in\mathcal{N}_2}\frac{1}{\delta_i} - \sum_{i\in\mathcal{N}_1}\frac{p_i^*}{1-\delta_i}\\
\ge& 2|\mathcal{N}_2| - 2\sum_{i\in\mathcal{N}_1} p_i^* \ge 0,
\end{aligned}$$

where the first inequality follows from $\delta_i \le \frac{1}{2}$. This concludes the proof. ■

Appendix B: Optimal Recognition Mechanism Holding Fixed the Voting Rule k

We provide an example to show that the results established in the contest design literature may fail to hold within our context when the voting rule is nondictatorial.

Example 3 *Suppose that* $n = 3$, $k = 2$, $h_i(x_i) = x_i$, *and* $c_i(x_i) = c_i x_i$ *with* $(c_1, c_2, c_3) = (1, 1, c)$. *Let* $(\delta_1, \delta_2, \delta_3) = (\frac{3}{8}, \frac{1}{2}, \frac{12}{13})$. *The designer choose* $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ *for a given* k *to maximize* $\Lambda(\boldsymbol{x}, \boldsymbol{p}) = \sum_{i\in\mathcal{N}} x_i - \lambda \sum_{i\in\mathcal{N}} \left|p_i - \frac{1}{n}\right|$, *with* $\lambda > 0$. *Assume that* λ *is sufficiently large and* c *is sufficiently small, with* $\lambda \gg \frac{1}{c} \gg 1$.

The objective function can be maximized by a recognition mechanism with $\boldsymbol{\alpha}^* = (\frac{62Y}{35}, \frac{62Y}{37}, \frac{62Yc}{39})$ *and* $\boldsymbol{\beta}^* = (0, \frac{17Y}{222}, 0)$, *where* $Y > 0$ *is an arbitrary positive constant. The game yields an equilibrium outcome of* $\boldsymbol{x} = (\frac{70}{372}, \frac{57}{372}, \frac{78}{372c})$ *and* $\boldsymbol{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. *The designer's payoff is* $\Lambda = \frac{127}{372} + \frac{78}{372c}$.

	Agent 1	Agent 2	Agent 3
Equilibrium efforts	70/372	57/372	$78/(372c)$
Winning probability	1/3	1/3	1/3
Equilibrium payoff	56/372	72/372	39/372
Winning coalition	$\{1, 2\}$	$\{1, 2\}$	$\{1, 3\}$

Table 2: Equilibrium Outcomes in Example 3.

Notably, the designer in Example 3 awards a positive headstart to agent 2. In contrast, with a fixed contest prize and an objective function $\Lambda(\boldsymbol{x}, \boldsymbol{p})$ weakly increasing with x_i for each $i \in \mathcal{N}$, Fu and Wu (2020) establish the suboptimality of a headstart and show that adjusting $\boldsymbol{\alpha}$ alone suffices to maximize $\Lambda(\boldsymbol{x}, \boldsymbol{p})$.

With $c_1 = c_2 > c_3$ and $\delta_1 < \delta_2 < \delta_3$, agent 3 is ex ante the strongest contender, followed by agent 2, then agent 1. The designer would benefit if agent 3 can be sufficiently incentivized given his low effort cost, which requires a larger prize spread for the agent. For this purpose, the designer can seek to reduce agent 1's continuation value, which decreases agent 3's vote-buying cost—i.e., w_3—given that by Table 2, agent 3 would include agent 1 in his winning coalition.

Further, by Table 2, agent 1 would buy agent 2's vote upon being the proposer. The designer can increase agent 2's continuation value to render agent 1 worse off, which can be achieved by awarding agent 2 either a headstart $\beta_2 > 0$ or a larger

α_2. The former is more effective: Both increase agent 2's recognition probabilities and improve his payoffs. However, a larger α_2 increases the marginal benefit of effort, which promotes his effort supply; effort is costly and, in turn, reduces agent 2's payoff, which (partially) offsets the payoff-improving effect of a larger α_2.

Multiplicative biases $\boldsymbol{\alpha}$ can more effectively motivate efforts due to their direct impact on the marginal benefits of efforts, which renders headstarts $\boldsymbol{\beta}$ redundant in Fu and Wu (2020). In contrast, varying $\boldsymbol{\beta}$ creates an opportunity to exploit the endogenous payoff structure of the game in our context due to the indirect prize effect. Such nuances complicate the roles played by biased contest rules.

Derivation for the Optimal Recognition Mechanism in Example 3

Proof. We demonstrate the optimality of $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ in Example 3. When the designer sufficiently cares about the profile of agents' recognition probabilities—i.e., when $\lambda \gg 1/c$—the optimal equilibrium winning probability profile must be $\boldsymbol{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and the designer's payoff at $\boldsymbol{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ reduces to $\Lambda = x_1 + x_2 + x_3$. When c is sufficiently small, agent 3 is excessively strong and the designer's payoff is mainly determined by x_3. Therefore, it suffices to show that $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ maximizes x_3 among all rules $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ that induce $\boldsymbol{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Fix $\boldsymbol{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. We first rewrite the equilibrium conditions in the proof of Theorem 1—i.e., conditions (13)-(18). Evidently, condition (14) is satisfied and condition (13) becomes

$$\alpha_i^* x_i + \beta_i^* = \frac{Y}{3}, \ \forall\, i \in \{1, 2, 3\}. \tag{33}$$

Next, consider condition (15). The condition holds with equality for $x_i > 0$. Further, if $x_i = 0$ for some $i \in \mathcal{N}$ and the strict inequality holds, we can increase α_i until the equality holds and at the same time keep unchanged the equilibrium effort profile $\boldsymbol{x}$ and recognition probabilities $\boldsymbol{p}$. Therefore, we can assume that equality holds for all agents and the condition becomes

$$\frac{Y c_i}{\alpha_i^*} = \frac{2(V_L + V^{\Delta})}{3} - \mu_i V^{\Delta}, \ \forall\, i \in \{1, 2, 3\}. \tag{34}$$

Substituting (34) into (33) yields

$$3 c_i x_i \leq \frac{2(V_L + V^{\Delta})}{3} - \mu_i V^{\Delta}, \ \forall\, i \in \{1, 2, 3\}, \tag{35}$$

with equality holding if $\beta_i^* = 0$. To establish the optimality of headstarts, it suffices to show that the inequality is strict for at least one agent.

Conditions (16), (17), and (18) are

$$\mu_i = \begin{cases} \frac{2}{3} \le \frac{1}{\delta_i} - \frac{1}{3} - \frac{V_L}{3V^\Delta} + \frac{c_i x_i}{V^\Delta}, & i \in \mathcal{N}_1, \\ \frac{1}{\delta_i} - \frac{1}{3} - \frac{V_L}{3V^\Delta} + \frac{c_i x_i}{V^\Delta} \in \left[0, \frac{2}{3}\right], & i \in \mathcal{N}_2, \\ 0 \ge \frac{1}{\delta_i} - \frac{1}{3} - \frac{V_L}{3V^\Delta} + \frac{c_i x_i}{V^\Delta}, & i \in \mathcal{N}_3, \end{cases} \tag{36}$$

$$\mu_1 + \mu_2 + \mu_3 = 1, \tag{37}$$

and

$$\sum_{i \in \mathcal{N}_1} \frac{\delta_i}{1-\delta_i} \left(\frac{V_L}{3} - c_i x_i \right) + \left(2 - |\mathcal{N}_1|\right) V^\Delta + V_L = 1. \tag{38}$$

Substituting (36) in (35) yields that

$$c_i x_i \le \begin{cases} \frac{2}{9} V_L, & i \in \mathcal{N}_1, \\ \frac{1}{4} \left[V_L - (\frac{1}{\delta_i} - 1) V^\Delta \right], & i \in \mathcal{N}_2, \\ \frac{2}{9}(V_L + V^\Delta), & i \in \mathcal{N}_3, \end{cases} \tag{39}$$

from which we can conclude $c_i x_i \le \frac{2V_L}{9}$ for $i \in \mathcal{N}_1$; together with (38), we have that

$$\sum_{i \in \mathcal{N}_1} \frac{\delta_i}{1-\delta_i} \times \frac{V_L}{9} + \left(2 - |\mathcal{N}_1|\right) V^\Delta + V_L \le 1. \tag{40}$$

In what follows, we will show that $c_3 x_3 \le \frac{30}{144-\delta_3}$, and the equality holds if and only if $\boldsymbol{\alpha}^* = (\frac{62Y}{35}, \frac{62Y}{37}, \frac{62Yc}{39})$ and $\boldsymbol{\beta}^* = (0, \frac{17Y}{222}, 0)$. Consider the following three cases.

Case I: $3 \in \boldsymbol{\mathcal{N}_1}$. Note that $|\mathcal{N}_1| \le k - 1 = 1$, we have that $\mathcal{N}_1 = \{3\}$. By (40), we can obtain that

$$\left[1 + \frac{\delta_3}{9(1-\delta_3)}\right] V_L + V^\Delta \le 1;$$

together with (38), we can obtain that

$$c_3 x_3 \le \frac{2V_L}{9} \le \frac{2(1-\delta_3)}{9 - 8\delta_3} < \frac{30}{144 - \delta_3}.$$

Case II: $3 \in \mathcal{N}_2$. By (36) and (39), we have that

$$0 \leq \frac{1}{\delta_3} - \frac{1}{3} - \frac{V_L}{3V^\Delta} + \frac{c_3 x_3}{V^\Delta} \leq \frac{1}{\delta_3} - \frac{1}{3} - \frac{V_L}{3V^\Delta} + \frac{V_L - (\frac{1}{\delta_3} - 1)V^\Delta}{4V^\Delta}.$$

Carrying out the algebra, we can obtain that

$$V_L \leq \left(\frac{9}{\delta_3} - 1\right) V^\Delta = \frac{35}{4} V^\Delta. \tag{41}$$

Further, $3 \notin \mathcal{N}_1$ implies that $\mathcal{N}_1 \in \big\{\{1\}, \{2\}, \emptyset\big\}$, and thus (40) becomes

$$1 \geq \left\{ \begin{array}{ll} \frac{16}{15} V_L + V^\Delta, & \text{if } \mathcal{N}_1 = \{1\} \\ \frac{10}{9} V_L + V^\Delta, & \text{if } \mathcal{N}_1 = \{2\} \\ V_L + 2V^\Delta, & \text{if } \mathcal{N}_1 = \emptyset \end{array} \right\} \geq \frac{16}{15} V_L + V^\Delta, \tag{42}$$

where the last inequality follows from (41).

Combining (39), (41), and (42), we have that

$$c_3 x_3 \leq \frac{1}{4}\left[V_L - (\frac{1}{\delta_3} - 1)V^\Delta\right] \leq \frac{30}{144 - \delta_3} = \frac{13}{62}.$$

Note that equality holds in condition (39) if and only if $\beta_3^* = 0$. Further, equality holds in condition (41) only if $\mu_3 = 0$. Last, equality holds in condition (42) if and only if $\mathcal{N}_1 = \{1\}$ and $\beta_1^* = 0$.

Because $\mathcal{N}_1 = \{1\}$ and $\mu_3 = 0$, we have that $\mu_1 = \frac{2}{3}$ from (36); together with (37), we have $\mu_2 = \frac{1}{3}$. Moreover, by (36), we can conclude $2 \in \mathcal{N}_2$, which implies that $\mathcal{N}_2 = \{2, 3\}$ and $\mathcal{N}_3 = \emptyset$.

Combining (41) and (42) (recall that equality holds in these two conditions), we can obtain $V_L = \frac{105}{124}$ and $V^\Delta = \frac{3}{31}$; together with (39), we have $x_1 = \frac{2V_L}{9} = \frac{35}{186}$. Substituting $\mu_2 = \frac{1}{3}$, $V_L = \frac{105}{124}$ and $V^\Delta = \frac{3}{31}$ in (36), we can obtain that $x_2 = \frac{V_L - 4V^\Delta}{3} = \frac{19}{124}$.

Last, we solve for $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$. Recall that $\beta_i^* = 0$ for $i \in \{1, 3\}$. Therefore, $\alpha_i^* = \frac{Y}{3x_i}$ from (33). For $i = 2$, we have $x_2 = \frac{19}{124}$. Further, by (34), we have $\frac{Y}{\alpha_2^*} = \frac{2V_L + V^\Delta}{3} = \frac{37}{62}$, which implies that $\alpha_2^* = \frac{62Y}{37}$; together with (33), we can conclude that $\beta_2^* = \frac{Y}{3} - \alpha_2^* x_2 = \frac{17Y}{222}$.

In summary, the equality holds in $c_3 x_3 \leq \frac{30}{144 - \delta_3}$ if and only if $\boldsymbol{\alpha}^* = (\frac{62Y}{35}, \frac{62Y}{37}, \frac{62Yc}{39})$

and $\boldsymbol{\beta}^* = (0, \frac{17Y}{222}, 0)$, under which the equilibrium is $\boldsymbol{x} = (\frac{35}{186}, \frac{19}{124}, \frac{39}{186c})$, $\boldsymbol{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $\boldsymbol{\mu} = (\frac{2}{3}, \frac{1}{3}, 0)$, $V_L = \frac{105}{124}$, and $V^\Delta = \frac{3}{31}$.

Case III: $\boldsymbol{3 \in \mathcal{N}_3}$. Condition (36), together with the postulated $3 \in \mathcal{N}_3$, implies that $\mu_3 = 0$. Analogous to the derivation of (41), we can obtain that

$$V_L > \left(\frac{9}{\delta_3} - 1\right) V^\Delta = \frac{35}{4} V^\Delta. \tag{43}$$

First, suppose $\mathcal{N}_1 \neq \emptyset$. By (42), we have that

$$1 \geq \frac{16V_L}{15} + V^\Delta. \tag{44}$$

Combining (35), (43), and (44) yields that

$$c_3 x_3 \leq \frac{2(V_L + V^\Delta)}{9} < \frac{30}{144 - \delta_3}.$$

Next, suppose $\mathcal{N}_1 = \emptyset$; together with $3 \in \mathcal{N}_3$ and $k = 2$, we can conclude that $\mathcal{N}_2 = \{1, 2\}$. It follows from (38) that

$$V_L + 2V^\Delta = 1. \tag{45}$$

Recall $\mu_3 = 0$. Combining (36), (37), and (39), we can obtain that

$$1 = \mu_1 + \mu_2 = \frac{1}{\delta_1} + \frac{1}{\delta_2} - \frac{2}{3} - \frac{2V_L}{3V^\Delta} + \frac{x_1 + x_2}{V^\Delta} \leq 4 - \frac{2V_L}{3V^\Delta} + \frac{V_L}{2V^\Delta} - \frac{2}{3},$$

which in turn implies that

$$V_L \leq 14 V^\Delta. \tag{46}$$

Therefore,

$$c_3 x_3 \leq \frac{2(V_L + V^\Delta)}{9} \leq \frac{5}{24} < \frac{30}{144 - \delta_3},$$

where the first inequality follows from (35) and the second inequality from (45) and (46). ■

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