Decentralized Contest Design in Networks*

Qiang Fu† Zijia Wang‡ Zenan Wu§

Zeyu Xing¶ Yuxuan Zhu‖

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Abstract

Heterogeneous players are matched into interconnected pairwise contests across multiple battlefields. Each organizer independently sets her contest rules to maximize effort provision on her respective battlefield. The conventional wisdom of leveling the playing field may fail in this environment. However, an even-odds equilibrium always exists, in which all contests are resolved with equal winning probabilities. Further, we identify sufficient conditions—concerning contest technologies and network structure—that mitigate network externalities and restore the level-playing-field principle, such that each organizer prefers a fully balanced contest regardless of others' choices. We provide alternative sufficient conditions under which the even-odds equilibrium remains unique, even when an organizer does not necessarily prefer a fully balanced contest.

Keywords: Competing Contests; Contest Design; Networks.

JEL Classification Codes: C72, D74, D85.

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†Department of Strategy and Policy, National University of Singapore, Email: bizfq@nus.edu.sg.

‡School of Economics, Renmin University of China, Email: wangzijia@ruc.edu.cn.

§School of Economics, Peking University, Email: zenan@pku.edu.cn.

¶School of Economics, Peking University, Email: xingzeyu@pku.edu.cn.

‖School of Economics, Peking University, Email: zhuyuxuan@pku.edu.cn.

1 Introduction

Economic agents often compete across multiple fronts and are connected—directly or indirectly—through networks. One's action on a battlefield not only influences local outcomes but also generates spillover remotely and globally. Consider, for example, a tech firm participating in multiple R&D challenges. If the firm assigns its top research team to a prioritized project, its competitor on a parallel project may face less pressure and reallocate resources elsewhere; the firm's choice may thus affect competition in contests in which it is not directly involved and against rivals it does not confront head-to-head. A similar dynamic would arise, for instance, when professional athletes plan their seasons, with each conserving stamina for preferred tournaments. These interactions—among multiple agents across multiple battlefields—form a network of interconnected contests with complex externalities.

Significant scholarly efforts have been devoted to examining the strategic interactions over networks and exploring how the nature of the underlying game—e.g., strategic substitutability or complementarity—determine the equilibrium and optimal intervention (Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv, 2010; Galeotti, Golub, and Goyal, 2020). Contests in networks deserve to be examined systematically, given the nonmonotone best responses often inherent in contest games (Dixit, 1987), which yield unique and important implications for strategic analysis of the game and optimal design of contest rules.[1] This property, for instance, underpins the conventional wisdom of leveling the playing field: A more balanced contest—one that encourages the underdog to challenge the favorite—incentivizes greater efforts, which sheds light on a plethora of practices that aim to promote closer competitions. Consider, for instance, the handicap systems in golf tournaments and horse racing, as well as the various measures to support small and medium-sized enterprises (SME) set by the U.S. Small Business Administration (SBA) in federal procurement.[2]

The economics literature has formally established that, to incentivize effort, a contest should prevent the emergence of dominant players in equilibrium—thus en-

[1] As Dixit (1987) demonstrates, players' best responses are often nonmonotone in contest games: In contrast to Cournot or Bertrand competitions, one's effort choice is a strategic complement to that of his opponent when he is in the lead, while it is a strategic substitute when he is behind.

[2] Many horse-racing tournaments—e.g., the Grand National or the Melbourne Cup—require that horses with higher initial ratings carry heavier weight. A similar mechanism—aero handicap—is implemented in Formula One (F1) championships, in which teams that performed better in previous seasons are allocated reduced aerodynamic testing time.

suring that no contender can win with a probability exceeding 1/2, which implies that a two-player contest must be resolved with equal winning odds.[3] However, this level-playing-field principle is established in settings of standalone contests. Imagine instead an organizer who sets the contest rules on her own turf to attract effort and attention from contenders who face competing opportunities elsewhere—e.g., a buyer running an R&D challenge to seek for an innovative technical solution when other projects are available to a pool of overlapped bidders. She needs to account for contenders' strategic trade-offs across multiple contests, anticipate the choices of competing organizers, and understand how her choice of local contest rules is transmitted through the network. This complexity casts doubt on the conventional wisdom of leveling the playing field.

Consider a simple case with each competing player subject to a resource constraint. Suppose that an organizer handicaps the frontrunner. This generates a direct *local effect* that intensifies competitions in her own contest. Meanwhile, a player's increased effort on her battlefield may come at the expense of reduced effort elsewhere. This causes an indirect *network effect*, because the local shift in efforts alters contenders' marginal benefits and costs of efforts across all other contests. [4] The global impact requires that all contenders reset their effort choices everywhere. These spillovers may feed back to the original battlefield through the network, and the overall outcome is a priori unclear. Section 2.2 provides an intuitive example in which the indirect network effect prevails: An organizer may prefer an imbalanced contest over an even race, which violates the level-playing-field principle.

To our knowledge, this paper presents the first formal analysis of *decentralized* contest design in a networked contest game to explore the boundaries of the level-playing-field principle and shed further light on its nature in a broader context.

Snapshot of the Model Interactions are modeled as a two-stage game. In the first stage, multiple organizers simultaneously set the contest rules for their respective battlefields. In the second stage, economic agents—whom we refer to as "players"—are

[3]See, e.g., Fu and Wu (2020), who show in a broad context that the optimal two-player contest always yields equal winning probabilities in equilibrium; when the contest involves three or more players, the optimum requires that each player's equilibrium winning odds be strictly below 1/2.

[4]König, Rohner, Thoenig, and Zilibotti (2017) and Amarasinghe, Raschky, Zenou, and Zhou (2026), for example, provide empirical evidence of spillovers from changes in local conditions in networked conflicts. Cortes-Corrales and Gorny (2025) examine the unintended knock-on effects of varying one player's strength throughout a network.

matched into pairwise contests, with each modeled as a two-player generalized Tullock contest. Players simultaneously allocate efforts across the contests they participate in. Each player either incurs a convex cost based on the aggregate effort exerted across all contests he participates in (a "pure-cost" case) or is subject to a budget constraint (a "pure-budget" case). As a result, increasing effort on one battlefield limits the player's ability to contribute to others.

Players differ in their contest technologies, cost functions, or resource endowments. This creates room for each organizer to structure her contest in order to exploit players' heterogeneity. The organizer imposes a multiplicative bias on each player's impact function. This function translates effort into output, and the choice of biases determines players' relative competitiveness, which tilts the balance of the contest. Such biases encapsulate diverse mechanisms observed in real-world competition. It can be a weighted scoring rule, such as handicap systems in golf tournaments. It can also reflect the various measures of actual productivity interventions, including technical support (e.g., mentorship in Mozilla's Open Innovation Challenge); resource subsidies (e.g., the U.S. Department of Defense's Small Business Innovation Research program (Lichtenberg, 1990)); capacity-building for SMEs (e.g., counseling and training in federal procurement bids); and preferential industrial policies that target specific firms (Franke, Kanzow, Leininger, and Schwartz, 2013).

We characterize the subgame perfect Nash equilibrium (SPNE) of the two-stage game. Specifically, this paper addresses (i) whether and when an organizer prefers to set biases to level her playing field in a networked environment and (ii) whether balanced competitions emerge in equilibrium, as they do in standalone contests.

Technical Nature of the Analysis Analyzing the SPNE of this two-stage networked contest game presents several technical challenges. Players compete in parallel contests, and organizers are linked indirectly through the overlapped competitions. Understanding these intertwined decisions requires a comprehensive account of the network externalities and nuanced strategic interdependence at and across two different levels of strategic interactions—i.e., the networked contests and interactions among competing organizers who set rules for subsequent contests.

First, given a profile of biases set in the first stage, there is no closed-form solution for the second-stage equilibrium of the networked contest. Organizers' objective functions are only implicitly characterized by equilibrium conditions and cannot be directly used to construct explicit best-response mappings.

Second, to establish a given profile of contest rules as an SPNE, we have to verify that each organizer's choice constitutes a *global* best response to others' chosen biases. However, organizers' payoff are generally non-explicit and non-concave due to cross-battlefield externalities and strategic interdependencies. Standard first-order conditions are thus insufficient for proving global first-stage optimality.

Third, even if equilibrium existence can be established by verifying that every organizer has no profitable deviation from a candidate profile, proving uniqueness presents substantially greater analytical challenges. Ruling out all potential alternative equilibria across every subgame is analytically infeasible, given the lack of closed-form solutions and the complex interdependencies among battlefields.

Summary of Findings We develop novel methods to address these difficulties and obtain lucid results. We first identify a unique profile of contest rules under which all contests are resolved with equal winning probabilities; we show that this profile constitutes an SPNE, which we term the *even-odds equilibrium* (Theorem 1). This demonstrates that balanced competition remains an equilibrium outcome despite the complexity introduced by the network structure. To verify the existence of such an equilibrium, we resort to a global deviation argument: For any organizer, we consider a non-local deviation in her choice of biases and compare the resulting second-stage equilibria before and after the deviation. This approach allows us to verify global optimality without relying on closed-form solutions or concavity in organizers' payoff.

We then examine the uniqueness of the even-odds equilibrium. The analysis consists of two layers. First, we ask whether the level-playing-field principle holds in a network—i.e., whether an organizer would choose to level her battlefield regardless of contest rules elsewhere. We provide sufficient conditions for this to be the case, which automatically guarantee equilibrium uniqueness (Theorem 2). Specifically, this follows if either (i) each player's impact function (i.e., the function that maps effort into contest output) is sufficiently concave or (ii) the network is acyclic. These conditions mitigate the intricate indirect effects of a rule change on a battlefield: The former limits spillovers across battlefields, while the latter shuts down feedback loops. However, even when these conditions are not satisfied, uniqueness may still hold. We provide a set of weaker conditions under which equal winning odds arise on all battlefields, even if an individual organizer does not unconditionally prefer a level playing field (Theorem 3). The concavity of impact functions can be milder, and the network may contain cycles. These results demonstrate the critical roles played by contest technologies and prevailing network structures in shaping equilibrium outcomes, which we discuss in more detail after presenting the formal analysis (Section 3.3).

It is noteworthy that it is infeasible to examine all subgames to rule out non-even-odds equilibria in this framework. We develop a targeted approach to overcome the analytical challenge. Suppose that an alternative equilibrium exists with uneven winning odds. For any given bias profile, we identify the least balanced battlefield—which will be formally defined later—and assess whether the organizer of that battlefield can profitably deviate. Though the second-stage equilibrium is implicit, we are able to compare outcomes before and after any hypothetical deviation and show that a profitable deviation always exists. This construction rules out all candidate equilibria that do not induce even winning odds everywhere.

We also consider two extensions. First, we analyze a sequential-move version of the game in which organizers set contest rules in a predetermined order. Whenever the sufficient conditions for the level-playing-field principle hold in the baseline model, the equilibrium outcome remains invariant to the timing of moves: A unique SPNE exists, replicating the equilibrium rule profile of the simultaneous-move game (Theorem 4). Second, we consider a centralized organizer who sets contest rules for all battlefields to maximize an objective that strictly increases in each player's total effort. Again , a unique equilibrium emerges with equal winning rates across contests (Theorem 5). This yields a useful policy insight: Decentralized rule-setting by self-interested organizers can—under certain conditions—achieve the same outcome as centralized planning, which suggests the potential efficiency of delegation in contest design.

Link to Literature Our paper belongs to the extensive literature on strategic interactions among economic agents within networks, such as Ballester, Calvó-Armengol, and Zenou (2006); Bramoullé, Kranton, and D'amours (2014); Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv (2010); and Galeotti, Golub, and Goyal (2020). A growing body of literature focuses on contests and conflicts within networks (Dziubiński, Goyal, and Vigier, 2016). These studies vary significantly in their assumptions regarding the mechanisms that link players and structure contests or conflicts.

Dziubiński, Goyal, and Minarsch (2021), for instance, examine a conflict network in which a ruler attacks connected "nodes" successively to acquire and accumulate resources. Dziubiński, Goyal, and Zhou (2025) assume that each player's effort in one battlefield generates spillovers and helps him win the battles on neighboring battlefields. Goyal and Vigier (2014) study an attacker-defender game, in which the attacker and defender allocate combative efforts across nodes. König, Rohner, Thoenig, and Zilibotti (2017) assume that each player's entry in his (single) contest is determined by his own effort, positive spillovers from allies, and negative spillovers from enemies. In contrast, Hiller (2017) allows players to form either positive links (alliances that aid in conflict) or negative links (direct conflicts).

Our paper more closely relates to Franke and Öztürk (2015); Xu, Zenou, and Zhou (2022); and Li and Zhou (2025), in which multiple players are matched into contests across a network and allocate efforts among battlefields. Franke and Öztürk (2015) assume bilateral contests on each battlefield, focusing on specific network structures such as regular, complete bipartite, and star-shaped configurations. They adopt convex cost functions, in which increased effort on one battlefield raises effort costs elsewhere. Based on a conflict network à la Franke and Öztürk (2015), Cortes-Corrales and Gorny (2025) demonstrate how a change in a player's strength triggers spillovers throughout the network. Xu et al. (2022) substantially generalize the framework of Franke and Öztürk (2015) by allowing multilateral contests, budget constraints, and unrestricted network structures. They employ variational inequalities to overcome technical challenges, establish equilibrium existence, and provide conditions for equilibrium uniqueness. Under quadratic costs and bilateral contests on each battlefield, Li and Zhou (2025) examine comparative statics within acyclic networks, and analyze how shocks propagate throughout the network. Despite the lack of closed-form solutions, they remarkably demonstrate that comparative statics can be pinned down using sign functions.[5] [6]

Our paper differs from this literature in two significant respects. First, our model incorporates strategic interactions across two layers: Players compete within a network, while organizers indirectly interact through players' strategic effort choices in response to independently set contest rules. Second, we specifically focus on orga-

[5] Matros and Rietzke (2024) and Sun, Xu, and Zhou (2023) also contribute to this strand of the literature. Unlike Franke and Öztürk (2015), Matros and Rietzke (2024) require that each player commit to a single effort level that applies identically across all contests they participate in. Sun et al. (2023) analyze both constrained (uniform effort) and unconstrained effort allocation regimes. They show that, for Tullock contest success functions in semi-symmetric networks, the two regimes produce the same total effort and equilibrium payoffs.

[6] In addition to the theoretical contributions, several notable studies empirically examine conflicts in networks, including Jackson and Nei (2015), Berman and Couttenier (2015), König et al. (2017), Harari and Ferrara (2018), Berman et al. (2021), and Amarasinghe et al. (2026).

nizers' strategic choices of contest rules. Contest design is not considered by Franke and Öztürk (2015) or Xu et al. (2022). Li and Zhou's (2025) comparative statics offer useful insights for contest design. However, their analysis primarily concerns externalities that arise from interventions on an individual battlefield. Our analysis accommodates both decentralized and centralized contest designs, and enable simultaneous and sequential rule-setting across all battlefields. Dziubiński, Goyal, and Zhou (2025) explore the design of network structure to maximize efforts or utilities. They focus on a centralized design problem, while we assume a given network structure and let the contest rule on each battlefield be set by a respective organizer.

Our paper naturally connects to the literature on multi-battle contests (e.g., Kovenock and Roberson, 2012; Snyder, 1989; Klumpp and Polborn, 2006; Konrad and Kovenock, 2009; Fu, Lu, and Pan, 2015). In particular, this study is closely related to the research stream that examines players' allocation of scarce resources across battlefields, a line of work that dates back to Borel (1921) and Borel and Ville (1938). The discrete version of the game is known as the Colonel Blotto game, with notable contributions including Friedman (1958), Roberson (2006), Kvasov (2007), Kovenock and Roberson (2012), Roberson and Kvasov (2012), and Fu and Iyer (2019).[7]

Our paper distinguishes itself from these studies along three dimensions. First, we allow for multiple players to be matched in contests across a network, whereas the literature predominantly assumes that two players compete in every battlefield. Second, our model accommodates both a pure-cost case with strictly convex cost functions and a pure-budget case, as seen in most Colonel Blotto game studies. Third, and most importantly, the networked contests in our model are subgames that follow decentralized decisions by contest organizers; none of the aforementioned studies consider contest rule setting.[8]

Finally, each organizer in our model competes for players' effort investment on her own battlefield. This renders our paper conceptually linked to the literature on competing contests, with Azmat and Möller (2009, 2018) and Morgan, Sisak, and

[7] Friedman (1958) analyzes two firms that allocate fixed advertising budgets across multiple marketing areas. Roberson (2006) fully characterizes the equilibrium of a Colonel Blotto game. Kovenock and Roberson (2012) introduce asymmetric prize valuations. Kvasov (2007) and Roberson and Kvasov (2012) relax the zero-sum assumption and allow for alternative uses of resources. Fu and Iyer (2019) accommodate rent-augmenting investment other than rent-seeking efforts.

[8] Feng and Lu (2018) and Feng, Jiao, Kuang, and Lu (2024) also consider contest design. However, they adopt team-based contest structures as in Fu, Lu, and Pan (2015). Their focus lies in the decisions of a central planner who governs the entire contest architecture.

Várdy (2018) as leading contributions. Unlike our setting, these studies typically assume that each player chooses which contest to enter, so organizers compete for contestants' discrete entry decisions. Körpeoğlu, Korpeoglu, and Hafalır (2022) allow solvers to participate in multiple contests, but their focus lies in comparing exclusivity versus non-exclusivity in contest design.

2 Preliminaries

In this section, we first lay out the primitives of our model, then provide an example to illustrate the nuances caused by the network.

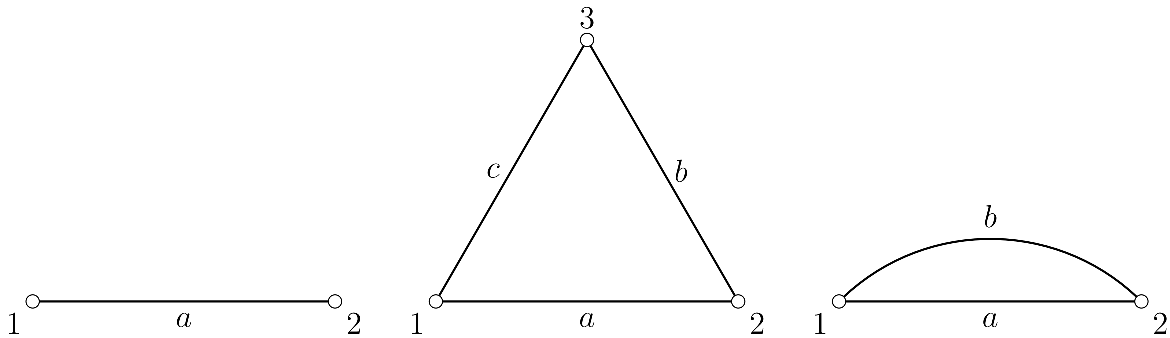
2.1 Model Setup

A finite set of risk-neutral players $\mathcal{N} \equiv \{1, 2, \ldots, N\}$ compete within a connected network.[9] For the sake of tractability, we focus on bilateral contests. Each player $i \in \mathcal{N}$ competes head-to-head with another on at least one battlefield. Let $\mathcal{E} \equiv \{a, b, \ldots\}$ denote the set of battlefields and $e \in \mathcal{E}$ an indicative battlefield. The network can then be represented by $\Gamma \subset \mathcal{N} \times \mathcal{E}$, where $(i, e) \in \Gamma$ if and only if player i is involved in the contest on battlefield e. Further, let $\mathcal{E}_i \equiv \{e \in \mathcal{E} : (i, e) \in \Gamma\}$ denote the set of battlefields with player i's participation and $\mathcal{N}^e \equiv \{i \in \mathcal{N} : (i, e) \in \Gamma\}$ the set of players who compete on battlefield e, with $|\mathcal{N}^e| = 2$ for all $e \in \mathcal{E}$.

The bilateral contest network Γ described above can model a rich class of interconnected contest games. Figure 1 depicts three examples. In each subfigure, the network is represented as a multigraph, whereby the vertices represent players and the edges between vertices represent battlefields. Figure 1a represents a stylized single-battle contest, in which players 1 and 2 fight on a battlefield a; Figure 1b depicts a triangular network structure in which three players are matched to three pairwise battles; Figure 1c represents a two-player multi-battle contest, in which two players compete against each other simultaneously on battlefields a and b.

Each battlefield $e \in \mathcal{E}$ is governed by an organizer and the game proceeds in two stages. In the first stage, organizers each set the rules for the contests on their own battlefields. In the second stage, having observed the rule set for each contest, players simultaneously exert their efforts to vie for wins.

[9] For disconnected networks, we can always decompose them into several connected components and our results remain intact.



(a) Standalone contest (b) Triangular contest (c) Multi-battle contest

Figure 1: Examples of Network Structures

Second Stage: Contests and Payoffs The contest between players i and j on battlefield e is modeled as a generalized Tullock contest. Put formally, fixing the profile of efforts $\boldsymbol{x}^e \equiv (x_i^e, x_j^e)$ the players exert on battlefield e, player i wins with a probability

$$p_i^e(x_i^e, x_j^e) = \begin{cases} \dfrac{\alpha_i^e f_i^e(x_i^e)}{\alpha_i^e f_i^e(x_i^e) + \alpha_j^e f_j^e(x_j^e)}, & x_i^e + x_j^e > 0, \\ \dfrac{1}{2}, & x_i^e + x_j^e = 0, \end{cases} \tag{1}$$

and player j wins with the complementary probability, i.e., $p_j^e(x_i^e, x_j^e) = 1 - p_i^e(x_i^e, x_j^e)$.

Following the tradition of the contest literature, we call $f_i^e(\cdot)$ player i's *impact function* on battlefield e, which indicates the player's contest technology on this battlefield and satisfies $f_i^e(0) = 0$, $(f_i^e)'(\cdot) > 0$, and $(f_i^e)''(\cdot) \leq 0$. Further, the parameters α_i^e and α_j^e, with $\alpha_i^e, \alpha_j^e > 0$ and $\alpha_i^e + \alpha_j^e = 1$, are the multiplicative biases the organizer on battlefield e assigns to players i and j, respectively, which determine their relative competitiveness in the contest.

Fixing player i's effort profile $\boldsymbol{x}_i \equiv (x_i^e)_{e \in \mathcal{E}_i}$ across all battlefields that involve him, let $X_i \equiv \sum_{e \in \mathcal{E}_i} x_i^e$ denote his total effort. The player bears a cost of $c_i(X_i)$. Players are subject to either resource constraints or regular cost functions.[10] In the former (*pure budget*) case, each player i's effort cost can technically take the form of $c_i(X_i) = 0$ for all $X_i \in [0, \overline{X}_i]$ and $c_i(X_i) = +\infty$ for all $X_i \in [\overline{X}_i, +\infty)$, where $\overline{X}_i \in (0, +\infty)$ is the

[10]This assumption is imposed for expositional convenience. Our analysis can easily be extended to the case in which some players are subject to a resource constraint while others have a regular effort cost function.

maximum effort at his disposal. In the latter (*pure cost*) case, we set $\overline{X}_i$ to $+\infty$ and let $c_i(\cdot)$ be twice differentiable and satisfy $c_i(0) = 0$, $c_i'(\cdot) > 0$, and $c_i''(\cdot) > 0$.

A victory on battlefield $e \in \mathcal{E}$ yields a prize value of $v^e > 0$ to the winner. A player i's expected payoff in the game is thus

$$\pi_i(\boldsymbol{x}_i, \boldsymbol{x}_{-i}) = \sum_{e \in \mathcal{E}_i} v^e p_i^e(\boldsymbol{x}^e) - c_i(X_i),$$

where $\boldsymbol{x}_{-i} \equiv (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{i-1}, \boldsymbol{x}_{i+1}, \ldots, \boldsymbol{x}_N)$ is the profile of effort strategies of all players other than i.

First Stage: Decentralized Contest Rule Setting in a Network In the beginning of the game, the organizer of each battlefield sets the rules for her battle. More formally, the organizer for each battlefield $e \in \mathcal{E}$ with $\mathcal{N}^e = \{i, j\}$ imposes multiplicative biases (α_i^e, α_j^e) on players' impact functions, with $\alpha_i^e, \alpha_j^e > 0$ and $\alpha_i^e + \alpha_j^e = 1$; they set the rules simultaneously, and all (α_i^e, α_j^e) become commonly known prior to the second stage of the game.

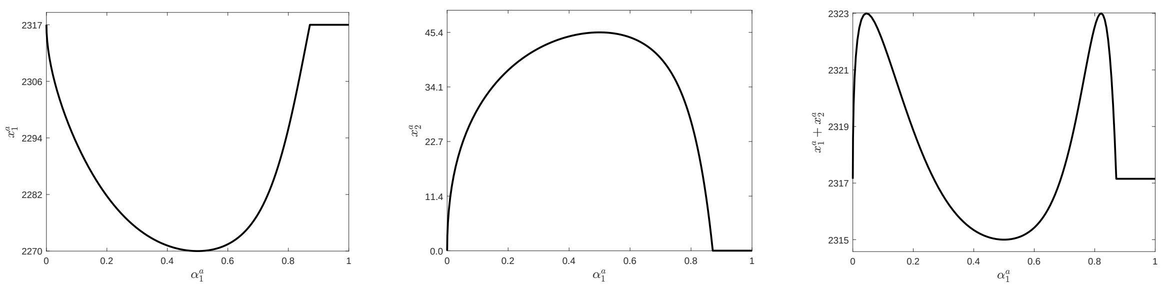
An organizer values the effort supplied by each player on her battlefield, so she chooses (α_i^e, α_j^e) to maximize an objective function $\Lambda^e(\boldsymbol{x}^e)$, which strictly increases with x_i^e for each $i \in \mathcal{N}^e$.

Summary The two-stage game can be described by $\mathcal{G} \equiv \left\langle \Gamma, \left(f_i^e(\cdot)\right)_{(i,e)\in\Gamma}, \left(c_i(\cdot)\right)_{i\in\mathcal{N}}, \left(\Lambda^e(\cdot)\right)_{e\in\mathcal{E}} \right\rangle$, where Γ represents the network structure, $\left(f_i^e(\cdot)\right)_{(i,e)\in\Gamma}$ the set of impact functions, $\left(c_i(\cdot)\right)_{i\in\mathcal{N}}$ the set of players' effort cost functions, and $\left(\Lambda^e(\cdot)\right)_{e\in\mathcal{E}}$ the set of organizers' objective functions. Organizers each set (α_i^e, α_j^e) in the first stage of the game, and players simultaneously sink their efforts afterward. We adopt subgame perfect Nash equilibrium (SPNE) in pure strategies as the solution concept.

2.2 An Illustrative Example

The literature espouses the merit of leveling the playing field in standalone contests (Dixit, 1987). Fu and Wu (2020) establish in a broad context that the optimal contest induces equal equilibrium winning odds in bilateral contests. We now provide a simple example to show that this level-playing-field principle may lose its bite when a contest is embedded in a network.

Example 1 ***(Optimality of Imbalanced Competitions in a Network)*** *Suppose that $\mathcal{N} = \{1, 2, 3\}$, $\mathcal{E} = \{a, b, c\}$, and Γ is a bilateral contest network with tri-*



(a) Player 1's effort: x_1^a (b) Player 2's effort: x_2^a (c) Total effort: Λ^a

Figure 2: Equilibrium effort and organizer's objective on battlefield a.

angular structure, as depicted in Figure 1b. Let $f_1^e(x_1^e) = 0.02x_1^e$ *for each* $e \in \mathcal{E}_1$ *and* $f_i^e(x_i^e) = x_i^e$ *for all* $i \in \{2,3\}$ *and* $e \in \mathcal{E}_i$*. Consider a pure budget case with* $(\overline{X}_1, \overline{X}_2, \overline{X}_3) = (2420, 62.4, 20)$*. The prizes for winning the battles are, respectively,* $(v^a, v^b, v^c) = (16.344, 17, 3)$*. Fix the biases on battlefields* b *and* c *at* $\boldsymbol{\alpha}^b = (\alpha_2^b, \alpha_3^b) = (0.1, 0.9)$ *and* $\boldsymbol{\alpha}^c = (\alpha_3^c, \alpha_1^c) = (0.1, 0.9)$*, respectively.*

A level playing field—i.e., with $p_1^a = p_2^b = 1/2$*—requires setting* $\alpha_1^a = \alpha_2^a = 1/2$*. Suppose that the organizer on battlefield* a *seeks to maximize total effort in the battle, i.e.,* $\Lambda^a = x_1^a + x_2^a$*. Figure 2c plots* Λ^a *as a function of* α_1^a*. It shows that total effort is minimized by setting* $\alpha_1^a = \alpha_2^a = 1/2$*, although the players win with equal probability.*

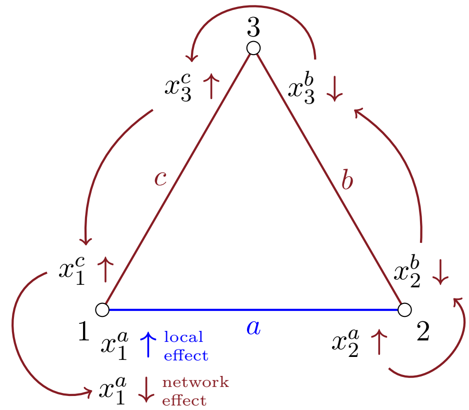
In this context, leveling the playing field maximizes player 2's effort (see Figure 2b), while minimizing player 1's (see Figure 2a). Given player 1's greater resource endowment, the total effort on this battlefield primarily relies on his input and is minimized when $p_1^a = p_2^b = 1/2$.

As α_1^a increases and approaches $1/2$, two effects are triggered. First, a more level playing field intensifies competition on battlefield a, prompting both players to increase their efforts—what we term the direct local effect, consistent with conventional wisdom. Second, this direct effect induces an indirect network effect: It propagates throughout the network, reshaping effort incentives on other battlefields; these shifts then feed back into battlefield a, further influencing x_1^a and x_2^a.

To illustrate these effects, consider the following thought experiment, focusing on how variations in α_1^a affect player 1's effort choice. Fix the biases on battlefields b and c—$(\alpha_2^b, \alpha_3^b) = (\alpha_3^c, \alpha_1^c) = (0.1, 0.9)$—and consider an initial case with $\alpha_1^a < 1/2 < \alpha_2^a$. Figure 3a shows players' relative standing on each battlefield under this set of biases. In this setting, player 2 is the frontrunner on battlefield a, since his winning

	player 1	player 2	player 3
battlefield a	weak	strong	-
battlefield b	-	weak	strong
battlefield c	strong	-	weak

(a) Relative strength between players



(b) Equilibrium effort incentives

Figure 3: Illustration of the direct local effect and the indirect network effect.

probability exceeds 1/2. Now suppose that α_1^a is increased toward 1/2. This change favors the underdog, player 1. As predicted by the direct local effect, both players intensify their efforts in response (see Figure 3b).

However, an increase in x_2^a would force player 2 to reduce his effort x_2^b on battlefield b, due to his budget constraint. By Figure 3a, player 2 is initially the underdog on battlefield b. A decrease in x_2^b gives the initial frontrunner on battlefield b—player 3—an easier win, which allows the player to scale back his effort x_3^b and redirect the saved resources to battlefield c. In turn, player 1—the initial leader on battlefield c—must respond to the more aggressive player 3 by raising his effort x_1^c. Ultimately, the increased demand on player 1's resources devoted to battlefield c forces him to reduce his effort x_1^a on battlefield a, as shown in Figure 3b.

The indirect network effect counteracts the direct local effect in shaping player 1's effort choice. As α_1^a increases and approaches 1/2, the indirect effect dominates, leading to a lower equilibrium effort x_1^a on battlefield a (see Figure 2a). In contrast, the direct and indirect effects reinforce each other for player 2, resulting in an increase in his effort on battlefield a as α_1^a approaches 1/2 (see Figure 2b).

3 Analysis

Assuming a nondegenerate conflict network—i.e., with $|\mathcal{E}| \geq 2$—we now solve the game by backward induction.[11] Section 3.1 characterizes and discusses the second-

[11]The analysis for the case of $|\mathcal{E}| = 1$ is straightforward.

stage equilibrium; Section 3.2 establishes an SPNE of the game in which all organizers level their own playing field such that players win each contest with equal probability. Section 3.3 examines equilibrium uniqueness.

3.1 Second-stage Equilibrium

The second-stage game is a collection of bilateral contests interconnected through a network. The equilibrium existence in this setting has been established by Xu, Zenou, and Zhou (2022), and we restate their result in our context as follows.

Lemma 1 ***(Xu, Zenou, and Zhou, 2022)*** *Fixing a profile of contest rules* $\boldsymbol{\alpha} = \{\boldsymbol{\alpha}^e\}_{e\in\mathcal{E}}$*, there exists a Nash equilibrium in the second-stage game. Specifically, the equilibrium effort profile* $\boldsymbol{x}^*(\boldsymbol{\alpha}) = \{\boldsymbol{x}^e(\boldsymbol{\alpha})\}_{e\in\mathcal{E}}$*, together with a set of parameters* $\{\lambda_i\}_{i\in\mathcal{N}}$*, satisfies the following first-order conditions:*

$$v^e p_i^e(\boldsymbol{x}^e)\left[1 - p_i^e(\boldsymbol{x}^e)\right] = \lambda_i g_i^e(x_i^e) \tag{2}$$

and

$$v^e \times \frac{\alpha_i^e (f_i^e)'(0)}{\alpha_j^e f_j^e(x_j^e)} \leq \lambda_i, \text{ whenever } x_i^e = 0, \tag{3}$$

where $g_i^e := f_i^e/(f_i^e)'$*. In the pure-cost case,* $\lambda_i = c_i'(X_i)$*; in the pure-budget case,* $X_i = \overline{X}_i$ *and* λ_i*s are the Lagrangian multipliers for budget constraints.*

Lemma 1 provides the necessary conditions that characterize equilibrium efforts. Specifically, (2) must be satisfied in the equilibrium whenever a player exerts a positive effort x_i^e. When a player exerts zero effort in a contest, condition (2) holds automatically, and the equilibrium further requires condition (3), which is obtained by substituting $x_i^e = 0$ into the complementary slackness conditions.

By Xu et al. (2022), if the second-stage equilibrium is interior—i.e., when each player exerts a positive effort in every contest he participates in—then the equilibrium must be unique. As a result, a unique equilibrium always emerges in the pure-cost case, since no player would completely forgo a contest. However, in the pure-budget case, multiple equilibria may arise. We adapt Example 4 of Xu et al. (2022) to illustrate this possibility in our context.

Example 2 ***(Multiple Equilibria in the Second-stage Game)*** *Suppose that* $\mathcal{N} = \{1, 2, 3\}$*,* $\mathcal{E} = \{a, b, c\}$*, and* Γ *is a triangular network as shown in Figure 1b.*

Further, set $f_i^e(x_i^e) = x_i^e$ *for each* $(i,e) \in \Gamma$*. Each player has a fixed budget, with* $(\overline{X}_1, \overline{X}_2, \overline{X}_3) = (\overline{X}_1, 1, 1)$ *and* $\overline{X}_1 > 8$*. The prize values are* $(v^a, v^b, v^c) = (1,1,1)$*. Fixing a set of neutral biases, with* $\boldsymbol{\alpha}^a = \boldsymbol{\alpha}^b = \boldsymbol{\alpha}^c = (1/2, 1/2)$*, there exists a continuum of equilibria* $\left\{(x_1^a, x_1^c, x_2^a, x_2^b, x_3^b, x_3^c) = (z, \overline{X}_1 - z, 0, 1, 1, 0) | 4 \leq z \leq \overline{X}_1 - 4\right\}$ *in the second-stage contest game.*

In this case, player 1 is endowed with an excessively large budget. His opponents simply forgo competing against him—i.e., player 2 on battlefield a and player 3 on battlefield c—and instead concentrate their limited resources on the competition between themselves—i.e., the contest on battlefield b. Player 1's effort allocation is optimal if it is sufficient to deter players 2 and 3. This condition can be satisfied by a range of effort levels, giving rise to multiple equilibria in the second-stage game. This complicates the overall equilibrium analysis, as the organizers' rule-setting decisions in the first stage may depend on which second-stage equilibrium is selected. However, our next result eliminates this concern.

For notational efficiency, let $\mathcal{X}(\boldsymbol{\alpha})$ denote the set of all second-stage equilibria corresponding to a given $\boldsymbol{\alpha}$.

Proposition 1 ***(Equilibrium Property)*** *Let* $\left(\boldsymbol{\alpha}^*, \boldsymbol{x}^*(\cdot)\right)$ *be an SPNE of the game* $\mathcal{G}$*. The following statements hold:*

(i) The second-stage equilibrium $\boldsymbol{x}^*(\boldsymbol{\alpha}^*)$ *on the equilibrium path is unique—i.e.,* $\left|\mathcal{X}(\boldsymbol{\alpha}^*)\right| = 1$*. Moreover, (3) holds with equality at* $\left(\boldsymbol{\alpha}^*, \boldsymbol{x}^*(\boldsymbol{\alpha}^*)\right)$*.*

(ii) Fix any $\boldsymbol{\alpha}' \neq \boldsymbol{\alpha}^*$ *off the equilibrium path and an arbitrary effort profile* $\boldsymbol{x}(\boldsymbol{\alpha}') \in \mathcal{X}(\boldsymbol{\alpha}')$*. Then* $\left(\boldsymbol{\alpha}^*, \{\boldsymbol{x}^*(\boldsymbol{\alpha}^*), \boldsymbol{x}(\boldsymbol{\alpha}')\}_{\boldsymbol{\alpha}' \neq \boldsymbol{\alpha}^*}\right)$ *also constitutes an SPNE of the whole game* $\mathcal{G}$*.*

This result is nontrivial. Despite the possibility of multiple equilibria for the second-stage contest game, Proposition 1(i) shows that such multiplicity does not arise *on the equilibrium path* of any SPNE. It is worth noting that multiple equilibria are unique to pure-budget cases and, as illustrated in Example 2, some players exert zero effort on certain battlefields in these equilibria. Multiple equilibria emerge when a dominant player (e.g., player 1 in Example 2) has enough resources to deter opponents across several battlefields, which affords him flexibility in how he allocates effort among them. However, this flexibility is at odds with the organizers' objective of eliciting effort. In the first stage of the game, an organizer will strategically adjust the contest rules (i.e., by heavily handicapping the dominant player) to avoid completely discouraging the weaker player and intensify the competition on her battlefield. This ensures a unique second-stage outcome on the equilibrium path.

Proposition 1(ii) further shows that an equilibrium outcome $\big(\boldsymbol{\alpha}^*, \boldsymbol{x}^*(\boldsymbol{\alpha}^*)\big)$ is robust even to equilibrium selection *off the path*: A profile $\big(\boldsymbol{\alpha}^*, \boldsymbol{x}^*(\boldsymbol{\alpha}^*)\big)$ can still be sustained as the equilibrium outcome of an SPNE even if multiple equilibria arise off-path, and for a profile of contest rules $\boldsymbol{\alpha}' \neq \boldsymbol{\alpha}^*$, an alternative second-stage equilibrium $\boldsymbol{x}(\boldsymbol{\alpha}') \neq \boldsymbol{x}^*(\boldsymbol{\alpha}')$ is selected. As a result, once we pin down an outcome $\big(\boldsymbol{\alpha}^*, \boldsymbol{x}^*(\boldsymbol{\alpha}^*)\big)$, we can construct an SPNE $\big(\boldsymbol{\alpha}^*, \boldsymbol{x}^*(\cdot)\big)$ of the game by arbitrarily selecting a second-stage equilibrium $\boldsymbol{x} \in \mathcal{X}(\boldsymbol{\alpha})$ for $\boldsymbol{\alpha} \neq \boldsymbol{\alpha}^*$.

The reasoning is as follows. Suppose, to the contrary, that an equilibrium outcome $\big(\boldsymbol{\alpha}^*, \boldsymbol{x}^*(\boldsymbol{\alpha}^*)\big)$ is sensitive to off-path equilibrium selection. Then there must exist some battlefield e_0 whose organizer can profitably deviate unilaterally to an alternative contest rule $(\boldsymbol{\alpha}^{e_0})'$; moreover, the bias profile $\boldsymbol{\alpha}' \equiv \big((\boldsymbol{\alpha}^{e_0})', (\boldsymbol{\alpha}^{-e_0})^*\big)$ induces multiple second-stage equilibria that differ in the effort profiles on battlefield e_0. However, this deviation is unlikely to be profitable for the organizer, since some player on the deviator's battlefield exerts zero effort. This contradiction implies that the choice of off-path equilibrium is irrelevant whenever an outcome can be sustained by an SPNE.

Proposition 1 paves the way for our equilibrium result. We can describe an SPNE simply by its equilibrium outcome $\big(\boldsymbol{\alpha}^*, \boldsymbol{x}^*(\boldsymbol{\alpha}^*)\big)$ without loss of generality.

3.2 Even-odds Equilibrium as SPNE

In this part, we construct an SPNE of the game. We call an SPNE an *even-odds equilibrium* if players win each contest with equal probability, i.e., $(p_i^e)^* = 1/2$ for all $(i, e) \in \Gamma$. Our first main result ensues.

Theorem 1 ***(Existence of Even-odds Equilibrium)*** *Fix a game $\mathcal{G}$. An even-odds equilibrium always exists. There is a unique profile of contest rules $\boldsymbol{\alpha}^{**}$ that leads to the equilibrium outcome of equal winning odds on every battlefield. As a result, the even-odds equilibrium can be described by a unique associated equilibrium outcome $\big(\boldsymbol{\alpha}^{**}, \boldsymbol{x}^{**}(\boldsymbol{\alpha}^{**})\big)$.*

Theorem 1 establishes that there always exists an SPNE in which the players in each contest win with equal probability. Further, the profile of contest rules that induces the even-odds outcome in the second-stage game is unique. Three remarks are in order. First, the result differs subtly from the conventional wisdom of leveling the playing field in the contest design literature. The literature typically considers a *centralized* design problem in which an organizer manipulates the competitive balance of a standalone contest (Lazear and Rosen, 1981; Dixit, 1987; Che and Gale, 1998). In contrast, we consider a *decentralized* design problem in which organizers independently manage their own battlefields within a network. An organizer's choice of biases factors in (i) the biases to be set by others and (ii) its implications for all players' second-stage effort choices $\boldsymbol{x}(\boldsymbol{\alpha})$ within the network—including those not involved in her battle—as Example 1 illustrates. Second, in the even-odds equilibrium, no single organizer is willing to unilaterally deviate from $\boldsymbol{\alpha}^{**}$; however, it is noteworthy that leveling the playing field is not necessarily optimal for an organizer if others do not level their playing fields. Third, while Theorem 1 establishes that the even-odds outcome can be sustained as a part of an SPNE, it does not verify its uniqueness. We discuss equilibrium uniqueness in Section 3.3.

Next, we delve into the fundamentals of the equilibrium and its analysis. As explained above, solving for the equilibrium is technically challenging. First, unlike a standalone contest, a closed-form solution to $\boldsymbol{x}(\boldsymbol{\alpha})$—the equilibrium efforts in the networked contest game—is unavailable. Second, the dynamic and reflexive interactions across the network causes irregularity to organizers' payoff functions, as illustrated in Figure 2c. As a result, fixing a potential candidate equilibrium—although the first-order conditions of players' equilibrium efforts with respect to the biases can be obtained by Lemma 1—it is almost impossible to verify the (local) second-order condition, let alone its global optimality.

We develop a novel approach that examines organizers' nonlocal deviations to verify the equilibrium instead of analyzing their best responses. We first establish the existence of a unique profile of biases $\boldsymbol{\alpha}^{**}$ that yields the even-odds outcome on every battlefield in the second stage; it remains to verify that $\boldsymbol{\alpha}^{**}$ indeed constitutes a first-stage equilibrium. In what follows, we provide a sketch of the proof. For expositional efficiency, we focus on the pure-cost case. The proof for the pure-budget case is similar, except that the expression of λ_i may differ. We will highlight these distinctions when necessary.

Suppose, to the contrary, that $\boldsymbol{\alpha}^{**}$ is not a part of an equilibrium. Then there exists a battlefield $e_0 \in \mathcal{E}$ whose organizer can set $\widetilde{\boldsymbol{\alpha}}^{e_0} \neq (\boldsymbol{\alpha}^{e_0})^{**}$ to induce greater effort from at least one player on her battlefield. Let $\widetilde{\boldsymbol{x}}$, $\widetilde{\boldsymbol{\lambda}}$, and $\widetilde{\boldsymbol{p}}$ denote the second-stage equilibrium efforts, marginal effort costs, and winning probabilities under the bias profile $\widetilde{\boldsymbol{\alpha}} \equiv \left(\widetilde{\boldsymbol{\alpha}}^{e_0}, (\boldsymbol{\alpha}^{-e_0})^{**}\right)$, respectively. Similarly, we denote equilibrium variables under the bias profile $\boldsymbol{\alpha}^{**}$ with double asterisks. The following lemma helps us predict how each player's equilibrium total effort changes in response to the organizer of battlefield e_0's deviation from $(\boldsymbol{\alpha}^{e_0})^{**}$.

Lemma 2 ***(Individual Player's Overall Effort Incentive)*** *For each $i \in \mathcal{N}$, $\widetilde{\lambda}_i \leq \lambda_i^{**}$.*

Under $\boldsymbol{\alpha}^{**}$, the competition on every battlefield is perfectly balanced. A deviation on battlefield e_0 disrupts not only the competitive balance of e_0 itself but also that of interconnected battlefields. Intuitively, this deviation generates negative overall incentives for players across the network. Lemma 2 confirms and formalizes this intuition: Every player's equilibrium marginal effort cost λ_i weakly decreases following the deviation. Given the strict convexity of the effort cost functions, their respective equilibrium total efforts in the contest also weakly decrease accordingly.

The next lemma concerns the spillover of the deviation to players' efforts on other individual battlefields.

Lemma 3 ***(Individual Player's Incentive on a Battlefield)*** *Fix an arbitrary battlefield $e \neq e_0$ with $\mathcal{N}^e = \{i, j\}$, and suppose $\widetilde{\lambda}_i/\lambda_i^{**} \leq \widetilde{\lambda}_j/\lambda_j^{**}$. Then $\widetilde{x}_i^e \geq (x_i^e)^{**}$.*

The deviation on battlefield e_0 triggers complex spillovers to interconnected contests, which alters players' marginal benefits and the marginal costs of efforts on other battlefields. While each player's total effort unambiguously decreases, it remains unclear how an individual adjusts effort on a specific battlefield other than e_0. Intuitively, the universal decline in marginal cost induced by the spillover tends to encourage players to increase effort on other battlefields. The ratio $\widetilde{\lambda}_i/\lambda_i^{**}$ captures the impact on player i's marginal cost: A lower ratio indicates a sharper decline in cost reduction and therefore a more significant increase in effort incentive from the cost side. Lemma 3 confirms that the player who experiences the greater cost reduction will indeed increase effort in response.

We are ready to prove Theorem 1. Let s denote the player with the lowest ratio $\widetilde{\lambda}_i/\lambda_i^{**}$ among all $i \in \mathcal{N}$. By Lemma 3, following the organizer's deviation on battlefield e_0, the player exerts weakly greater effort on all other battlefields—i.e., $\widetilde{x}_s^e \geq (x_s^e)^{**}$ for all $e \in \mathcal{E}_s$ and $e \neq e_0$. Two possible cases arise: Player s is either involved in the contest on battlefield e_0 or not.

Case (a): $s \notin \mathcal{N}^{e_0}$. Lemma 3 implies that player s's effort weakly increases in all contests he participates in (he is not involved in the contest on battlefield e_0). Therefore, his total effort weakly increases following the deviation on battlefield e_0. Meanwhile, Lemma 2 predicts that his total effort would weakly decrease. We can then conclude that his total effort remains unchanged, with $\widetilde{\lambda}_s = \lambda_s^{**}$. By definition, player s has the lowest ratio $\widetilde{\lambda}_s/\lambda_s^{**} = 1$; together with Lemma 2, we can establish that the marginal effort costs of all players remain unchanged, i.e., $\widetilde{\lambda}_i/\lambda_i^{**} = 1$ for all $i \in \mathcal{N}$.

Let i_0 and j_0 denote the players on battlefield e_0. Recall that players win with equality probability and thus exert positive effort in all their contests; therefore, the first-order condition (2)—i.e., $v^e p_i^e(\boldsymbol{x}^e)\left[1 - p_i^e(\boldsymbol{x}^e)\right] = \lambda_i g_i^e(x_i^e)$—holds for all $i \in \mathcal{N}$ and $e \in \mathcal{E}$ under the bias profile $\boldsymbol{\alpha}^{**}$. It follows that $\widetilde{x}_{i_0}^{e_0} \leq (x_{i_0}^{e_0})^{**}$ and $\widetilde{x}_{j_0}^{e_0} \leq (x_{j_0}^{e_0})^{**}$, because $\widetilde{p}_{i_0}^{e_0}(1-\widetilde{p}_{i_0}^{e_0}) \leq 1/4 = (p_{j_0}^{e_0})^{**}(1-(p_{j_0}^{e_0})^{**})$. This contradicts the assumption that setting $\widetilde{\boldsymbol{\alpha}}^{e_0}$ is a profitable deviation for the organizer on battlefield e_0.

Case (b): $s \in \mathcal{N}^{e_0}$. Assume, for contradiction, that setting $\widetilde{\boldsymbol{\alpha}}^{e_0}$ is a profitable deviation for the organizer on battlefield e_0. If $\widetilde{x}_s^{e_0} > (x_s^{e_0})^{**}$, then we must have $\widetilde{X}_s > X_s^{**}$ by Lemma 3. Consequently, $\widetilde{\lambda}_s = c_s'(\widetilde{X}_s) > c_s'(X_s^{**}) = \lambda_s^{**}$, which contradicts Lemma 2.

Alternatively, suppose $\widetilde{x}_s^{e_0} \leq (x_s^{e_0})^{**}$. Then the other player on battlefield e_0, denoted by j_0, must exert strictly more effort, i.e., $\widetilde{x}_{j_0}^{e_0} > (x_{j_0}^{e_0})^{**} > 0$. Moreover, since $\widetilde{x}_s^{e_0} \leq (x_s^{e_0})^{**}$, we have

$$\widetilde{g}_s^{e_0} \equiv \frac{f_s^{e_0}\left(\widetilde{x}_s^{e_0}\right)}{(f_s^{e_0})'\left(\widetilde{x}_s^{e_0}\right)} \leq \frac{f_s^{e_0}\left((x_s^{e_0})^{**}\right)}{(f_s^{e_0})'\left((x_s^{e_0})^{**}\right)} \equiv (g_s^{e_0})^{**}.$$

Similarly, we can obtain $\widetilde{g}_{j_0}^{e_0} > (g_{j_0}^{e_0})^{**}$. Together, these imply

$$\frac{\widetilde{\lambda}_s}{\lambda_s^{**}} \geq \frac{\widetilde{\lambda}_s \widetilde{g}_s^{e_0}}{\lambda_s^{**}(g_{i_0}^{e_0})^{**}} = \frac{\widetilde{\lambda}_{j_0} \widetilde{g}_{j_0}^{e_0}}{\lambda_{j_0}^{**}(g_{j_0}^{e_0})^{**}} > \frac{\widetilde{\lambda}_{j_0}}{\lambda_{j_0}^{**}},$$

where the equality follows from the first-order condition (2). This contradicts the definition whereby player s has the lowest $\widetilde{\lambda}_i/\lambda_i^{**}$ among all players.

This argument demonstrates that any unilateral deviation from $\boldsymbol{\alpha}^{**}$ would not render an organizer better off. Hence, the bias profile $\boldsymbol{\alpha}^{**}$ constitutes a first-stage equilibrium of the game $\mathcal{G}$. We thus establish that leveling the playing field can always be sustained as an SPNE, even in the networked environment. It remains unknown whether this is the unique outcome of the game, and to what extent the level-playing-field principle can be preserved in a network.

3.3 Equilibrium Uniqueness

This section addresses the uniqueness of the even-odds equilibrium established in Theorem 1. Our discussion unfolds on two levels. First, we investigate the boundary of the level-playing-field principle within the network—i.e., we identify the conditions under which an organizer always sets her contest rule to induce perfectly balanced competition, regardless of the rules adopted in competing contests. The even-odds equilibrium must be unique as long as the level-playing-field principle holds. Second, we explore whether, and under what conditions, this equilibrium remains unique even when the principle fails—i.e., when perfectly balanced competition does not automatically maximize an organizer's payoff.

We present the following preliminaries to ease subsequent analysis and exposition. Fixing $e \in \mathcal{E}$, define $w^e := v^e p_i^e(1-p_i^e) = v^e p_j^e(1-p_j^e)$, with $i, j \in \mathcal{N}^e$. The parameter w^e is an intuitive measure of the competitive balance on battlefield e: A larger w^e implies a more balanced playing field; it is maximized when competition on battlefield e is a perfectly even race, with $p_i^e = p_j^e = 1/2$ and $w^e = v^e/4$. Given the correspondence between w^{e_0} and (p_i^e, p_j^e), we obtain the following.

Lemma 4 ***(Reformulating Organizers' Design Problem)*** *The tuple $\big(\boldsymbol{\alpha}^*, \boldsymbol{x}^*(\cdot)\big)$ constitutes an SPNE if and only if for each battlefield $e_0 \in \mathcal{E}$, with $\mathcal{N}^{e_0} = \{i_0, j_0\}$, $(w^{e_0})^*$ solves the following maximization problem:*

$$\begin{aligned}\max_{\{w^{e_0},(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_N)\}} \ & \Lambda^{e_0}(x_{i_0}^{e_0}, x_{j_0}^{e_0}) \\ s.t.\ & (1) \text{ holds for } \boldsymbol{\alpha}^* \text{ in all } e \neq e_0, \\ & (2) \text{ holds for all } (i,e) \in \Gamma, \\ & p_{i_0}^{e_0}(1-p_{i_0}^{e_0}) = w^{e_0}/v^{e_0}. \end{aligned} \tag{4}$$

Lemma 4 establishes an equivalence between the decision problem of an organizer on battlefield e_0 who sets biases $\boldsymbol{\alpha}^{e_0}$ and that of the organizer who chooses w^{e_0}.[12] It

[12] The reformulation was first introduced by Fu and Wu (2020) to characterize optimal contest under a centralized organizer, and can naturally be adapted to our setting.

is worth noting that when $w^{e_0} < v^{e_0}/4$, there exist two probabilities $p_{i_0}^{e_0}$ that satisfy (4); so the mapping between $\boldsymbol{\alpha}^{e_0}$—which determine $p_{i_0}^{e_0}$—and w^{e_0} is not one-to-one. However, this nuance does not affect our analysis: The even-odds equilibrium requires $w^{e_0} = v^{e_0}/4$, which can only be attained when $p_{i_0}^{e_0} = 1/2$. In summary, to verify the uniqueness of the equilibrium, it suffices to focus on the optimality of $w^{e_0} = v^{e_0}/4$ in relevant contexts.

3.3.1 The Level-playing-field Principle in a Conflict Network

Example 1 demonstrates the complications introduced by the indirect network effect in an organizer contest-rule decision. Specifically, changes to the contest rules on a single battlefield may affect the equilibrium behavior on others, which in turn reflexively influence the original one. This feedback loop casts doubt on the level-playing-field principle well established in standalone contests.

In what follows, we examine the extent to which the conventional wisdom holds in a network. We begin by introducing two assumptions. The first concerns the impact function $f_i^e(\cdot)$, and the second the network topology Γ.

Assumption 1 *For each $(i, e) \in \Gamma$, f_i^e is 2-concave—i.e., $(f_i^e)^2$ is concave.*

Assumption 2 *The multigraph Γ reduces to a tree after we replace any set of multiple edges in Γ with a single edge.*

Assumption 1 requires sufficient concavity on the impact functions, which corresponds to a sufficiently noisy contest on each battlefield. Intuitively, greater concavity implies that a player's winning odds become less responsive to effort differentials. That is, outcomes depend more on random factors than on players' actions, so changes in efforts have limited influence on winning probabilities and therefore players' marginal benefits of efforts. Assumption 2 requires that the network be acyclic. This prevents recursive feedback loops and dampens indirect network effects: While changes on one battlefield may spill over to others and vice versa, the absence of cycles guarantees that these effects cannot feed back to their origin. We obtain the following.

Theorem 2 ***(The Level-playing-field Principle in a Conflict Network)*** *Suppose that Assumption 1 or 2 holds. Fixing an arbitrary battlefield $e \in \mathcal{E}$ and a bias* *profile $\boldsymbol{\alpha}^{-e}$, it is optimal for the battlefield organizer to set $\boldsymbol{\alpha}^e$ that induce $w^e = v^e/4$—i.e., a fully level playing field with players to win with equal probability—regardless of the contest rules set for other contests. As a result, $\big(\boldsymbol{\alpha}^{**}, \boldsymbol{x}^{**}(\boldsymbol{\alpha}^{**})\big)$ is the unique equilibrium outcome of game $\mathcal{G}$.*

Theorem 2 identifies sufficient conditions under which the level-playing-field principle can be sustained in a conflict network, such that an organizer always sets her rules to induce even winning odds, irrespective of others' choices.

We now present a sketch proof of the theorem to illustrate the logic underlying our result. For this purpose, we first introduce the term $\frac{d\lambda_i}{dw^{e_0}}\frac{dx_i^e}{dw^{e_0}}$, which plays a critical role in our analysis. We call it the *ripple effect* for a player i on a battlefield e. Recall that w^{e_0} measures the degree of competitive balance on battlefield e_0. The two components, $\frac{d\lambda_i}{dw^{e_0}}$ and $\frac{dx_i^e}{dw^{e_0}}$, respectively capture how a change in the competitive balance on e_0 affects player i's total effort and his effort on an arbitrary battlefield e. When $\frac{d\lambda_i}{dw^{e_0}}\frac{dx_i^e}{dw^{e_0}} < 0$, player i's total effort (as indicated by λ_i) and his effort on battlefield e move in opposite directions in response to a change in w^{e_0}. Conversely, when $\frac{d\lambda_i}{dw^{e_0}}\frac{dx_i^e}{dw^{e_0}} > 0$, the two forces are aligned.

We then present two lemmata about the general properties of ripple effects, with neither requiring Assumption 1 or 2.

Lemma 5 ***(Ripple Effects on Competing Battlefields)*** *Fix a battlefield $e_0 \in \mathcal{E}$, with $\mathcal{N}^{e_0} = \{i_0, j_0\}$. The following holds when competition on battlefield e_0 becomes more balanced (as w^{e_0} increases):*

(i) The ripple effect for each battlefield $e \neq e_0$ is non-positive, i.e.,

$$\frac{d\lambda_i}{dw^{e_0}}\frac{dx_i^e}{dw^{e_0}} + \frac{d\lambda_j}{dw^{e_0}}\frac{dx_j^e}{dw^{e_0}} \leq 0, \ \textit{with } \mathcal{N}^e = \{i, j\}. \tag{5}$$

(ii) The ripple effect for each player $i \in \mathcal{N}$ is non-negative, i.e.,

$$\sum_{e \in \mathcal{E}_i} \frac{d\lambda_i}{dw^{e_0}}\frac{dx_i^e}{dw^{e_0}} \geq 0. \tag{6}$$

Lemma 5(i) suggests that balancing the competition on battlefield e_0—i.e., increasing w^{e_0}—induces a negative aggregate ripple effect for each other contest. In contrast, by Lemma 5(ii), the aggregate ripple effect for each player is positive. The claim in Lemma 5(ii) follows directly from the definition. Recall that $\mathcal{E}_i$ denotes the set of battles that involve player i. The aggregate ripple effect for player i can be expressed as

$$\sum_{e\in\mathcal{E}_i}\frac{d\lambda_i}{dw^{e_0}}\frac{dx_i^e}{dw^{e_0}}=\frac{d\lambda_i}{dw^{e_0}}\sum_{e\in\mathcal{E}_i}\frac{dx_i^e}{dw^{e_0}}=\frac{d\lambda_i}{dw^{e_0}}\frac{dX_i}{dw^{e_0}},$$

where X_i denotes player i's total effort across all contests. This expression must be positive, because λ_i and X_i move in the same direction under convex cost functions.

We now develop a key thought experiment for our proof. Consider battlefield e_0 where players i_0 and j_0 compete under rules that initially yield equal winning probabilities. Recall that our goal is to establish Assumption 1 or 2 as the sufficient condition under which the level-playing-field principle holds in a network. For this purpose, we examine a hypothetical scenario in which the organizer could get better off by unilaterally tilting the competitive balance on battlefield e_0. The next lemma presents the properties of ripple effects under such a circumstance. Together with Lemma 5, it leads to contradiction when either of the two assumptions is satisfied.

Lemma 6 ***(Ripple Effects in the Deviating Battlefield)*** *Fix a battlefield $e_0 \in \mathcal{E}$, with $\mathcal{N}^{e_0}=\{i_0,j_0\}$, and a bias profile $\boldsymbol{\alpha}$ that leads to $w^{e_0}=v^e/4$. Suppose that the organizer of battlefield e_0 can benefit from resetting her current contest rule $\boldsymbol{\alpha}^{e_0}$, given the bias profile $\boldsymbol{\alpha}^{-e_0}$ for others. There always exists a degree of competitive balance $\widetilde{w}^{e_0}<v^{e_0}/4$, such that the following holds: Letting all derivatives be evaluated at $w^{e_0}=\widetilde{w}^{e_0}$ and assuming $\frac{dx_{i_0}^{e_0}}{dw^{e_0}}\geq\frac{dx_{j_0}^{e_0}}{dw^{e_0}}$ without loss of generality,*

(i) $\frac{dx_{i_0}^{e_0}}{dw^{e_0}}\geq 0\geq\frac{dx_{j_0}^{e_0}}{dw^{e_0}}$*;*

(ii) $\frac{\widetilde{w}^{e_0}}{\lambda_{j_0}}\frac{d\lambda_{j_0}}{dw^{e_0}}\geq 1\geq\frac{w^{e_0}}{\lambda_{i_0}}\frac{d\lambda_{i_0}}{dw^{e_0}}\geq 0$*;*

(iii) $\frac{dx_{i_0}^{e_0}}{dw^{e_0}}\frac{d\lambda_{i_0}}{dw^{e_0}}\geq 0\geq\frac{dx_{j_0}^{e_0}}{dw^{e_0}}\frac{d\lambda_{j_0}}{dw^{e_0}}$*;*

(iv) for $e'\neq e_0$, if $\mathcal{N}^{e'}=\mathcal{N}^{e_0}=\{i_0,j_0\}$, $\frac{dx_{j_0}^{e'}}{dw^{e_0}}\leq 0$ *and* $\frac{dx_{j_0}^{e'}}{dw^{e_0}}\frac{d\lambda_{j_0}}{dw^{e_0}}\leq 0.$

In summary, whenever the organizer of battlefield e_0 can benefit from an imbalanced contest, we can identify some value $\widetilde{w}^{e_0}$ for which the resulting ripple effects satisfy the properties in Lemma 6. Crucially, $\widetilde{w}^{e_0}$ does not have to be a profitable deviation from $w^{e_0}=v^{e_0}/4$ for the given contest rules $\boldsymbol{\alpha}^{-e_0}$ on other battlefields.

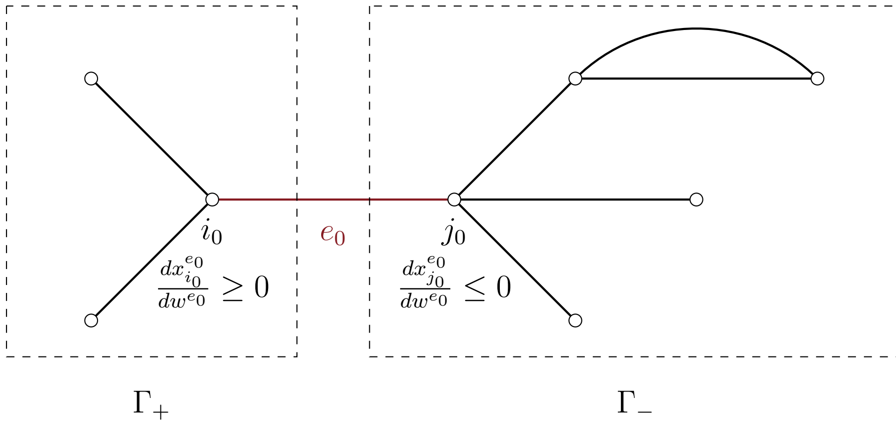


Figure 4: Network Structure under Assumption 2

Lemma 6(iii), which follows directly from (i) and (ii), establishes that the ripple effects at $\widetilde{w}^{e_0}$ diverge in sign between players i_0 and j_0: positive for i_0 but negative for j_0. Moreover, Lemma 6(iv) demonstrates that when i_0 and j_0 compete on additional battlefields, j_0's ripple effects remain negative on those battlefields as well.

We are now ready to prove Theorem 2. Suppose that the hypothetical scenario described above does exist. We derive a contradiction through Lemmata 5 and 6 under either Assumption 1 or Assumption 2.

Case (a): Assumption 1 holds. The proof is similar to that of Theorem 1. Let s denote the player with the highest value of $\left|\frac{w^{e_0}}{\lambda_i}\frac{d\lambda_i}{dw^{e_0}}\right|$, i.e., $\left|\frac{w^{e_0}}{\lambda_s}\frac{d\lambda_s}{dw^{e_0}}\right| = \max_{i\in\mathcal{N}}\left|\frac{w^{e_0}}{\lambda_i}\frac{d\lambda_i}{dw^{e_0}}\right|$. By Lemma 6(ii), $\frac{w^{e_0}}{\lambda_{j_0}}\frac{d\lambda_{j_0}}{dw^{e_0}} \geq \frac{w^{e_0}}{\lambda_{i_0}}\frac{d\lambda_{i_0}}{dw^{e_0}} \geq 0$. Therefore, either $s \notin \{i_0, j_0\}$ or we can set $s = j_0$ without loss of generality.

Next, fix a battlefield $e \in \mathcal{E}_s$ and consider the associated ripple effect. The case with $e = e_0$ and $s = j_0$ is straightforward. Lemma 6(iii) leads to $\frac{d\lambda_{j_0}}{dw^{e_0}}\frac{dx_{j_0}^{e_0}}{dw^{e_0}} \leq 0$. This, together with Lemma 6(iv), indicates that $\sum_{e\in\mathcal{E}_s}\frac{d\lambda_s}{dw^{e_0}}\frac{dx_s^e}{dw^{e_0}} \leq 0$, which contradicts Lemma 5(ii).[13]

If $e \neq e_0$, we prove in the Appendix that the ripple effect for player s on battlefield e is negative, i.e., $\frac{d\lambda_s}{dw^{e_0}}\frac{dx_s^e}{dw^{e_0}} < 0$. It is worth noting that Assumption 1 plays a critical role in verifying this inequality. This, again, enables us to conclude $\sum_{e\in\mathcal{E}_s}\frac{d\lambda_s}{dw^{e_0}}\frac{dx_s^e}{dw^{e_0}} \leq 0$.

Case (b): Assumption 2 holds. Under Assumption 2, the multigraph Γ decom-

[13]It can be verified that $\sum_{e\in\mathcal{E}_s}\frac{d\lambda_s}{dw^{e_0}}\frac{dx_s^e}{dw^{e_0}} = 0$ is impossible.

poses into two connected components upon removal of all edges between i_0 and j_0. For notational convenience, denote by Γ_+ and Γ_- the two connected components containing i_0 and j_0 (see Figure 4 for a graphical illustration). Further, let $\mathcal{N}_-$ and $\mathcal{E}_-$ represent the player and battlefield sets in Γ_-, respectively. The sets $\mathcal{N}_+$ and $\mathcal{E}_+$ can be similarly defined.

Following Lemma 6(i), we can assume $\frac{dx_{i_0}^{e_0}}{dw^{e_0}} \geq 0 \geq \frac{dx_{j_0}^{e_0}}{dw^{e_0}}$ without loss of generality. Next, consider the aggregate ripple effect in Γ_-, i.e.,

$$\mathcal{I}_- := \sum_{i \in \mathcal{N}_-} \sum_{e \in \mathcal{E}_i} \frac{d\lambda_i}{dw^{e_0}} \frac{dx_i^e}{dw^{e_0}}.$$

By Lemma 5(ii), the ripple effect for each player across all battlefields he participates in must be non-negative. Consequently, $\sum_{e \in \mathcal{E}_i} \frac{d\lambda_i}{dw^{e_0}} \frac{dx_i^e}{dw^{e_0}} \geq 0$ for all $i \in \mathcal{N}_-$, which in turn implies that $\mathcal{I}_- \geq 0$. Meanwhile, $\mathcal{I}_-$ can alternatively be expressed as

$$\mathcal{I}_- = \sum_{e \in \mathcal{E}_-} \sum_{i \in \mathcal{N}^e} \frac{d\lambda_i}{dw^{e_0}} \frac{dx_i^e}{dw^{e_0}} + \sum_{e \in \mathcal{E}_{i_0} \cap \mathcal{E}_{j_0}} \frac{d\lambda_{j_0}}{dw^{e_0}} \frac{dx_{j_0}^e}{dw^{e_0}}.$$

From Lemma 5(i), $\sum_{i \in \mathcal{N}^e} \frac{d\lambda_i}{dw^{e_0}} \frac{dx_i^e}{dw^{e_0}} \leq 0$ for each battlefield $e \in \mathcal{E}_-$. By Lemma 6(iii) and (iv), $\frac{d\lambda_{j_0}}{dw^{e_0}} \frac{dx_{j_0}^e}{dw^{e_0}} \leq 0$ for all $e \in \mathcal{E}_{i_0} \cap \mathcal{E}_{j_0}$. We can then conclude $\mathcal{I}_- \leq 0$, which is a contradiction.[14]

Theorem 2 establishes that when either Assumption 1 or Assumption 2 holds, each organizer's optimal strategy is to maintain equal winning probabilities on her battlefield, independent of other battlefields' contest rules. This revives the level-playing-field principle in a networked contest setting. The result reveals the respective roles played by contest technologies and prevailing network structure in shaping equilibrium outcome. Both serve to limit the indirect network effect caused by the change in the competitive balance on one battlefield, so each organizer can focus on the direct local effect when setting her contest rule.

First, Assumption 1 requires strongly concave impact functions $f_i^e(\cdot)$. For simplicity, our discussion focuses on the pure-cost case. The first-order conditions (2)

[14] It can be verified that $\mathcal{I}_- = 0$ is impossible.

that determine the equilibrium can be written as

$$\frac{(f_i^e)'}{f_i^e} w^e = \lambda_i.$$

The left-hand side indicates the marginal benefit of a player i's effort on battlefield e, while the right-hand side gives the marginal cost. Suppose that the competitive balance of a battlefield e_0 varies. Its spillover alters players' marginal benefits and marginal costs of efforts on all other battlefields. This requires that players adjust their efforts everywhere to rebalance their costs and benefits. A strongly concave impact function, as previously noted, limits the impact of a change in effort on winning probability and therefore the impact on the marginal benefit of effort.

In the proof for Case (a), we identify a player s with $\left|\frac{w^{e_0}}{\lambda_s}\frac{d\lambda_s}{dw^{e_0}}\right| = \max_{i\in\mathcal{N}}\left|\frac{w^{e_0}}{\lambda_i}\frac{d\lambda_i}{dw^{e_0}}\right|$. Note that the player, when being evaluated at $w^{e_0} = \widetilde{w}^{e_0}$, is most significantly affected by the change on battlefield e_0 in terms of marginal effort cost, since $\left|\frac{w^{e_0}}{\lambda_i}\frac{d\lambda_i}{dw^{e_0}}\right|$ is the elasticity of λ_i with respect to w^{e_0}. Our analysis verifies that the rebalancing is impossible for player s. The significant change in marginal effort cost cannot be matched by the limited change in the marginal benefit of the player's effort. Contradiction with Lemma 5(ii) thus ensues, because the aggregate ripple effect is negative—i.e., $\sum_{e\in\mathcal{E}_s}\frac{d\lambda_s}{dw^{e_0}}\frac{dx_s^e}{dw^{e_0}} \leq 0$. This implies that such a $\widetilde{w}^{e_0}$ does not exist, and hence the above hypothetical scenario is unlikely. More intuitively, strongly concave impact functions restrain the spillover caused by a change in the competitive balance on a battlefield. This limits the indirect network effect that would feed back to the origin and prevents it from reversing the direct local effect.

Second, Assumption 2 requires an acyclic network structure. As stated in the sketch proof and illustrated in Figure 4, the multigraph Γ can be split into two connected components if the edges between two vertices—i.e., players i_0 and j_0—are removed. The decomposition enables us to separate the respective impact of a change in the contest rule for battlefield e_0 on players i_0 and j_0, which paves the way for our analysis. Suppose that Assumption 2 is violated (see, e.g., Figure 1b). Their effort choices would be entangled: They not only engage in the direct competition on battlefield e_0, but are also connected via various indirect paths traversing other players and battlefields (see, e.g., Figure 3b). The direct local effect on x_{i_0} and x_{j_0} caused by a change in $\boldsymbol{\alpha}^{e_0}$ triggers indirect network effects that reflexively affect the choices of x_{i_0} and x_{j_0}, which causes complications. An acyclic network severs the linkages and keeps x_{i_0} and x_{j_0} immune to the shock of the indirect network effects.

3.3.2 Equilibrium Uniqueness when the Level-playing-field Principle Fails

Next, we examine to what extent the even-odds equilibrium remains unique when Assumptions 1 and 2 are not satisfied, in which case the level-playing-field principle may not hold. We present the following two assumptions.

Assumption 1′ *For each $(i, e) \in \Gamma$, f_i^e is ρ-concave with $\rho = \frac{1+\sqrt{2}}{2} \approx 1.2$—i.e., $(f_i^e)^\rho$ is concave.*

Assumption 2′ *The simple graph, obtained by replacing all parallel edges in Γ with single edges, has the following structure: Each edge is contained in at most one cycle, and all cycles in the simple graph have odd length.*

Assumptions 1′ and 2′ impose weaker restrictions and can respectively be implied by Assumptions 1 and 2. Specifically, Assumption 1′ demands a weaker notion of concavity for impact functions, while Assumption 2′ allows for cycles in the network.

Theorem 3 ***(Equilibrium Uniqueness when the Level-playing-field Principle Fails)*** *Suppose that Assumption 1′ or 2′ holds. Then $\big(\boldsymbol{\alpha}^{**}, \boldsymbol{x}^{**}(\boldsymbol{\alpha}^{**})\big)$ is the unique equilibrium outcome of game $\mathcal{G}$.*

We outline the key logic of the proof. Assume, for contradiction, that an alternative equilibrium exists in which at least one battlefield e exhibits imbalance ($p_i^e \neq p_j^e$). Consider the battlefield with *minimal* w^e value and perturb the contest rule on this battlefield. It can be verified in the proof that under either Assumption 1′ or 2′, the direct local effect dominates the indirect network effects. This implies that the organizer on this battlefield can improve the performance of her contest by increasing the value of w^e toward $v^e/4$—i.e., leveling the playing field. The hypothetical equilibrium thus dissolves.

Recall that $w^e \equiv v^e p_i^e p_j^e$. A small w^e implies either a small prize value v^e or a lopsided competition. Both cases imply limited effort incentives. The former means a small stake that discourages significant investment; the latter implies that one player faces a slim chance of winning, while the other expects an easy win, which disincentivizes both. Consequently, a marginal change in contest rules would not trigger substantial effort adjustments, thereby containing network spillovers and dampening indirect effects.

Assumption $1'$ fulfills a role comparable to that of Assumption 1: Strong concavity attenuates equilibrium effort levels and weakens cross-battlefield spillovers. Assumption $2'$ is similar to Assumption 2: The simplified network architecture constrains the channels for the indirect effects to be transmitted and compounded, which allows the local effect to prevail.

To further illustrate the result, we revisit Example 1. As previously noted, the level-playing-field principle does not hold under this network structure: Given $(\alpha_2^b, \alpha_3^b) = (0.1, 0.9)$ and $(\alpha_3^c, \alpha_1^c) = (0.1, 0.9)$, the organizer on battlefield a would not fully level the playing field. However, the set of biases provided in Example 1—i.e., $(\alpha_1^a, \alpha_2^a) = (0.5, 0.5)$, $(\alpha_2^b, \alpha_3^b) = (0.1, 0.9)$, and $(\alpha_3^c, \alpha_1^c) = (0.1, 0.9)$—cannot constitute a first-stage equilibrium. The triangular network fails Assumption 2 but satisfies Assumption $2'$. By Theorem 3, the game possesses a unique SPNE, in which players in every battlefield win with equal probabilities.

To close this section, it is useful to note that Assumption $1'$ or $2'$ is a sufficient but not necessary condition. The equilibrium uniqueness result established in Theorems 2 and 3 holds more broadly than the context defined by Assumption $1'$ or $2'$. Although an analytic result is difficult, our numerical exercises suggest that the even-odds equilibrium can remain unique even with a less concave impact function or in a network in which some edges are contained in multiple cycles of an arbitrary length.

4 Extensions

We now examine two extensions. Section 4.1 generalizes the model to incorporate richer timing architectures for battlefield organizers. Section 4.2 considers the design problem from a centralized organizer's perspective.

4.1 Sequential Move

In the baseline model, organizers set contest rules simultaneously. We now consider an alternative setting in which organizers act sequentially. Specifically, organizers are partitioned into T groups, and the first stage of the game unfolds over $T \geq 1$ period(s) accordingly. In each period, one group of organizers choose their contest rules simultaneously, with their choices observable to later movers. Fix $t \in \{1, \ldots, T\}$, and let $\mathcal{E}^t$ denote the set of battlefields whose organizers act in period t. Further, let $\boldsymbol{\alpha}^{<t} := (\boldsymbol{\alpha}^e)_{e \in \mathcal{E}^\tau, 1 \leq \tau \leq t-1}$ denote the bias profile chosen prior to period t. The following

result ensues.

Theorem 4 ***(Equilibrium with Sequential Move)*** *Suppose that Assumption 1 or 2 holds. Then there exists a unique SPNE, in which every organizer perfectly levels the playing field. That is, for each $t \in \{1, \ldots, T\}$, a biases profile $\boldsymbol{\alpha}^{<t}$, and a battlefield $e \in \mathcal{E}^t$, the organizer of battlefield e chooses $\boldsymbol{\alpha}^e(\boldsymbol{\alpha}^{<t})$ such that $p_i^e = p_j^e = 1/2$. As a result, $(\boldsymbol{\alpha}^{**}, \boldsymbol{x}^{**})$ is the unique equilibrium outcome.*

By Theorem 4, the conditions that sustain the level-playing-field principle in a simultaneous-move setting ensure that $(\boldsymbol{\alpha}^{**}, \boldsymbol{x}^{**})$ remains the unique equilibrium outcome under sequential moves.

The proof and logic are straightforward. Consider a simple example with two battlefields, as illustrated in Figure 1c. Let the organizer of battlefield a move first. The first-stage game can be solved by backward induction. The organizer on battlefield b—the second mover—will fully balance the playing field regardless of the contest rule for battlefield a, as implied by Theorem 2.

Now consider the organizer of battlefield a, the first mover. She anticipates that the late mover will fully level the playing field. In other words, the contest rule for battlefield b is endogenously pegged to her choice for battlefield a to induce a fully balanced competition on battlefield b. This dynamic strategic linkage neutralizes the indirect network effect caused by her own choice, which allows her to focus on the direct local effect. Thus, she would also set a contest rule to fully level the playing field on her battlefield.

4.2 Centralized Contest Design

We now let a central organizer set contest rules $\boldsymbol{\alpha} = \{\boldsymbol{\alpha}^e\}_{e \in \mathcal{E}}$ for all battlefields in the first stage of the game. Upon observing the contest rules, players simultaneously exert their efforts in the second stage.

We assume that the organizer maximizes an objective function determined by the profile of players' individual total effort profile, i.e., $\Lambda(\boldsymbol{x}) := \Lambda(X_1, X_2, \ldots, X_N)$. It is noteworthy that the central organizer is only concerned about each player's individual total effort X_i instead of his entire effort profile $\boldsymbol{x}_i$. This assumption ensures the existence of an optimum.[15] Clearly, varying contest rules has no effect on the

[15] Otherwise, an optimum may not exist. To see this, consider a setting with $\mathcal{N} = \{1, 2\}$, $\mathcal{E} = \{a, b\}$, and $\Gamma = \{(1, a), (1, b), (2, a), (2, b)\}$, as in Figure 1c. Set $f_i^e(x_i^e) = x_i^e$ for all $(i, e) \in \Gamma$, $c_i(X_i) = (X_i)^2$,

organizer's payoff in the pure-budget case. We therefore focus on the pure-cost case. Moreover, we assume that the objective function $\Lambda(\boldsymbol{x})$ is strictly increasing in X_i for all $i \in \mathcal{N}$. That is, the organizer strictly benefits from each player's effort contribution. A simple example is the aggregate effort over the network— i.e., $\Lambda(\boldsymbol{x}) = \sum_{i\in\mathcal{N}} X_i$. The following result ensues.

Theorem 5 ***(Centralized Contest Design within A Network)*** *Suppose that the central organizer's objective function $\Lambda(\boldsymbol{x}) = \Lambda(X_1, \ldots, X_N)$ is strictly increasing in X_i for all $i \in \mathcal{N}$. The optimal contest is unique, in which the organizer sets $\boldsymbol{\alpha} = \boldsymbol{\alpha}^{**}$ and players win with equal probability on every battlefield.*

Theorem 5 shows that a central organizer always benefits from leveling the playing fields. Her choices of $\boldsymbol{\alpha}$ internalize the externalities each battlefield's contest rule imposes on the others. Leveling the playing field maximizes every player's effort incentives. Recall that $\boldsymbol{\alpha}^{**}$ is the unique bias profile that would induce even winning odds on all battlefields. This leads to the following result.

Corollary 1 ***(Centralization versus Decentralization)*** *The optimal contest rules for the central designer, $\boldsymbol{\alpha}^{**}$, also constitute a first-stage equilibrium of the decentralized contest design game in which the organizer on each battlefield unilaterally maximizes total effort in her own battle.*

Corollary 1 yields useful practical implications. Notably, the central organizer's interests are not aligned with the organizers in our original decentralized contest design game. Nevertheless, Corollary 1 predicts that if the central organizer simply delegates the task of setting contest rules to a set of self-interested agents—each managing a single battlefield—the resulting equilibrium outcome may still replicate the centrally determined optimum.

5 Conclusion

In this paper, we analyze a game of decentralized contest design in which multiple players engage in pairwise contests within a network. Each battlefield is managed by an organizer who sets contest rules to incentivize effort supply for her own contest.

and $(v^a, v^b) = (1, 1)$. Suppose that the organizer's objective is to maximize total effort on battlefield a, i.e., $\Lambda = x_1^a + x_2^a$. We can verify that no optimal biases exist. The organizer can generate total effort arbitrarily close to the supremum—equal to 1/4—by setting $\boldsymbol{\alpha}^a = (1, 1)$ and $\boldsymbol{\alpha}^b = (\varepsilon, 1 - \varepsilon)$, where ε is an infinitesimal positive parameter.

We investigate the subgame perfect Nash equilibrium of the game and examine the extent to which the well-known level-playing-field principle continues to hold in this networked setting, given the complex externalities that arise when contest rules are set independently for individual battlefields. We show that an even-odds equilibrium always exists, in which the contest on every battlefield is resolved with equal probability. We further identify sufficient conditions under which the level-playing-field principle remains valid—i.e., conditions under which each organizer prefers a fully balanced contest regardless. We also demonstrate that the even-odds equilibrium may remain unique even when these conditions are not satisfied, and a fully balanced contest is not necessarily an organizer's unconditional best response.

Our paper is the first to analyze decentralized contest design in a networked context. The analysis sheds new light on the game-theoretic structure of networked contest games and contributes novel insights to our understanding of the conventional wisdom in the contest literature of leveling the playing field.

Ample opportunities for future research remain. For instance, Section 3.3 identifies sufficient conditions for the uniqueness of the even-odds equilibrium, even when the level-playing-field principle fails to hold. These conditions, however, are not necessary: In all cases examined, our numerical exercises reveal that the equilibrium remains unique even when the conditions are violated. This observation naturally implies that uniqueness can hold under more general conditions, e.g., as long as the impact functions are strictly concave. Although this conjecture is analytically difficult to establish, it warrants serious research attention going forward.

Our paper assumes bilateral contests on each battlefield. A natural extension would be to allow multilateral competitions, which would introduce formidable technical challenges. First, in a battlefield e with n^e players, the organizer's choice of contest rules becomes a vector of $(n^e - 1)$ dimenions, rather than a single variable as in our current setup. This greatly increases the dimensionality of the decision problem, with the complications further compounded in a networked environment. Second, in a multilateral setting, defining and measuring competitive balance on a battlefield can be considerably more elusive.

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Appendix: Proofs

Proof of Proposition 1. By Xu, Zenou, and Zhou (2022), the second-stage equilibrium is unique for an arbitrary first-stage biases profile for the pure-cost case, and it suffices to consider the pure-budget case. We state several intermediate results (whose proof can be found in the Supplemental Appendix):

Lemma A1 *Fix $\boldsymbol{\alpha}$ and $e \in \mathcal{E}$, with $\mathcal{N}^e = \{i, j\}$. If there exists $\boldsymbol{x} \in \mathcal{X}(\boldsymbol{\alpha})$ such that $x_j^e > 0$, then $(x_i^e)' = x_i^e$ for all $\boldsymbol{x}' \in \mathcal{X}(\boldsymbol{\alpha})$. As a corollary, fixing $\boldsymbol{x} \in \mathcal{X}(\boldsymbol{\alpha})$, if $x_i^e = 0$ for some (i, e)—which implies $x_j^e > 0$—then $(x_i^e)' = 0$ for all $\boldsymbol{x}' \in \mathcal{X}(\boldsymbol{\alpha})$.*

Lemma A2 *Fix $\boldsymbol{\alpha}$, $\boldsymbol{x}' \in \mathcal{X}(\boldsymbol{\alpha})$, and battlefield e_1, with $\mathcal{N}^{e_1} = \{i_0, j_1\}$. If $(x_{j_1}^{e_1})' = 0$, then for any $\boldsymbol{x}'' \in \mathcal{X}(\boldsymbol{\alpha})$, we have $(x_{j_u}^{e_u})'' = 0$ for all $e_u \in \mathcal{E}_{i_0}$, with $\mathcal{N}^{e_u} = \{i_0, j_u\}$. Moreover, $\lambda_{i_0}(\boldsymbol{x}'') = 0$.*

Lemma A3 *Fixing $\boldsymbol{\alpha}$, $\lambda_i(\boldsymbol{x}') = \lambda_i(\boldsymbol{x}'') =: \lambda_i$ for all $\boldsymbol{x}', \boldsymbol{x}'' \in \mathcal{X}(\boldsymbol{\alpha})$ and all $i \in \mathcal{N}$.*

We first prove part (i) of the proposition. Let $(\boldsymbol{\alpha}^*, \boldsymbol{x}^*(\cdot))$ be an SPNE and let $\boldsymbol{x}' := \boldsymbol{x}^*(\boldsymbol{\alpha}^*)$. Suppose, to the contrary, that there exists $\boldsymbol{x}'' \neq \boldsymbol{x}'$ such that $\boldsymbol{x}'' \in \mathcal{X}(\boldsymbol{\alpha}^*)$. Then there exists $i_0 \in \mathcal{N}$ and $e_1 \in \mathcal{E}_{i_0}$ such that $(x_{i_0}^{e_1})' \neq (x_{i_0}^{e_1})''$. By Lemma A1, $(x_{j_1}^{e_1})' = 0$ and $(x_{j_1}^{e_1})'' = 0$, where $j_1 \in \mathcal{N}^{e_1}$ and $j_1 \neq i_0$; otherwise, player i_0 would choose the same effort in the battlefield across all equilibria, which contradicts $(x_{i_0}^{e_1})' \neq (x_{i_0}^{e_1})''$. Further, by Lemma A2, $(x_{j_u}^{e_u})' = (x_{j_u}^{e_u})'' = 0$ for all $e_u \in \mathcal{E}_{i_0}$, with $\mathcal{N}^{e_u} = \{i_0, j_u\}$, and $\lambda_{i_0}(\boldsymbol{x}') = \lambda_{i_0}(\boldsymbol{x}'') = 0$.

Consider an arbitrary battlefield $e_u \in \mathcal{E}_{i_0}$ and player i_0's opponent, player j_u. By Lemma A3, λ_{j_u} is the same across all equilibria; together with the fact that $(x_{j_u}^{e_u})' = (x_{j_u}^{e_u})'' = 0$, we can conclude that

$$\lambda_{j_u} \geq \frac{(\alpha_{j_u}^{e_u})^* (f_{j_u}^{e_u})'(0)}{(\alpha_{i_0}^{e_u})^* f_{i_0}^{e_u}(x_{i_0}^{e_u})},$$

where $x_{i_0}^{e_u}$ is player i_0's equilibrium effort on battlefield e_u. By the monotonicity of $f_{i_0}^{e_u}(\cdot)$, there exists a unique $x_{i_0}^{e_u}$ such that the above inequality holds with equality, and denote it by $\hat{x}_{i_0}^{e_u}$. It follows immediately that $\hat{x}_{i_0}^{e_u} \leq (x_{i_0}^{e_u})'$ and $\hat{x}_{i_0}^{e_u} \leq (x_{i_0}^{e_u})''$, which in turn implies that

$$\sum_{e_u \in \mathcal{E}_{i_0}} \hat{x}_{i_0}^{e_u} \leq \sum_{e_u \in \mathcal{E}_{i_0}} (x_{i_0}^{e_u})' \leq \overline{X}_{i_0} \text{ and } \sum_{e_u \in \mathcal{E}_{i_0}} \hat{x}_{i_0}^{e_u} \leq \sum_{e_u \in \mathcal{E}_{i_0}} (x_{i_0}^{e_u})'' \leq \overline{X}_{i_0}.$$

If $\sum_{e_u\in\mathcal{E}_{i_0}}\hat{x}_{i_0}^{e_u}=\overline{X}_{i_0}$, then all inequalities above hold with equality, which implies that $(x_{i_0}^{e_u})'=(x_{i_0}^{e_u})''=\hat{x}_{i_0}^{e_u}$ for all $e_u\in\mathcal{E}_{i_0}$. This contradicts the postulated $(x_{i_0}^{e_1})'\neq(x_{i_0}^{e_1})''$. If $\sum_{e_u\in\mathcal{E}_{i_0}}\hat{x}_{i_0}^{e_u}<\overline{X}_{i_0}$, we consider the following alternative strategy for player i_0: $\boldsymbol{x}_{i_0}=\big(x_{i_0}^{e_1},(\hat{x}_{i_0}^{e_u})_{e_u\in\mathcal{E}_{i_0}\setminus\{e_1\}}\big)$, where $x_{i_0}^{e_1}=\overline{X}_{i_0}-\sum_{e_u\in\mathcal{E}_{i_0}\setminus\{e_1\}}\hat{x}_{i_0}^{e_u}\geq 0$. It follows immediately that $x_{i_0}^{e_1}\geq(x_{i_0}^{e_1})'$ and $x_{i_0}^{e_1}\geq(x_{i_0}^{e_1})''$. Recall the postulated $(x_{i_0}^{e_1})'\neq(x_{i_0}^{e_1})''$. We can thus assume $x_{i_0}^{e_1}>(x_{i_0}^{e_1})'$ without loss.

Set $\boldsymbol{\alpha}^{e_1}=(\alpha_{i_0}^{e_1},\alpha_{j_1}^{e_1})$ such that $\lambda_{j_1}=\frac{\alpha_{j_1}^{e_1}(f_{j_1}^{e_1})'(0)}{\alpha_{i_0}^{e_1}f_{i_0}^{e_1}(x_{i_0}^{e_1})}$. It is straightforward to verify that $(\boldsymbol{x}_{i_0},(\boldsymbol{x}_{-i_0})')$ satisfies (2) and (3) in Lemma 1 under $\big(\boldsymbol{\alpha}^{e_1},\{(\boldsymbol{\alpha}^{e})^*\}_{e\in\mathcal{E}\setminus\{e_1\}}\big)$, and thus constitutes a second-stage equilibrium. Following a similar argument as in the previous analysis, we can conclude that for any second-stage equilibrium under this biases profile, player i_0's equilibrium effort on battlefield e_1 is $x_{i_0}^{e_1}$. Note that $x_{i_0}^{e_1}>(x_{i_0}^{e_1})'$. Therefore, fixing $\{(\boldsymbol{\alpha}^{e})^*\}_{e\in\mathcal{E}\setminus\{e_1\}}$, the organizer of battlefield e_1 is better off by deviating from $(\boldsymbol{\alpha}^{e_1})^*$ to $\boldsymbol{\alpha}^{e_1}$, which contradicts the postulate that $\boldsymbol{\alpha}^*$ constitutes a first-stage equilibrium.

Next, we prove part (ii) of the proposition. Suppose, to the contrary, that there exists $\boldsymbol{x}(\cdot)$, with $\boldsymbol{x}(\boldsymbol{\alpha})\in\mathcal{X}(\boldsymbol{\alpha})$, such that $(\boldsymbol{\alpha}^*,\boldsymbol{x}(\cdot))$ is not an SPNE. Therefore, fixing $\boldsymbol{x}(\cdot)$ and $(\boldsymbol{\alpha}^{-e})^*$, there exists a battlefield e such that setting $(\boldsymbol{\alpha}^{e})^*$ is suboptimal to its organizer. Denote the most profitable deviation for the organizer by $(\boldsymbol{\alpha}^{e})'$ and let $\boldsymbol{\alpha}'=\big((\boldsymbol{\alpha}^{e})',(\boldsymbol{\alpha}^{-e})^*\big)$. By assumption, $(\boldsymbol{\alpha}^*,\boldsymbol{x}^*(\cdot))$ is an SPNE. Therefore, the organizer of battlefield e is better off in $\boldsymbol{x}^*(\boldsymbol{\alpha}^*)$ than in $\boldsymbol{x}^*(\boldsymbol{\alpha}')$. Thus, she is strictly better off in $\boldsymbol{x}(\boldsymbol{\alpha}')$ than in $\boldsymbol{x}^*(\boldsymbol{\alpha}')$.

By Lemma A1, it is impossible that the two players on battlefield e are active. Otherwise, the equilibrium efforts on battlefield e under $\boldsymbol{x}(\boldsymbol{\alpha}')$ coincide with those under $\boldsymbol{x}^*(\boldsymbol{\alpha}')$. Meanwhile, it is evident that at least one player exerts positive effort in each battlefield in the second stage. Therefore, it must be the case that one player remains active and the other inactive on battlefield e, and the active player's effort is strictly higher in $\boldsymbol{x}(\boldsymbol{\alpha}')$ than in $\boldsymbol{x}^*(\boldsymbol{\alpha}')$. Following a similar argument as in the proof of part (i), this is impossible given that $(\boldsymbol{\alpha}^{e})'$ is the most profitable deviation for the organizer of battlefield e. This concludes the proof. □

Proof of Lemma 2. For the pure-cost case, similar to (8), we can obtain that

$$\widetilde{\lambda}_i=c_i'\Big(\sum_{e\in\mathcal{E}_i}(g_i^e)^{-1}\Big(\frac{v^e\widetilde{p}_i^e(1-\widetilde{p}_i^e)}{\widetilde{\lambda}_i}\Big)\Big)\leq c_i'\Big(\sum_{e\in\mathcal{E}_i}(g_i^e)^{-1}\Big(\frac{v^e}{4\widetilde{\lambda}_i}\Big)\Big),$$

where the inequality follows from the monotonicity of c_i' and $(g_i^e)^{-1}$ and $\widetilde{p}_i^e(1-\widetilde{p}_i^e) \leq 1/4$. The above inequality, together with (8), implies $\widetilde{\lambda}_i \leq \lambda_i^{**}$. The proof for the pure-budget case is similar and omitted for brevity. □

Proof of Lemma 3. For notational convenience, define $(f_i^e)^{**} := f_i^e((x_i^e)^{**})$, $(g_i^e)^{**} := g_i^e((x_i^e)^{**})$, $\widetilde{f}_i^e := f_i^e(\widetilde{x}_i^e)$, and $\widetilde{g}_i^e := g_i^e(\widetilde{x}_i^e)$. Suppose, to the contrary, that $\widetilde{x}_i^e < (x_i^e)^{**}$. By (2), we have

$$\lambda_i^{**}(g_i^e)^{**} = \lambda_j^{**}(g_j^e)^{**} = \frac{v^e}{4} \text{ and } \widetilde{\lambda}_i\widetilde{g}_i^e = \widetilde{\lambda}_j\widetilde{g}_j^e = v^e\widetilde{p}_i^e(1-\widetilde{p}_i^e). \tag{7}$$

Evidently, at least one player is active on battlefield e in the equilibrium. We consider three cases:

(a) Suppose $\widetilde{x}_i^e = 0$ and $\widetilde{x}_j^e > 0$. The postulated $\widetilde{\lambda}_i/\lambda_i^{**} \leq \widetilde{\lambda}_j/\lambda_j^{**}$ implies that $\widetilde{\lambda}_i = 0$. Meanwhile, it follows from (3) that $\widetilde{\lambda}_i \geq \frac{(\alpha_i^e)^{**}(f_i^e)'(0)}{(\alpha_j^e)^{**}\widetilde{f}_j^e} > 0$. A contraction.

(b) Suppose $\widetilde{x}_j^e = 0$ and $\widetilde{x}_i^e > 0$. From (3), we have $\widetilde{\lambda}_j \geq \frac{(\alpha_j^e)^{**}(f_j^e)'(0)}{(\alpha_i^e)^{**}\widetilde{f}_i^e}$. Further, $(x_j^e)^{**} > 0$ implies $\lambda_j^{**} < \frac{(\alpha_j^e)^{**}(f_j^e)'(0)}{(\alpha_i^e)^{**}(f_i^e)^{**}}$. Note that $\widetilde{\lambda}_j \leq \lambda_j^{**}$ by Lemma 2. Together, these indicate that $(f_i^e)^{**} < \widetilde{f}_i^e$, which implies that $(x_i^e)^{**} < \widetilde{x}_i^e$. A contradiction.

(c) Now suppose, $\widetilde{x}_j^e > 0$ and $\widetilde{x}_i^e > 0$. Let $a_i := \widetilde{f}_i^e/(f_i^e)^{**}$ and $a_j := \widetilde{f}_j^e/(f_j^e)^{**}$. Note that $g_i^e(\cdot) \equiv f_i^e(\cdot)/(f_i^e(\cdot))'$ is strictly increasing; together with the postulated $\widetilde{x}_i^e < (x_i^e)^{**}$, we have that $\widetilde{g}_i^e < (g_i^e)^{**}$. Further, by (7), we have $\frac{\widetilde{\lambda}_i\widetilde{g}_i^e}{\lambda_i^{**}(g_i^e)^{**}} = \frac{\widetilde{\lambda}_j\widetilde{g}_j^e}{\lambda_j^{**}(g_j^e)^{**}}$; together with $\widetilde{g}_i^e < (g_i^e)^{**}$ and the postulated $\widetilde{\lambda}_i/\lambda_i^{**} \leq \widetilde{\lambda}_j/\lambda_j^{**}$, we can obtain that $\widetilde{g}_j^e < (g_j^e)^{**}$ and $\widetilde{x}_j^e < (x_j^e)^{**}$, which implies that $0 < a_i, a_j < 1$.

By (1), we have that $1 = \frac{(p_i^e)^{**}}{(p_j^e)^{**}} = \frac{(\alpha_i^e)^{**}(f_i^e)^{**}}{(\alpha_j^e)^{**}(f_j^e)^{**}}$, which implies $\frac{\widetilde{p}_i^e}{\widetilde{p}_j^e} = \frac{(\alpha_i^e)^{**}\widetilde{f}_i^e}{(\alpha_j^e)^{**}\widetilde{f}_j^e} = \frac{\widetilde{f}_i^e/(f_i^e)^{**}}{\widetilde{f}_j^e/(f_j^e)^{**}} = \frac{a_i}{a_j}$ and thus $(\widetilde{p}_i^e, \widetilde{p}_j^e) = (\frac{a_i}{a_i+a_j}, \frac{a_j}{a_i+a_j})$; together with (7), we have that $\frac{\widetilde{g}_i^e}{(g_i^e)^{**}} = 4\widetilde{p}_i^e(1-\widetilde{p}_i^e) \times \frac{\lambda_i^{**}}{\widetilde{\lambda}_i} \geq \frac{4a_ia_j}{(a_i+a_j)^2}$, where the inequality follows from Lemma 2. Further, from the concavity of $f_i^e(\cdot)$ and the postulated $\widetilde{x}_i^e < (x_i^e)^{**}$, we have

$$a_i = \frac{\widetilde{f}_i^e}{(f_i^e)^{**}} = \frac{\widetilde{g}_i^e}{(g_i^e)^{**}} \times \frac{(f_i^e)'(\widetilde{x}_i^e)}{(f_i^e)'((x_i^e)^{**})} \geq \frac{\widetilde{g}_i^e}{(g_i^e)^{**}} \geq \frac{4a_ia_j}{(a_i+a_j)^2},$$

which implies $\frac{4a_j}{(a_i+a_j)^2} \leq 1$. Similarly, we can obtain that $\frac{4a_i}{(a_i+a_j)^2} \leq 1$. Summing the two inequalities yields $a_i + a_j \geq 2$, which contradicts $a_i, a_j < 1$.

This completes the proof. □

Proof of Theorem 1. It suffices to show that there is a unique profile of contest rules $\boldsymbol{\alpha}^{**}$ that leads to the equilibrium outcome of equal winning odds on every battlefield. For the pure-cost case, we first solve for the second-stage equilibrium profile $\boldsymbol{x}^{**}$ that leads to equal winning odds in all battlefields. Plugging $(p_i^e)^{**} = 1/2$ into (2) yields $\frac{v^e}{4} = \lambda_i^{**} g_i^e\big((x_i^e)^{**}\big)$; together with the monotonicity of $g_i^e \equiv f_i^e/(f_i^e)'$, we have $(x_i^e)^{**} = (g_i^e)^{-1}\big(\frac{v^e}{4\lambda_i^{**}}\big)$ and

$$\lambda_i^{**} = c_i'(X_i^{**}) = c_i'\Bigg(\sum_{e\in\mathcal{E}_i}(g_i^e)^{-1}\Big(\frac{v^e}{4\lambda_i^{**}}\Big)\Bigg), \tag{8}$$

from which we can solve for λ_i^{**} and pin down $\boldsymbol{x}^{**}$. The first-stage biases profile $\boldsymbol{\alpha}^{**}$ is uniquely determined by (1).

The proof of the pure-budget case closely follows that of the pure-cost case after we replace (8) with $\overline{X}_i = \sum_{e\in\mathcal{E}_i}(g_i^e)^{-1}\big(\frac{v^e}{4\lambda_i^{**}}\big)$. This concludes the proof. □

Proof of Lemma 4. It suffices to show that for any $\boldsymbol{\alpha}^{e_0}$, the organizer of battlefield e_0 can choose w^{e_0} to induce the same equilibrium effort profile $\boldsymbol{x}$ and vice versa.

First, fix an arbitrary $\boldsymbol{\alpha}^{e_0}$ and a second-stage equilibrium $\boldsymbol{x}^*$, which yields $(w^{e_0})^*$. Evidently, the organizer can set $w^{e_0} = (w^{e_0})^*$ to induce $\boldsymbol{x}^*$. Second, fixing an arbitrary $w^{e_0} \le v^{e_0}/4$—which induces $\boldsymbol{x}^*$—the winning probability on battlefield e_0 can be solved from (4). The corresponding biases $\boldsymbol{\alpha}^{e_0}$ can then be derived from (1). □

Proof of Lemma 5. We first state an intermediate result (whose proof can be found in the Supplemental Appendix).

Lemma A4 *Fix a battlefield $e_0 \in \mathcal{E}$, with $\mathcal{N}^{e_0} = \{i_0, j_0\}$. The following statements hold in the second-stage equilibrium:*

(i) Fix $e \neq e_0$, with $\mathcal{N}^e = \{i, j\}$. If $x_i^e, x_j^e > 0$, then

$$\frac{dx_i^e}{dw^{e_0}} = -\frac{w^e m_i^e}{\lambda_i} \times \frac{\big[1-(2p_i^e-1)m_j^e\big]\frac{1}{\lambda_i}\frac{d\lambda_i}{dw^{e_0}} + (2p_i^e-1)m_j^e\frac{1}{\lambda_j}\frac{d\lambda_j}{dw^{e_0}}}{1+(m_i^e-m_j^e)(p_i^e-p_j^e)}, \tag{9}$$

where $g_i^e := g_i^e(x_i^e)$ and $m_i^e := \frac{(f_i^e)'(x_i^e)^2}{(f_i^e)'(x_i^e)^2 - f_i^e(x_i^e)(f_i^e)''(x_i^e)} \in [0,1]$.

(ii) For battlefield e_0, we have that

$$\frac{dx_{i_0}^{e_0}}{dw^{e_0}} = g_{i_0}^{e_0} m_{i_0}^{e_0} \left[\frac{1}{w^{e_0}} - \frac{1}{\lambda_{i_0}} \frac{d\lambda_{i_0}}{dw^{e_0}} \right] \text{ and } \frac{dx_{j_0}^{e_0}}{dw^{e_0}} = g_{j_0}^{e_0} m_{j_0}^{e_0} \left[\frac{1}{w^{e_0}} - \frac{1}{\lambda_{j_0}} \frac{d\lambda_{j_0}}{dw^{e_0}} \right]. \quad (10)$$

(iii) For each $i \in \mathcal{N}$, we have that

$$\delta_i \frac{d\lambda_i}{dw^{e_0}} = \sum_{e \in \mathcal{E}_i} \frac{dx_i^e}{dw^{e_0}}, \quad (11)$$

where $\delta_i = 0$ in the pure-budget case and $\delta_i = \frac{1}{c_i''(X_i)}$ in the pure-cost case.

We are ready to prove the lemma. For part (i), first consider the case of $x_i^e, x_j^e > 0$. By (9), we have that

$$\begin{aligned} &\frac{d\lambda_i}{dw^{e_0}} \frac{dx_i^e}{dw^{e_0}} + \frac{d\lambda_j}{dw^{e_0}} \frac{dx_j^e}{dw^{e_0}} \\ =& - w^e \frac{m_i^e \left[1 - (2p_i^e - 1)m_j^e\right] \left(\frac{1}{\lambda_i} \frac{d\lambda_i}{dw^{e_0}}\right)^2 + m_j^e \left[1 + (2p_i^e - 1)m_i^e\right] \left(\frac{1}{\lambda_j} \frac{d\lambda_j}{dw^{e_0}}\right)^2}{1 + (m_i^e - m_j^e)(p_i^e - p_j^e)}. \end{aligned}$$

Note that $m_i^e \in [0, 1]$. Moreover, $x_i^e, x_j^e > 0$ implies that $p_i^e \in (0, 1)$. Therefore, $1 - (2p_i^e - 1)m_j^e > 0$, $1 + (2p_i^e - 1)m_i^e > 0$, and $1 + (m_i^e - m_j^e)(p_i^e - p_j^e) > 0$, which implies (5).

Next, suppose $x_i^e = 0$ and $x_j^e > 0$. If $x_i^e = 0$ in a neighborhood of w^{e_0}, then by Lemma A2, $\lambda_j = 0$ in this neighborhood. Therefore, $\frac{dx_i^e}{dw^{e_0}} = \frac{d\lambda_j}{dw^{e_0}} = 0$, which also implies (5). Otherwise, if $x_i^e > 0$ in a neighborhood of w^{e_0}, then (5) holds in the neighborhood and is satisfied at w^{e_0} by continuity.

Next, we prove part (ii) of the lemma. By (11), we have that $\sum_{e \in \mathcal{E}_i} \frac{d\lambda_i}{dw^{e_0}} \frac{dx_i^e}{dw^{e_0}} = \delta_i \left(\frac{d\lambda_i}{dw^{e_0}}\right)^2 \geq 0$. This concludes the proof. □

Proof of Lemma 6. For part (i) of the lemma, we first show that there exists $\widetilde{w}^{e_0} < v^{e_0}/4$ such that $\left.\frac{dx_{j_0}^{e_0}}{dw^{e_0}}\right|_{w^{e_0} = \widetilde{w}^{e_0}} \leq 0$. By assumption, there exists $\hat{w}^{e_0} < v^{e_0}/4$ such that $\Lambda^{e_0}(x_{i_0}^{e_0}, x_{j_0}^{e_0})\Big|_{w^{e_0} = \hat{w}^{e_0}} \geq \Lambda^{e_0}(x_{i_0}^{e_0}, x_{j_0}^{e_0})\Big|_{w^{e_0} = v^{e_0}/4}$. Recall that Λ^{e_0} is strictly increasing in $x_{i_0}^{e_0}$ and $x_{j_0}^{e_0}$. There exists $\widetilde{w}^{e_0} \in (\hat{w}^{e_0}, v^{e_0}/4)$ to satisfy $\left.\frac{dx_{i_0}^{e_0}}{dw^{e_0}}\right|_{w^{e_0} = \widetilde{w}^{e_0}} \leq 0$ or $\left.\frac{dx_{j_0}^{e_0}}{dw^{e_0}}\right|_{w^{e_0} = \widetilde{w}^{e_0}} \leq 0$. Further, $\left.\frac{dx_{i_0}^{e_0}}{dw^{e_0}}\right|_{w^{e_0} = \widetilde{w}^{e_0}} \geq \left.\frac{dx_{j_0}^{e_0}}{dw^{e_0}}\right|_{w^{e_0} = \widetilde{w}^{e_0}}$ by assumption. Therefore,

$\left.\frac{dx_{j_0}^{e_0}}{dw^{e_0}}\right|_{w^{e_0}=\widetilde{w}^{e_0}} \leq 0$. In the rest of the proof, all derivatives are evaluated at $w^{e_0} = \widetilde{w}^{e_0}$.

Next, we show that $\frac{dx_{i_0}^{e_0}}{dw^{e_0}} \geq 0$. Suppose, to the contrary, that $\frac{dx_{i_0}^{e_0}}{dw^{e_0}} < 0$. By (10), we have that $\frac{d\lambda_{i_0}}{dw^{e_0}} > 0$ and $\frac{d\lambda_{j_0}}{dw^{e_0}} > 0$. Therefore,

$$\frac{d\lambda_{i_0}}{dw^{e_0}} \frac{dx_{i_0}^{e_0}}{dw^{e_0}} + \frac{d\lambda_{j_0}}{dw^{e_0}} \frac{dx_{j_0}^{e_0}}{dw^{e_0}} < 0. \tag{12}$$

Note that

$$\mathcal{I} = \sum_{i\in\mathcal{N}} \sum_{e\in\mathcal{E}_i} \frac{d\lambda_i}{dw^{e_0}} \frac{dx_i^e}{dw^{e_0}} \geq 0, \tag{13}$$

where the inequality follows from Lemma 5(ii). However, (5) and (12) imply that

$$\mathcal{I} = \sum_{e\neq e_0} \sum_{i\in\mathcal{N}^e} \frac{d\lambda_i}{dw^{e_0}} \frac{dx_i^e}{dw^{e_0}} + \frac{d\lambda_{i_0}}{dw^{e_0}} \frac{dx_{i_0}^{e_0}}{dw^{e_0}} + \frac{d\lambda_{j_0}}{dw^{e_0}} \frac{dx_{j_0}^{e_0}}{dw^{e_0}} < 0,$$

which is a contradiction.

Next, we prove part (ii) of the lemma. The first two inequalities in part (ii) follow immediately from part (i) and (10), and it remains to prove $\frac{d\lambda_{i_0}}{dw^{e_0}} > 0$. Suppose, to the contrary, that $\frac{d\lambda_{i_0}}{dw^{e_0}} < 0$; together with $\frac{dx_{i_0}^{e_0}}{dw^{e_0}} > 0$ as shown in part (i), we can obtain (12). From (12) and Lemma 5(ii), we have that $\mathcal{I} < 0$, which contradicts (13).

Part (iii) of the lemma follows immediately from parts (i) and (ii), and it remains to prove part (iv). It suffices to show $\frac{dx_{j_0}^{e'}}{dw^{e_0}} \leq 0$; $\frac{dx_{j_0}^{e'}}{dw^{e_0}} \frac{d\lambda_{j_0}}{dw^{e_0}} \leq 0$ can be implied by $\frac{dx_{j_0}^{e'}}{dw^{e_0}} \leq 0$ and (ii).

Fix $e' \neq e_0$, with $\mathcal{N}^{e'} = \{i_0, j_0\}$. By (9), we have that

$$\frac{dx_{j_0}^{e'}}{dw^{e_0}} = -\frac{w^{e'} m_{j_0}^{e'}}{\lambda_{j_0}} \times \frac{\left[1 - (2p_{j_0}^{e'} - 1)m_{i_0}^{e'}\right] \frac{1}{\lambda_{j_0}} \frac{d\lambda_{j_0}}{dw^{e_0}} + (2p_{j_0}^{e'} - 1)m_{i_0}^{e'} \frac{1}{\lambda_{i_0}} \frac{d\lambda_{i_0}}{dw^{e_0}}}{1 + (m_{i_0}^{e'} - m_{j_0}^{e'})(p_{i_0}^{e'} - p_{j_0}^{e'})}.$$

Recall that we have shown $1 + (m_{i_0}^{e'} - m_{j_0}^{e'})(p_{i_0}^{e'} - p_{j_0}^{e'}) > 0$ in Lemma 5. Further, simple algebra would verify that

$$\begin{aligned}
&\left[1 - (2p_{j_0}^{e'} - 1)m_{i_0}^{e'}\right] \frac{1}{\lambda_{j_0}} \frac{d\lambda_{j_0}}{dw^{e_0}} + (2p_{j_0}^{e'} - 1)m_{i_0}^{e'} \frac{1}{\lambda_{i_0}} \frac{d\lambda_{i_0}}{dw^{e_0}} \\
=&\frac{1}{\lambda_{j_0}} \frac{d\lambda_{j_0}}{dw^{e_0}} - (2p_{j_0}^{e'} - 1)m_{i_0}^{e'} \left[\frac{1}{\lambda_{j_0}} \frac{d\lambda_{j_0}}{dw^{e_0}} - \frac{1}{\lambda_{i_0}} \frac{d\lambda_{i_0}}{dw^{e_0}}\right]
\end{aligned}$$

$$\geq \frac{1}{\lambda_{j_0}} \frac{d\lambda_{j_0}}{dw^{e_0}} - \left[\frac{1}{\lambda_{j_0}} \frac{d\lambda_{j_0}}{dw^{e_0}} - \frac{1}{\lambda_{i_0}} \frac{d\lambda_{i_0}}{dw^{e_0}} \right] \geq 0,$$

where the first inequality follows from $p_{j_0}^{e'} \in [0,1]$, $m_{i_0}^{e'} \in [0,1]$, and $\frac{1}{\lambda_{j_0}} \frac{d\lambda_{j_0}}{dw^{e_0}} - \frac{1}{\lambda_{i_0}} \frac{d\lambda_{i_0}}{dw^{e_0}} \geq 0$; and the second inequality follows from part (ii) of the lemma. Therefore, $\frac{dx_{j_0}^{e'}}{dw^{e_0}} \leq 0$, which concludes the proof. □

Proof of Theorem 2. It remains to show that under Assumption 1, $\frac{d\lambda_s}{dw^{e_0}} \frac{dx_s^e}{dw^{e_0}} < 0$ for each $e \in \mathcal{E}_s$, with $e \neq e_0$. We first show that $\frac{d\lambda_s}{dw^{e_0}} \neq 0$. Suppose, to the contrary, that $\frac{d\lambda_s}{dw^{e_0}} = 0$. By the definition of s, $\frac{d\lambda_i}{dw^{e_0}} = 0$ for each $i \in \mathcal{N}$; together with (10), we have that $\frac{dx_{i_0}^{e_0}}{dw^{e_0}} < 0$ and $\frac{dx_{j_0}^{e_0}}{dw^{e_0}} < 0$, which contradicts Lemma 6(i).

Next, suppose $\frac{d\lambda_s}{dw^{e_0}} > 0$ (the analysis for the case with $\frac{d\lambda_s}{dw^{e_0}} < 0$ follows analogously). Fix $e \in \mathcal{E}_s$, with $e \neq e_0$ and $\mathcal{N}^e = \{s, j\}$. By (9), we have that

$$\frac{dx_s^e}{dw^{e_0}} = -\frac{w^e m_s^e}{\lambda_s} \times \frac{\left[1 - (2p_s^e - 1)m_j^e\right] \frac{1}{\lambda_s} \frac{d\lambda_s}{dw^{e_0}} + (2p_s^e - 1)m_j^e \frac{1}{\lambda_j} \frac{d\lambda_j}{dw^{e_0}}}{1 + (m_s^e - m_j^e)(p_s^e - p_j^e)}.$$

Assumption 1 implies that $(f_j^e)'(0) = (f_s^e)'(0) = +\infty$ and thus $p_s^e \in (0,1)$. Further, the assumption implies $m_j^e \in [0, 1/2]$. Carrying out the algebra, we can obtain that

$$\begin{aligned}
&\left[1 - (2p_s^e - 1)m_j^e\right] \frac{1}{\lambda_s} \frac{d\lambda_s}{dw^{e_0}} + (2p_s^e - 1)m_j^e \frac{1}{\lambda_j} \frac{d\lambda_j}{dw^{e_0}} \\
=& \frac{1}{\lambda_s} \frac{d\lambda_s}{dw^{e_0}} - (2p_s^e - 1)m_j^e \left[\frac{1}{\lambda_s} \frac{d\lambda_s}{dw^{e_0}} - \frac{1}{\lambda_j} \frac{d\lambda_j}{dw^{e_0}} \right] \\
>& \frac{1}{\lambda_s} \frac{d\lambda_s}{dw^{e_0}} - \frac{1}{2} \times \left[\frac{1}{\lambda_s} \frac{d\lambda_s}{dw^{e_0}} - \frac{1}{\lambda_j} \frac{d\lambda_j}{dw^{e_0}} \right] \geq 0,
\end{aligned}$$

where the first inequality follows from $m_j^e \in [0, 1/2]$ and $p_s^e \in (0,1)$; and the last inequality follows from $\frac{1}{\lambda_s} \frac{d\lambda_s}{dw^{e_0}} \geq \left| \frac{1}{\lambda_j} \frac{d\lambda_j}{dw^{e_0}} \right|$. This concludes the proof. □

Proof of Theorem 3 Consider the following two cases depending on whether Assumption 1′ or 2′ is satisfied.

Case (a): Assumption 1′ holds. Note that Assumption 1′ implies that $(f_i^e)'(0) = \infty$ and thus $x_i^e > 0$ for all $(i, e) \in \Gamma$. Suppose, to the contrary, that there exists an alternative equilibrium $\big(\boldsymbol{\alpha}^\dagger, \boldsymbol{x}^\dagger(\cdot)\big) \neq \big(\boldsymbol{\alpha}^{**}, \boldsymbol{x}^{**}(\cdot)\big)$. Then there exists (i,e) such that $(p_i^e)^\dagger \neq 1/2$, which implies that $(w^e)^\dagger < v^e/4$. Let e_0 be the battlefield with minimal w^e among all battlefields that satisfy $(w^e)^\dagger < v^e/4$—i.e., $(w^{e_0})^\dagger = \min\big\{(w^e)^\dagger : (w^e)^\dagger < v^e/4\big\}$—and $\mathcal{N}^{e_0} = \{i_0, j_0\}$. By (13), we have that

$$\mathcal{I} \equiv \sum_{(i,e)\in\Gamma} \frac{d\lambda_i}{dw^{e_0}} \frac{dx_i^e}{dw^{e_0}} \geq 0. \tag{14}$$

Note that $\mathcal{I}$ can be divided into three parts:

$$\mathcal{I} = \underbrace{\frac{d\lambda_{i_0}}{dw^{e_0}} \frac{dx_{i_0}^{e_0}}{dw^{e_0}}}_{\mathcal{I}_1} + \underbrace{\frac{d\lambda_{j_0}}{dw^{e_0}} \frac{dx_{j_0}^{e_0}}{dw^{e_0}}}_{\mathcal{I}_2} + \underbrace{\sum_{e\neq e_0} \sum_{i\in\mathcal{N}^e} \frac{d\lambda_i}{dw^{e_0}} \frac{dx_i^e}{dw^{e_0}}}_{\mathcal{I}_3}. \tag{15}$$

Let $s \in \mathcal{N}$ such that $\left|\frac{1}{\lambda_s^\dagger}\frac{d\lambda_s}{dw^{e_0}}\right| = \max_{i\in\mathcal{N}}\left|\frac{1}{\lambda_i^\dagger}\frac{d\lambda_i}{dw^{e_0}}\right|$. By Lemma 6, $s \neq i_0$. Suppose $\frac{1}{\lambda_s^\dagger}\frac{d\lambda_s}{dw^{e_0}} > 0$ (the analysis for the case with $\frac{1}{\lambda_s^\dagger}\frac{d\lambda_s}{dw^{e_0}} < 0$ follows analogously). By (6), there exists $e^\dagger \in \mathcal{E}_s$ such that $\frac{dx_s^{e^\dagger}}{dw^{e_0}} \geq 0$. Denote player s's opponent on battlefield $e^\dagger$ by $j^\dagger$. Evidently, $e^\dagger \neq e_0$. Otherwise, if $e^\dagger = e_0$, then $s = j_0$, which contradicts Lemma 6(iii).

The following intermediate result ensues (whose proof can be found in the Supplemental Appendix).

Lemma A5 *The following statements hold:*

$$\mathcal{I}_1 \leq \frac{(w^{e_0})^\dagger}{4\rho} \times \left(\frac{1}{\lambda_{j_0}^\dagger}\frac{d\lambda_{j_0}}{dw^{e_0}}\right)^2, \tag{16}$$

$$\mathcal{I}_2 \leq 0, \tag{17}$$

$$\mathcal{I}_3 \leq -(w^{e_0})^\dagger \frac{1-\big(2(p_s^{e^\dagger})^\dagger - 1\big)(m_{j^\dagger}^{e^\dagger})^\dagger}{\big(2(p_s^{e^\dagger})^\dagger - 1\big)^2 (m_{j^\dagger}^{e^\dagger})^\dagger} \left(\frac{1}{\lambda_{j_0}^\dagger}\frac{d\lambda_{j_0}}{dw^{e_0}}\right)^2. \tag{18}$$

Plugging (16), (17), and (18) into (15), we can obtain that

$$\mathcal{I} \leq (w^{e_0})^\dagger \times \left(\frac{1}{\lambda_{j_0}^\dagger}\frac{d\lambda_{j_0}}{dw^{e_0}}\right)^2 \times \left[\frac{1}{4\rho} - \frac{1-\big(2(p_s^{e^\dagger})^\dagger - 1\big)(m_{j^\dagger}^{e^\dagger})^\dagger}{\big(2(p_s^{e^\dagger})^\dagger - 1\big)^2 (m_{j^\dagger}^{e^\dagger})^\dagger}\right] < 0,$$

where the last inequality follows from $2(p_s^{e^\dagger})^\dagger - 1 \in (0,1)$, $(m_{j^\dagger}^{e^\dagger})^\dagger \in [0, 1/\rho]$, and $\rho = \frac{1+\sqrt{2}}{2}$. This contradicts (14).

Case (b): Assumption 2′ holds. Recall e_0 as defined in Case (a). The proof is the same as that of Theorem 2 if e_0 is not in a cycle, and it suffices to consider the case in which e_0 is contained in a unique cycle with odd length. Denote the set of players on the cycle by $\{i_0, \ldots, i_{2\ell}\}$—where $2\ell + 1$ gives the length of the cycle—and let $\mathcal{N}^{e_0} = \{i_0, i_1\}$.

By Assumption 2′, if we remove all edges on this cycle—i.e., all battlefields e with $\mathcal{N}^e = \{i_k, i_{k+1}\}$ for some $k \in \{0, \ldots, 2\ell\}$—the network is divided into $2\ell + 1$ connected components, and each contains exactly one player on the cycle. Denote the connected component that contains player i_k by $\Gamma(k)$. Further, denote the players and battlefields in $\Gamma(k)$ by $\mathcal{N}(k)$ and $\mathcal{E}(k)$, respectively. For each $k \in \{0, \ldots, 2\ell\}$, define

$$\mathcal{I}_k^+ := \sum_{e \in \mathcal{E}: \mathcal{N}^e = \{i_k, i_{k+1}\}} \frac{dx_{i_k}^e}{dw^{e_0}} \frac{d\lambda_{i_k}}{dw^{e_0}} \quad \text{and} \quad \mathcal{I}_k^- := \sum_{e \in \mathcal{E}: \mathcal{N}^e = \{i_k, i_{k-1}\}} \frac{dx_{i_k}^e}{dw^{e_0}} \frac{d\lambda_{i_k}}{dw^{e_0}}.$$

The following intermediate result ensues (whose proof can be found in the Supplemental Appendix).

Lemma A6 *The following holds:*

(i) For each $k \in \{0, \ldots, 2\ell\}$, $\mathcal{I}_k^+ + \mathcal{I}_k^- \geq 0$.

(ii) For each $k \in \{1, \ldots, 2\ell\}$, $\mathcal{I}_k^+ + \mathcal{I}_{k+1}^- < 0$.

(iii) The signs of $\mathcal{I}_0^+$ and $\mathcal{I}_1^-$ are different.

(iv) The signs of $\mathcal{I}_k^+$ are the same among all $k \in \{0, \ldots, 2\ell\}$ and those of $\mathcal{I}_k^-$ are the same among all $k \in \{0, \ldots, 2\ell\}$.

By Lemma A6, there are two cases: (i) $\mathcal{I}_k^+ > 0$ and $\mathcal{I}_k^- < 0$ for all $k \in \{0, \ldots, 2\ell\}$; and (ii) $\mathcal{I}_k^+ < 0$ and $\mathcal{I}_k^- > 0$ for all $k \in \{0, \ldots, 2\ell\}$. In what follows, we focus on the former (the analysis for the latter case is similar).

For notational convenience, define $\mu_k := \left| \frac{1}{\lambda_{i_k}} \frac{d\lambda_{i_k}}{dw^{e_0}} \right|$ and the following:

$$\mathcal{M}_{k,k}^+ := \sum_{e \in \mathcal{E}: \mathcal{N}^e = \{i_k, i_{k+1}\}} -w^e m_{i_k}^e \frac{1 - (2p_{i_k}^e - 1) m_{i_{k+1}}^e}{1 + (m_{i_k}^e - m_{i_{k+1}}^e)(p_{i_k}^e - p_{i_{k+1}}^e)}, \tag{19}$$

$$\mathcal{M}^{-}_{k+1,k+1} := \sum_{e\in\mathcal{E}:\mathcal{N}^e=\{i_k,i_{k+1}\}} -w^e m^e_{i_{k+1}} \frac{1-(2p^e_{i_{k+1}}-1)m^e_{i_k}}{1+(m^e_{i_k}-m^e_{i_{k+1}})(p^e_{i_k}-p^e_{i_{k+1}})}, \tag{20}$$

$$\mathcal{M}_{k,k+1} := \sum_{e\in\mathcal{E}:\mathcal{N}^e=\{i_k,i_{k+1}\}} -w^e m^e_{i_k} \frac{(2p^e_{i_k}-1)m^e_{i_{k+1}}}{1+(m^e_{i_k}-m^e_{i_{k+1}})(p^e_{i_k}-p^e_{i_{k+1}})}. \tag{21}$$

We state several intermediate results (whose proof can be found in the Supplemental Appendix).

Lemma A7 *The signs of $\mathcal{M}_{k,k+1}$ are the same among all $k \in \{0,\dots,2\ell\}$.*

Lemma A8 *Suppose that $\mathcal{I}^+_k > 0$ for all $k \in \{0,\dots,2\ell\}$. Fixing $k \in \{1,\dots,2\ell\}$, the following holds:*

(i) If $\mathcal{M}_{k,k+1} > 0$, then $\frac{1}{\lambda_{i_{k+1}}}\frac{d\lambda_{i_{k+1}}}{dw^{e_0}}$ and $\frac{1}{\lambda_{i_k}}\frac{d\lambda_{i_k}}{dw^{e_0}}$ have the same sign. Moreover,

$$\mu_{k+1} > 2\mu_k. \tag{22}$$

(ii) If $\mathcal{M}_{k,k+1} < 0$, then $\frac{1}{\lambda_{i_{k+1}}}\frac{d\lambda_{i_{k+1}}}{dw^{e_0}}$ and $\frac{1}{\lambda_{i_k}}\frac{d\lambda_{i_k}}{dw^{e_0}}$ have different signs. Moreover,

$$\mathcal{I}^+_k \le (-\mathcal{I}^-_{k+1}) \times \frac{\mu_k}{\mu_k + 2\mu_{k+1}}, \tag{23}$$

and

$$-\mathcal{I}^-_{k+1} \ge \frac{2\mu_{k+1}}{\mu_k}\mathcal{I}^+_k + w^{e_0}\mu_k\mu_{k+1}. \tag{24}$$

Now we can prove the equilibrium uniqueness. By Lemma A7, for all $k \in \{0,\dots,2\ell\}$, either $\mathcal{M}_{k,k+1} > 0$ or $\mathcal{M}_{k,k+1} < 0$. In the former case, from (22) in Lemma A8(i), we have that $\mu_0 = \mu_{2\ell+1} > 2\mu_{2\ell} > \cdots > 2^{2\ell}\mu_1$. Meanwhile, by Lemma 6(ii), we have that $\frac{1}{\lambda_{i_1}}\frac{d\lambda_{i_1}}{dw^{e_0}} \ge \frac{1}{\lambda_{i_0}}\frac{d\lambda_{i_0}}{dw^{e_0}} \ge 0$, which implies that $\mu_1 \ge \mu_0$. A contradiction.

In the latter case, it follows from (23) and Lemma A6(i) that

$$\mathcal{I}^+_k \le (-\mathcal{I}^-_{k+1}) \times \frac{\mu_k}{\mu_k + 2\mu_{k+1}} \le \mathcal{I}^+_{k+1} \times \frac{\mu_k}{\mu_k + 2\mu_{k+1}}, \forall\, k \in \{1,\dots,2\ell\},$$

which implies that

$$\mathcal{I}^+_2 \le \mathcal{I}^+_0 \times \prod_{k=2}^{2\ell} \frac{\mu_k}{\mu_k + 2\mu_{k+1}}. \tag{25}$$

Next, combining Lemma A6(ii) and (24) in Lemma A8(ii) yields that

$$\mathcal{I}_2^+ \geq \frac{2\mu_2}{\mu_1}\mathcal{I}_1^+ + \max\left\{\mathcal{I}_1^+, w^{e_0}\mu_1\mu_2\right\}. \tag{26}$$

Moreover, we have that

$$\begin{aligned}\mathcal{I}_0^+ - \mathcal{I}_1^+ \leq \mathcal{I}_0^+ + \mathcal{I}_1^- =& \frac{dx_{i_0}^{e_0}}{dw^{e_0}}\frac{d\lambda_{i_0}}{dw^{e_0}} + \frac{dx_{i_1}^{e_0}}{dw^{e_0}}\frac{d\lambda_{i_1}}{dw^{e_0}} + \sum_{e:\mathcal{N}^e=\{i_0,i_1\},e\neq e_0}\sum_{i\in\mathcal{N}^e}\frac{dx_i^e}{dw^{e_0}}\frac{d\lambda_i}{dw^{e_0}} \\ \leq& \frac{dx_{i_0}^{e_0}}{dw^{e_0}}\frac{d\lambda_{i_0}}{dw^{e_0}} = w^{e_0}m_{i_0}^{e_0}\mu_0\left(\frac{1}{w^{e_0}} - \mu_0\right) \leq w^{e_0}\mu_0(\mu_1 - \mu_0), \end{aligned} \tag{27}$$

where the first inequality follows from Lemma A6(i); the second inequality follows from (5) and Lemma 6(iii); the second equality follows from (10); and the last inequality follows from Lemma 6(ii) and $m_{i_0}^{e_0} \leq 1$.

Combining (25), (26) and (27) yields that

$$\mathcal{H}(\mathcal{I}_1^+) := \frac{2\mu_2}{\mu_1}\mathcal{I}_1^+ + \max\left\{\mathcal{I}_1^+, w^{e_0}\mu_1\mu_2\right\} - \left[\mathcal{I}_1^+ + w^{e_0}\mu_0(\mu_1-\mu_0)\right] \times \prod_{k=2}^{2\ell}\frac{\mu_k}{\mu_k + 2\mu_{k+1}} \leq 0.$$

Note that $\mathcal{H}(\cdot)$ is linear in $\mathcal{I}_1^+$ on $\left[0, w^{e_0}\mu_1\mu_2\right]$ and $\left[w^{e_0}\mu_1\mu_2, +\infty\right)$. Simple algebra would verify that $\mathcal{H}(0) > 0$, $\mathcal{H}(w^{e_0}\mu_1\mu_2) > 0$, and $\mathcal{H}(\infty) > 0$. Therefore, $\mathcal{H}(\mathcal{I}_1^+) > 0$, and we arrive at the contradiction. This completes the proof. □

Proof of Theorem 4. We prove Theorem 4 by induction on t.

Base case: Consider the last period $t = T$. Fixing $e \in \mathcal{E}^T$, the organizer chooses $\boldsymbol{\alpha}^e$ to maximize $\Lambda^e(\boldsymbol{x}^e)$, holding fixed $\boldsymbol{\alpha}^{-e}$. By Theorem 2, the organizer chooses $\boldsymbol{\alpha}^e$ to induce $p_i^e = p_j^e = 1/2$, with $\mathcal{N}^e = \{i, j\}$, in the equilibrium.

Inductive step: For each $t \in \{1, \ldots, T-1\}$, suppose that the statement holds for each $\tau > t$. We show that for each battlefield $e_0 \in \mathcal{E}^t$ and each $\boldsymbol{\alpha}^{<t}$, the battlefield organizer chooses $\boldsymbol{\alpha}^e$ to induce $p_{i_0}^{e_0} = p_{j_0}^{e_0} = 1/2$, with $\mathcal{N}^{e_0} = \{i_0, j_0\}$.

First, following a similar argument as in the proof of Lemma A4, we can show that (9) holds for each $e \in \cup_{\tau\leq t}\mathcal{E}^\tau$ and $e \neq e_0$. Second, by the induction hypothesis, for each $e \in \cup_{\tau>t}\mathcal{E}^\tau$ and each $i \in \mathcal{N}^e$, we have that $p_i^e \equiv 1/2$ and thus $\frac{dp_i^e}{dw^{e_0}} = 0$. This implies that $\frac{dx_i^e}{dw^{e_0}} = -\frac{g_i^e m_i^e}{\lambda_i}\frac{d\lambda_i}{dw^{e_0}}$ and thus $\frac{dx_i^e}{dw^{e_0}}\frac{d\lambda_i}{dw^{e_0}} \leq 0$.

Fixing the biases profile up to period $t-1$—i.e., fixing $\boldsymbol{\alpha}^{<t}$—the battlefield organizers in period t choose their biases, anticipating the biases set by their followers. Note that the condition $\frac{dx_i^e}{dw^{e_0}}\frac{d\lambda_i}{dw^{e_0}} \leq 0$ we prove above ensures that Lemmata 5 and 6 continue to hold. Following a similar argument as in the proof of Theorem 2, we can show that all battlefield organizers in period t will again choose biases to induce equal winning probabilities in their battlefields. This completes the inductive step.

Conclusion: By the principle of induction, for each $t \in \{1, \ldots, T\}$, a biases profile $\boldsymbol{\alpha}^{<t}$, and a battlefield $e \in \mathcal{E}^t$, the organizer chooses $\boldsymbol{\alpha}^e(\boldsymbol{\alpha}^{<t})$ such that $p_i^e = p_j^e = 1/2$. This completes the proof. □

Proof of Theorem 5. Fix an arbitrary biases profile $\boldsymbol{\alpha}$ and $\boldsymbol{x} \in \mathcal{X}(\boldsymbol{\alpha})$. It follows from (2) that

$$X_i = \sum_{e\in\mathcal{E}_i} x_i^e = \sum_{e\in\mathcal{E}_i} (g_i^e)^{-1}\left(\frac{v^e p_i^e(1-p_i^e)}{\lambda_i}\right) \leq \sum_{e\in\mathcal{E}_i} (g_i^e)^{-1}\left(\frac{v^e}{4c_i'(X_i)}\right). \tag{28}$$

Further, by the definition of $\boldsymbol{\alpha}^{**}$, we have that $(p_i^e)^{**} = 1/2$ for all $(i,e) \in \Gamma$. Similar to (28), we have that

$$X_i^{**} = \sum_{e\in\mathcal{E}_i} (g_i^e)^{-1}\left(\frac{v^e (p_i^e)^{**}(1-(p_i^e)^{**})}{(\lambda_i)^{**}}\right) = \sum_{e\in\mathcal{E}_i} (g_i^e)^{-1}\left(\frac{v^e}{4c_i'(X_i^{**})}\right). \tag{29}$$

A closer look at (28) and (29) reveals that $X_i \leq X_i^{**}$, where the inequality holds with equality if and only if $p_i^e = 1/2$ for all $(i,e) \in \Gamma$. Further, by Theorem 1, we must have $\boldsymbol{\alpha} = \boldsymbol{\alpha}^{**}$. This completes the proof. □

Supplemental Appendix: Omitted Proofs

Proof of Lemma A1

Proof. Suppose, to the contrary, that there exists $e_0 \in \mathcal{E}$, with $\mathcal{N}^{e_0} = \{i_0, j_0\}$, and two second-stage equilibria $\boldsymbol{x}', \boldsymbol{x}'' \in \mathcal{X}(\boldsymbol{\alpha})$, with $(x_{i_0}^{e_0})' \neq (x_{i_0}^{e_0})''$ and $(x_{j_0}^{e_0})' > 0$.

Note that equilibrium requires that $\frac{\partial \pi_i(\boldsymbol{x}')}{\partial x_i^e} \leq \lambda_i(\boldsymbol{x}')$, where the inequality holds with equality if $(x_i^e)' > 0$. Similarly, $\frac{\partial \pi_i(\boldsymbol{x}'')}{\partial x_i^e} \leq \lambda_i(\boldsymbol{x}'')$, where the inequality holds with equality if $(x_i^e)'' > 0$. Together, these indicate that

$$\sum_{(i,e)\in\Gamma} \big((x_i^e)' - (x_i^e)''\big) \times \frac{\partial \pi_i(\boldsymbol{x}')}{\partial x_i^e} \geq 0,$$

and

$$\sum_{(i,e)\in\Gamma} \big((x_i^e)' - (x_i^e)''\big) \times \frac{\partial \pi_i(\boldsymbol{x}'')}{\partial x_i^e} \leq 0.$$

Combining the above two inequalities yields that

$$\sum_{(i,e)\in\Gamma} \big((x_i^e)' - (x_i^e)''\big) \times \left[\frac{\partial \pi_i(\boldsymbol{x}')}{\partial x_i^e} - \frac{\partial \pi_i(\boldsymbol{x}'')}{\partial x_i^e}\right] \geq 0. \tag{A1}$$

Define $\boldsymbol{x}(z) := z\boldsymbol{x}' + (1-z)\boldsymbol{x}''$, with $z \in [0,1]$, and

$$\omega(z) := \sum_{(i,e)\in\Gamma} \big((x_i^e)' - (x_i^e)''\big) \times \frac{\partial \pi_i\big(\boldsymbol{x}(z)\big)}{\partial x_i^e}.$$

Evidently, (A1) is equivalent to $\omega(1) \geq \omega(0)$. Meanwhile, $\omega(z)$ can be rewritten as

$$\begin{aligned}
\omega(z) &= \sum_{e\in\mathcal{E},\, \mathcal{N}^e=\{i,j\}} v^e \times \left[\big((x_i^e)' - (x_i^e)''\big) \times \frac{\partial p_i^e\big(\boldsymbol{x}(z)\big)}{\partial x_i^e} + \big((x_j^e)' - (x_j^e)''\big) \times \frac{\partial p_j^e\big(\boldsymbol{x}(z)\big)}{\partial x_j^e}\right] \\
&= \sum_{e\in\mathcal{E},\, \mathcal{N}^e=\{i,j\}} v^e \times \left[\big((x_i^e)' - (x_i^e)''\big) \times \frac{\partial p_i^e\big(\boldsymbol{x}(z)\big)}{\partial x_i^e} - \big((x_j^e)' - (x_j^e)''\big) \times \frac{\partial p_i^e\big(\boldsymbol{x}(z)\big)}{\partial x_j^e}\right],
\end{aligned}$$

which yields that

$$\omega'(z) = \sum_{e\in\mathcal{E},\, \mathcal{N}^e=\{i,j\}} v^e \times \left[\big((x_i^e)' - (x_i^e)''\big)^2 \times \frac{\partial^2 p_i^e\big(\boldsymbol{x}(z)\big)}{(\partial x_i^e)^2} + \big((x_j^e)' - (x_j^e)''\big)^2 \times \frac{\partial^2 p_j^e\big(\boldsymbol{x}(z)\big)}{(\partial x_j^e)^2}\right].$$

Recall that $\frac{\partial p_i^e}{\partial x_i^e} = p_i^e(1-p_i^e)\frac{(f_i^e)'}{f_i^e}$, which in turn implies that

$$\begin{aligned}\frac{\partial^2 p_i^e}{(\partial x_i^e)^2} &= (1-2p_i^e)p_i^e(1-p_i^e)\left[\frac{(f_i^e)'}{f_i^e}\right]^2 + p_i^e(1-p_i^e)\frac{(f_i^e)'' f_i^e - (f_i^e)'(f_i^e)'}{(f_i^e)^2} \\ &= \frac{p_i^e(1-p_i^e)}{(f_i^e)^2} \times \Big[(1-2p_i^e)(f_i^e)'(f_i^e)' + (f_i^e)'' f_i^e - (f_i^e)'(f_i^e)'\Big] \\ &= \frac{\alpha_i^e \alpha_j^e f_j^e}{(\alpha_i^e f_i^e + \alpha_j^e f_j^e)^3} \times \Big[(f_i^e)''(\alpha_i^e f_i^e + \alpha_j^e f_j^e) - 2\alpha_i^e (f_i^e)'(f_i^e)'\Big] \leq 0,\end{aligned}$$

where the equality holds if and only if $f_j^e = 0$, or equivalently, $x_j^e = 0$. Similarly, we have that $\frac{\partial^2 p_j^e}{(\partial x_j^e)^2} \leq 0$. Together, these indicate that $\omega'(z) \leq 0$. Moreover, from $(x_{i_0}^{e_0})' \neq (x_{i_0}^{e_0})''$ and the postulated $(x_{j_0}^{e_0})' > 0$, we have that

$$\big((x_{i_0}^{e_0})' - (x_{i_0}^{e_0})''\big)^2 \times \frac{\partial^2 p_{i_0}^{e_0}\big(\boldsymbol{x}(z)\big)}{(\partial x_{i_0}^{e_0})^2} < 0.$$

Therefore, $\omega'(z) < 0$ for all $z \in (0,1)$, which implies that $\omega(1) < \omega(0)$. This contradicts (A1). ■

Proof of Lemma A2

Proof. Suppose $(x_{j_1}^{e_1})' = 0$. By Lemma A1, $(x_{j_1}^{e_1})'' = 0$ for all $\boldsymbol{x}'' \in \mathcal{X}(\boldsymbol{\alpha})$. Evidently, each battlefield has at least one active player in the second-stage equilibrium, which implies that $(x_{i_0}^{e_1})'' > 0$.

Suppose, to the contrary, that $(x_{j_u}^{e_u})'' > 0$ for some $e_u \in \mathcal{E}_{i_0}$, with $\mathcal{N}^{e_u} = \{i_0, j_u\}$. Then player i_0 has a profitable deviation. Specifically, suppose that he slightly reduces $(x_{i_0}^{e_1})''$ and increases $(x_{i_0}^{e_u})''$ by the same amount. This does not change his winning odds on battlefield e_1 but strictly increases his winning odds on battlefield e_u. Therefore, for all $\boldsymbol{x}'' \in \mathcal{X}(\boldsymbol{\alpha})$, $(x_{j_u}^{e_u})'' = 0$ for all $e_u \in \mathcal{E}_{i_0}$.

It remains to show $\lambda_{i_0}(\boldsymbol{x}'') = 0$. Thus far, we have shown that $(x_{j_1}^{e_1})'' = 0$ and $(x_{i_0}^{e_1})'' > 0$; together with (1) and (2), we can conclude $\lambda_{i_0}(\boldsymbol{x}'') = 0$. ■

Proof of Lemma A3

Proof. Evidently, each battlefield has at least one active player in the second-stage equilibrium. There are two cases:

(a) player i has an inactive opponent in some battlefield in one equilibrium. In this case, by Lemma A2, we have that $\lambda_i = 0$ across all equilibria.

(b) player i has an active opponent in all his battlefields in all equilibria. Note that, for the pure-budget case, it is impossible that player i remains inactive in all his battlefields in any equilibrium. This implies that in any equilibrium, player i must be active in at lease one battlefield. Fix an equilibrium and consider one such battlefield, say battlefield e, with $\mathcal{N}^e = \{i, j\}$. It follows immediately that $x_i^e, x_j^e > 0$. By Lemma A1, both players i and j exert the same effort—i.e., x_i^e and x_j^e—in all equilibria. Note that λ_i is uniquely determined by (2) once we know x_i^e and x_j^e. This implies that λ_i must be the same across all equilibria.

This concludes proof of the lemma. ■

Proof of Lemma A4

Proof. We first prove part (i) of the lemma. Fixing $e \neq e_0$, with $\mathcal{N}^e = \{i, j\}$, (1) can be rewritten as

$$p_i^e = \frac{\alpha_i^e f_i^e}{\alpha_i^e f_i^e + \alpha_j^e f_j^e} \text{ and } p_j^e = 1 - p_i^e = \frac{\alpha_j^e f_j^e}{\alpha_i^e f_i^e + \alpha_j^e f_j^e},$$

from which we can conclude that

$$\frac{p_i^e}{1 - p_i^e} = \frac{\alpha_i^e f_i^e}{\alpha_j^e f_j^e}.$$

Taking the logarithm of both sides of the above equation and differentiating it with respect to w^{e_0} gives

$$\frac{1}{p_i^e(1 - p_i^e)} \frac{dp_i^e}{dw^{e_0}} = \frac{1}{g_i^e} \frac{dx_i^e}{dw^{e_0}} - \frac{1}{g_j^e} \frac{dx_j^e}{dw^{e_0}}. \tag{A2}$$

Suppose $x_i^e, x_j^e > 0$ at w^{e_0}. Then, by continuity, $x_i^e, x_j^e > 0$ in a neighborhood of w^{e_0}. Therefore, the first-order condition (2) holds in the neighborhood, which gives

$$v^e p_i^e (1 - p_i^e) = \lambda_i g_i^e = \lambda_j g_j^e.$$

Again, taking the logarithm of both sides of the above equation and differentiating it with respect to w^{e_0} gives

$$\frac{1-2p_i^e}{p_i^e(1-p_i^e)}\frac{dp_i^e}{dw^{e_0}} = \frac{1}{\lambda_i}\frac{d\lambda_i}{dw^{e_0}} + \frac{1}{g_i^e m_i^e}\frac{dx_i^e}{dw^{e_0}} = \frac{1}{\lambda_j}\frac{d\lambda_j}{dw^{e_0}} + \frac{1}{g_j^e m_j^e}\frac{dx_j^e}{dw^{e_0}}. \tag{A3}$$

Combining (A2) and (A3), we have that

$$\begin{aligned}\frac{1}{\lambda_i}\frac{d\lambda_i}{dw^{e_0}} =& \frac{1-2p_i^e}{p_i^e(1-p_i^e)}\frac{dp_i^e}{dw^{e_0}} - \frac{1}{g_i^e m_i^e}\frac{dx_i^e}{dw^{e_0}} \\ =& -\Big[1-(1-2p_i^e)m_i^e\Big]\frac{1}{g_i^e m_i^e}\frac{dx_i^e}{dw^{e_0}} - (1-2p_i^e)m_j^e\frac{1}{g_j^e m_j^e}\frac{dx_j^e}{dw^{e_0}}.\end{aligned} \tag{A4}$$

Similarly, we have that

$$\frac{1}{\lambda_j}\frac{d\lambda_j}{dw^{e_0}} = -\Big[1+(1-2p_i^e)m_j^e\Big]\frac{1}{g_j^e m_j^e}\frac{dx_j^e}{dw^{e_0}} + (1-2p_i^e)m_i^e\frac{1}{g_i^e m_i^e}\frac{dx_i^e}{dw^{e_0}}. \tag{A5}$$

Combining (A4) and (A5) yields that

$$\frac{dx_i^e}{dw^{e_0}} = -g_i^e m_i^e \frac{\Big[1-(2p_i^e-1)m_j^e\Big]\frac{1}{\lambda_i}\frac{d\lambda_i}{dw^{e_0}} + (2p_i^e-1)m_j^e\frac{1}{\lambda_j}\frac{d\lambda_j}{dw^{e_0}}}{1+(m_i^e-m_j^e)(p_i^e-p_j^e)}.$$

Substituting (2) into the above equation gives (9).

Next, we prove part (ii) of the lemma. The first-order condition (2) on battlefield e_0 becomes

$$w^{e_0} = \lambda_{i_0} g_{i_0}^{e_0} = \lambda_{j_0} g_{j_0}^{e_0}.$$

Note that this condition holds in a neighborhood of w^{e_0}. Taking the logarithm of both sides of the above condition and differentiating it with respect to w^{e_0} gives (10).

Last, we prove part (iii) of the lemma. For the pure-budget case, it is evident that the left-hand side of (11) is zero because $\delta_i = 0$. Further, $\sum_{e\in\mathcal{N}_i} x_i^e = \overline{X}^i$ implies that the right-hand side—i.e., $\sum_{e\in\mathcal{E}_i}\frac{dx_i^e}{dw^{e_0}}$—is also zero, and thus (11) holds. For the pure-cost case, we have that $\lambda_i = c_i'(X_i) = c_i'\big(\sum_{e\in\mathcal{N}_i} x_i^e\big)$. Differentiating both sides with respect to w^{e_0} gives (11). This completes the proof. ■

Proof of Lemma A5

Proof. First, consider $\mathcal{I}_1$. It is straightforward to verify that all inequalities in Lemma 6(i)-(iii) are strict under Assumption 1′; together with (10), $\mathcal{I}_1$ can bounded from above by

$$\mathcal{I}_1 = \frac{d\lambda_{i_0}}{dw^{e_0}} \frac{dx_{i_0}^{e_0}}{dw^{e_0}} = \lambda_{i_0}^{\dagger} (g_{i_0}^{e_0})^{\dagger} (m_{i_0}^{e_0})^{\dagger} \left[\frac{1}{(w^{e_0})^{\dagger}} - \frac{1}{\lambda_{i_0}^{\dagger}} \frac{d\lambda_{i_0}}{dw^{e_0}} \right] \times \frac{1}{\lambda_{i_0}^{\dagger}} \frac{d\lambda_{i_0}}{dw^{e_0}}$$
$$\leq \frac{\lambda_{i_0}^{\dagger} (g_{i_0}^{e_0})^{\dagger} (m_{i_0}^{e_0})^{\dagger}}{4} \times \left(\frac{1}{(w^{e_0})^{\dagger}} \right)^2$$
$$= \frac{(w^{e_0})^{\dagger} (m_{i_0}^{e_0})^{\dagger}}{4} \times \left(\frac{1}{(w^{e_0})^{\dagger}} \right)^2$$
$$\leq \frac{(w^{e_0})^{\dagger} (m_{i_0}^{e_0})^{\dagger}}{4} \times \left(\frac{1}{\lambda_{j_0}^{\dagger}} \frac{d\lambda_{j_0}}{dw^{e_0}} \right)^2 \leq \frac{(w^{e_0})^{\dagger}}{4\rho} \times \left(\frac{1}{\lambda_{j_0}^{\dagger}} \frac{d\lambda_{j_0}}{dw^{e_0}} \right)^2,$$

where the first inequality follows from the AM-GM inequality; the third equality follows from (2); the second inequality follows from Lemma 6(ii); and the last inequality follows from Assumption 1′. This gives (16).

Next, consider $\mathcal{I}_2$. By Lemma 6(iii), we have that $\mathcal{I}_2 \leq 0$. This gives (17).

Last, consider $\mathcal{I}_3$. By (5), $\mathcal{I}_3$ can be bounded from above by

$$\mathcal{I}_3 = \sum_{e \neq e_0} \sum_{i \in \mathcal{N}^e} \frac{d\lambda_i}{dw^{e_0}} \frac{dx_i^e}{dw^{e_0}} \leq \sum_{i \in \mathcal{N}^{e^\dagger}} \frac{d\lambda_i}{dw^{e_0}} \frac{dx_i^{e^\dagger}}{dw^{e_0}} = \frac{d\lambda_s}{dw^{e_0}} \frac{dx_s^{e^\dagger}}{dw^{e_0}} + \frac{d\lambda_{j^\dagger}}{dw^{e_0}} \frac{dx_{j^\dagger}^{e^\dagger}}{dw^{e_0}};$$

together with (9), we can obtain that

$$\mathcal{I}_3 \leq \frac{d\lambda_s}{dw^{e_0}} \frac{dx_s^{e^\dagger}}{dw^{e_0}} + \frac{d\lambda_{j^\dagger}}{dw^{e_0}} \frac{dx_{j^\dagger}^{e^\dagger}}{dw^{e_0}}$$
$$= -\frac{(w^{e^\dagger})^{\dagger} (m_s^{e^\dagger})^{\dagger} \left[1 - \left(2(p_s^{e^\dagger})^{\dagger} - 1 \right) (m_{j^\dagger}^{e^\dagger})^{\dagger} \right] \left(\frac{1}{\lambda_s^{\dagger}} \frac{d\lambda_s}{dw^{e_0}} \right)^2}{1 + \left[(m_s^{e^\dagger})^{\dagger} - (m_{j^\dagger}^{e^\dagger})^{\dagger} \right] \left[(p_s^{e^\dagger})^{\dagger} - (p_{j^\dagger}^{e^\dagger})^{\dagger} \right]}$$
$$- \frac{(w^{e^\dagger})^{\dagger} (m_{j^\dagger}^{e^\dagger})^{\dagger} \left[1 - \left(2(p_{j^\dagger}^{e^\dagger})^{\dagger} - 1 \right) (m_s^{e^\dagger})^{\dagger} \right] \left(\frac{1}{\lambda_{j^\dagger}^{\dagger}} \frac{d\lambda_{j^\dagger}}{dw^{e_0}} \right)^2}{1 + \left[(m_s^{e^\dagger})^{\dagger} - (m_{j^\dagger}^{e^\dagger})^{\dagger} \right] \left[(p_s^{e^\dagger})^{\dagger} - (p_{j^\dagger}^{e^\dagger})^{\dagger} \right]}. \tag{A6}$$

Next, we provide an estimate of the term $(\frac{1}{\lambda_{j^\dagger}^{\dagger}} \frac{d\lambda_{j^\dagger}}{dw^{e_0}})^2$. Recall that $\frac{dx_s^{e^\dagger}}{dw^{e_0}} \geq 0$; together with (9), we can obtain that

$$0 \le \frac{dx_s^{e^\dagger}}{dw^{e_0}} = -\frac{(w^{e^\dagger})^\dagger (m_s^{e^\dagger})^\dagger}{\lambda_s^\dagger} \times \frac{\left[1-(2(p_s^{e^\dagger})^\dagger-1)(m_{j^\dagger}^{e^\dagger})^\dagger\right]\frac{1}{\lambda_s^\dagger}\frac{d\lambda_s}{dw^{e_0}} + (2(p_s^{e^\dagger})^\dagger-1)(m_{j^\dagger}^{e^\dagger})^\dagger \frac{1}{\lambda_{j^\dagger}^\dagger}\frac{d\lambda_{j^\dagger}}{dw^{e_0}}}{1+\left[(m_s^{e^\dagger})^\dagger-(m_{j^\dagger}^{e^\dagger})^\dagger\right]\left[(p_s^{e^\dagger})^\dagger-(p_{j^\dagger}^{e^\dagger})^\dagger\right]}.$$

Recall from the definition of s that $\frac{1}{\lambda_s^\dagger}\frac{d\lambda_s}{dw^{e_0}} \ge \left|\frac{1}{\lambda_{j^\dagger}^\dagger}\frac{d\lambda_{j^\dagger}}{dw^{e_0}}\right|$. Simple algebra would verify that $\frac{1}{\lambda_{j^\dagger}^\dagger}\frac{d\lambda_{j^\dagger}}{dw^{e_0}} < 0$, $2(p_s^{e^\dagger})^\dagger - 1 > 0$, and

$$\frac{1}{\lambda_{j^\dagger}^\dagger}\frac{d\lambda_{j^\dagger}}{dw^{e_0}} \le -\frac{1-\big(2(p_s^{e^\dagger})^\dagger-1\big)(m_{j^\dagger}^{e^\dagger})^\dagger}{\big(2(p_s^{e^\dagger})^\dagger-1\big)(m_{j^\dagger}^{e^\dagger})^\dagger} \times \frac{1}{\lambda_s^\dagger}\frac{d\lambda_s}{dw^{e_0}}. \tag{A7}$$

Substituting (A7) into (A6) yields that

$$\mathcal{I}_3 \le \frac{d\lambda_s}{dw^{e_0}}\frac{dx_s^{e^\dagger}}{dw^{e_0}} + \frac{d\lambda_{j^\dagger}}{dw^{e_0}}\frac{dx_{j^\dagger}^{e^\dagger}}{dw^{e_0}} \le -(w^{e^\dagger})^\dagger \frac{1-\big(2(p_s^{e^\dagger})^\dagger-1\big)(m_{j^\dagger}^{e^\dagger})^\dagger}{\big(2(p_s^{e^\dagger})^\dagger-1\big)^2(m_{j^\dagger}^{e^\dagger})^\dagger}\left(\frac{1}{\lambda_s^\dagger}\frac{d\lambda_s}{dw^{e_0}}\right)^2.$$

To proceed, first note that $2(p_s^{e^\dagger})^\dagger - 1 > 0$ implies that $(w^{e^\dagger})^\dagger < v^{e^\dagger}/4$. Further, recall from the definition of e_0 that $(w^{e_0})^\dagger \le (w^e)^\dagger$ for each battlefield e that satisfies $(w^e)^\dagger < v^e/4$. This implies that $(w^{e_0})^\dagger \le (w^{e^\dagger})^\dagger$. Second, recall, from the definition of s, we have that $\left|\frac{1}{\lambda_{j_0}^\dagger}\frac{d\lambda_{j_0}}{dw^{e_0}}\right| \le \frac{1}{\lambda_s^\dagger}\frac{d\lambda_s}{dw^{e_0}}$.

Combining $(w^{e_0})^\dagger \le (w^e)^\dagger$, $\left|\frac{1}{\lambda_{j_0}^\dagger}\frac{d\lambda_{j_0}}{dw^{e_0}}\right| \le \frac{1}{\lambda_s^\dagger}\frac{d\lambda_s}{dw^{e_0}}$, and the above inequality gives (18). This completes the proof. ■

Proof of Lemma A6

Proof. We first prove part (i) of the lemma. Carrying out the algebra, we can obtain that

$$\sum_{i\in\mathcal{N}(k)}\underbrace{\sum_{e\in\mathcal{E}_i}\frac{dx_i^e}{dw^{e_0}}\frac{d\lambda_i}{dw^{e_0}}}_{\ge 0} = \sum_{e\in\mathcal{E}(k)}\underbrace{\sum_{i\in\mathcal{N}^e}\frac{dx_i^e}{dw^{e_0}}\frac{d\lambda_i}{dw^{e_0}}}_{\le 0} + \mathcal{I}_k^+ + \mathcal{I}_k^-,$$

where the first inequality follows from (6), and the second inequality follows from (5). Therefore, we can conclude that $\mathcal{I}_k^+ + \mathcal{I}_k^- \ge 0$.

Part (ii) of the lemma follows immediately from (5). For part (iii), by parts (i) and (ii) of the lemma, we have that $\mathcal{I}_0^+ + \mathcal{I}_1^- > 0$. Therefore, at least one of $\mathcal{I}_0^+$

and $\mathcal{I}_1^-$ is positive. Meanwhile, by Lemma 6, at least one of $\mathcal{I}_0^+$ and $\mathcal{I}_1^-$ is negative. Therefore, $\mathcal{I}_0^+$ and $\mathcal{I}_1^-$ have different signs.

Last, we prove part (iv). We consider the case of $\mathcal{I}_0^+ > 0 > \mathcal{I}_1^-$ (the analysis for the case of $\mathcal{I}_1^- > 0 > \mathcal{I}_0^+$ is similar). Suppose that $\mathcal{I}_k^- < 0$ for some $k \in \{1, \ldots, 2\ell\}$. By part (i) of the lemma, we have that $\mathcal{I}_k^+ > 0$. Further, by part (ii) of the lemma, we have that $\mathcal{I}_{k+1}^- < 0$. By the principle of mathematical induction, we can conclude that $\mathcal{I}_k^+ > 0$ and $\mathcal{I}_k^- < 0$ for each $k \in \{0, \ldots, 2\ell\}$. This completes the proof. ■

Proof of Lemma A7

Proof. Recall $\mathcal{M}_{k,k}^+$, $\mathcal{M}_{k+1,k+1}^-$, and $\mathcal{M}_{k,k+1}$ as defined in (19), (20), and (21). By (9), for each $k \in \{1, \ldots, 2\ell\}$, we have that

$$\mathcal{I}_k^+ = \mathcal{M}_{k,k}^+ \left[\frac{1}{\lambda_{i_k}} \frac{d\lambda_{i_k}}{dw^{e_0}}\right]^2 + \mathcal{M}_{k,k+1} \left[\frac{1}{\lambda_{i_k}} \frac{d\lambda_{i_k}}{dw^{e_0}}\right] \left[\frac{1}{\lambda_{i_{k+1}}} \frac{d\lambda_{i_{k+1}}}{dw^{e_0}}\right], \tag{A8}$$

and

$$\mathcal{I}_{k+1}^- = \mathcal{M}_{k+1,k+1}^- \left[\frac{1}{\lambda_{i_{k+1}}} \frac{d\lambda_{i_{k+1}}}{dw^{e_0}}\right]^2 - \mathcal{M}_{k,k+1} \left[\frac{1}{\lambda_{i_k}} \frac{d\lambda_{i_k}}{dw^{e_0}}\right] \left[\frac{1}{\lambda_{i_{k+1}}} \frac{d\lambda_{i_{k+1}}}{dw^{e_0}}\right]. \tag{A9}$$

For notational convenience, define $\mathcal{M} := \prod_{k=0}^{2\ell} \mathcal{M}_{k,k+1}$. We first show that $\mathcal{M}_{0,1}$ and $\mathcal{M}$ have the same sign. By Lemma A6(iii) and (iv), either we have that $\mathcal{I}_k^+ > 0$ and $\mathcal{I}_k^- < 0$ for all $k \in \{0, \ldots, 2\ell\}$ or we have that $\mathcal{I}_k^+ < 0$ and $\mathcal{I}_k^- > 0$ for all $k \in \{0, \ldots, 2\ell\}$. In what follows, we focus on the former case (the analysis for the latter case is similar). Evidently, we have $\mathcal{M}_{k,k}^+ < 0$ from (19); together with (A8), we have that

$$\mathcal{M}_{k,k+1} \left[\frac{1}{\lambda_{i_k}} \frac{d\lambda_{i_k}}{dw^{e_0}}\right] \left[\frac{1}{\lambda_{i_{k+1}}} \frac{d\lambda_{i_{k+1}}}{dw^{e_0}}\right] > 0, \ \forall\, k \in \{1, \ldots, 2\ell\}, \tag{A10}$$

which in turn implies that

$$\prod_{k=1}^{2\ell} \left\{ \mathcal{M}_{k,k+1} \left[\frac{1}{\lambda_{i_k}} \frac{d\lambda_{i_k}}{dw^{e_0}}\right] \left[\frac{1}{\lambda_{i_{k+1}}} \frac{d\lambda_{i_{k+1}}}{dw^{e_0}}\right] \right\} > 0.$$

The above inequality implies that $\mathcal{M}\mathcal{M}_{0,1} \left[\frac{1}{\lambda_{i_0}} \frac{d\lambda_{i_0}}{dw^{e_0}}\right] \left[\frac{1}{\lambda_{i_1}} \frac{d\lambda_{i_1}}{dw^{e_0}}\right] > 0$. Further, by

Lemma 6, we have that $\frac{1}{\lambda_{i_0}}\frac{d\lambda_{i_0}}{dw^{e_0}} > 0$ and $\frac{1}{\lambda_{i_1}}\frac{d\lambda_{i_1}}{dw^{e_0}} > 0$. Together, these indicate that $\mathcal{M}\mathcal{M}_{0,1} > 0$.

Next, we show that $\mathcal{M}_{k,k+1}$ and $\mathcal{M}$ have the same sign for all $k \in \{1, \ldots, 2\ell\}$. By (A10), we have that $\mathcal{M}_{k,k+1} \neq 0$. Therefore, there exists a battlefield e such that $\mathcal{N}^e = \{i_k, i_{k+1}\}$ and $p^e_{i_k} \neq 1/2$. Following a similar argument as in the previous analysis for battlefield e_0, we can show that $\mathcal{M}\mathcal{M}_{k,k+1} > 0$. This implies that $\mathcal{M}_{k,k+1}$ and $\mathcal{M}_{0,1}$ have the same sign, which concludes the proof. ■

Proof of Lemma A8

Proof. We first prove part (i) of the lemma. By (A10), $\frac{1}{\lambda_{i_{k+1}}}\frac{d\lambda_{i_{k+1}}}{dw^{e_0}}$ and $\frac{1}{\lambda_{i_k}}\frac{d\lambda_{i_k}}{dw^{e_0}}$ have the same sign. Further, it follows from (A8) and the postulated $\mathcal{I}^+_k > 0$ that

$$\mu_{k+1} > \frac{-\mathcal{M}^+_{k,k}}{\mathcal{M}_{k,k+1}} \times \mu_k. \tag{A11}$$

By (19) and (21), we have that

$$-\mathcal{M}^+_{k,k} - 2\mathcal{M}_{k,k+1} = \sum_{e\in\mathcal{E}:\mathcal{N}^e=\{i_k,i_{k+1}\}} w^e m^e_{i_k} \frac{1 + (2p^e_{i_k} - 1)m^e_{i_{k+1}}}{1 + (m^e_{i_k} - m^e_{i_{k+1}})(p^e_{i_k} - p^e_{i_{k+1}})} \geq 0,$$

which implies that $\frac{-\mathcal{M}^+_{k,k}}{\mathcal{M}_{k,k+1}} \geq 2$. Substituting the inequality into (A11) gives (22).

Next, we prove part (ii) of the lemma. By (A10), $\frac{1}{\lambda_{i_{k+1}}}\frac{d\lambda_{i_{k+1}}}{dw^{e_0}}$ and $\frac{1}{\lambda_{i_k}}\frac{d\lambda_{i_k}}{dw^{e_0}}$ have different signs. Further, by (20) and (21), we have that

$$\mathcal{M}^-_{k+1,k+1} - 2\mathcal{M}_{k,k+1} = \sum_{e\in\mathcal{E}:\mathcal{N}^e=\{i_k,i_{k+1}\}} -w^e m^e_{i_{k+1}} \frac{1 - (2p^e_{i_k} - 1)m^e_{i_k}}{1 + (m^e_{i_k} - m^e_{i_{k+1}})(p^e_{i_k} - p^e_{i_{k+1}})} \leq 0.$$

Note that $\mathcal{M}^+_{k,k} \leq 0$ from (19). Together, these indicate that

$$\mathcal{M}^+_{k,k}\mu_k\left(\mu_k + 2\mu_{k+1}\right) + \left(\mathcal{M}^-_{k+1,k+1} - 2\mathcal{M}_{k,k+1}\right)\mu^2_{k+1} \leq 0.$$

Combining the above inequality and (A8) and (A9) gives (23).

It remains to prove (24). For each e with $\mathcal{N}^e = \{i_k, i_{k+1}\}$, define

$$\mathcal{I}^+_k(e) := \frac{dx_{i_k}}{dw^{e_0}}\frac{d\lambda_{i_k}}{dw^{e_0}} \text{ and } \mathcal{I}^-_{k+1}(e) := \frac{dx_{i_{k+1}}}{dw^{e_0}}\frac{d\lambda_{i_{k+1}}}{dw^{e_0}}.$$

Further, define

$$
\begin{aligned}
\mathcal{E}_{k,k+1} &:= \left\{e : \mathcal{N}^e = \{i_k, i_{k+1}\}\right\}, \\
\mathcal{E}^1_{k,k+1} &:= \left\{e : \mathcal{N}^e = \{i_k, i_{k+1}\}, \mathcal{I}^+_k(e) > 0, \mathcal{I}^-_{k+1}(e) < 0\right\}, \\
\mathcal{E}^2_{k,k+1} &:= \left\{e : \mathcal{N}^e = \{i_k, i_{k+1}\}, \mathcal{I}^+_k(e) < 0, \mathcal{I}^-_{k+1}(e) > 0\right\}, \\
\mathcal{E}^3_{k,k+1} &:= \left\{e : \mathcal{N}^e = \{i_k, i_{k+1}\}, \mathcal{I}^+_k(e) \le 0, \mathcal{I}^-_{k+1}(e) \le 0\right\}.
\end{aligned}
$$

By (5), we have that $\mathcal{I}^+_k(e) + \mathcal{I}^-_{k+1}(e) \le 0$, which implies that it is impossible to have both $\mathcal{I}^+_k(e) > 0$ and $\mathcal{I}^-_{k+1}(e) > 0$. Therefore, $\mathcal{E}_{k,k+1} = \mathcal{E}^1_{k,k+1} \cup \mathcal{E}^2_{k,k+1} \cup \mathcal{E}^3_{k,k+1}$. Further, $\sum_{e \in \mathcal{E}_{k,k+1}} \mathcal{I}^+_k(e) = \mathcal{I}^+_k > 0$ implies that $\mathcal{E}^1_{k,k+1}$ is non-empty.

Next, we claim that

$$
-\mathcal{I}^-_{k+1}(e) \ge \frac{2\mu_{k+1}}{\mu_k} \mathcal{I}^+_k(e) + w^e \mu_k \mu_{k+1}, \; e \in \mathcal{E}^1_{k,k+1}. \tag{A12}
$$

Carrying out the algebra, (A12) is equivalent to

$$
\mu_{k+1} \ge \mu_k \times \frac{1 - 2m^e_{i_k} + (2p^e_{i_k} - 1)(m^e_{i_k} - m^e_{i_{k+1}} + m^e_{i_k} m^e_{i_{k+1}})}{m^e_{i_{k+1}}(1 + (2p^e_{i_k} - 1)m^e_{i_k})}.
$$

For each $e \in \mathcal{E}^1_{k,k+1}$, we have that $\mathcal{I}^+_k(e) > 0$, which implies that $p^e_{i_k} > 1/2$ and

$$
\mu_{k+1} \ge \mu_k \times \frac{1 - (2p^e_{i_k} - 1)m^e_{i_{k+1}}}{(2p^e_{i_k} - 1)m^e_{i_{k+1}}}.
$$

Therefore, (A12) holds if

$$
\frac{1 - (2p^e_{i_k} - 1)m^e_{i_{k+1}}}{(2p^e_{i_k} - 1)m^e_{i_{k+1}}} \ge \frac{1 - 2m^e_{i_k} + (2p^e_{i_k} - 1)(m^e_{i_k} - m^e_{i_{k+1}} + m^e_{i_k} m^e_{i_{k+1}})}{m^e_{i_{k+1}}(1 + (2p^e_{i_k} - 1)m^e_{i_k})},
$$

which is equivalent to

$$
2(1 - p^e_{i_k}) \left[1 + (2p^e_{i_k} - 1)(m^e_{i_k} - m^e_{i_{k+1}})\right] \ge 0.
$$

The above inequality obviously holds, since $m^e_{i_k}, m^e_{i_{k+1}} \in (0, 1]$ and $p^e_{i_k} \in [0, 1]$.

Moreover, we claim that

$$-\mathcal{I}_{k+1}^{-}(e) \geq \frac{2\mu_{k+1}}{\mu_k}\mathcal{I}_k^{+}(e),\ e \in \mathcal{E}_{k,k+1}^2 \cup \mathcal{E}_{k,k+1}^3. \tag{A13}$$

If $e \in \mathcal{E}_{k,k+1}^3$, then (A13) obviously holds. If otherwise $e \in \mathcal{E}_{k,k+1}^2$, following a similar argument as in the proof of (23), we can obtain that $\mathcal{I}_{k+1}^{-}(e) \leq -\mathcal{I}_k^{+}(e) \times \frac{\mu_{k+1}}{\mu_{k+1}+2\mu_k} \leq -\mathcal{I}_k^{+}(e) \times \frac{2\mu_{k+1}}{\mu_k}$, which also yields (A13).

In summary, we have (A12) and (A13). This in turn implies that

$$\begin{aligned}
-\mathcal{I}_{k+1}^{-} &= \sum_{e\in\mathcal{E}_{k,k+1}^1} -\mathcal{I}_{k+1}^{-}(e) + \sum_{e\in\mathcal{E}_{k,k+1}^2\cup\mathcal{E}_{k,k+1}^3} -\mathcal{I}_{k+1}^{-}(e) \\
&\geq \sum_{e\in\mathcal{E}_{k,k+1}^1} \left[\frac{2\mu_{k+1}}{\mu_k}\mathcal{I}_k^{+}(e) + w^e\mu_k\mu_{k+1}\right] + \sum_{e\in\mathcal{E}_{k,k+1}^2\cup\mathcal{E}_{k,k+1}^3} \frac{2\mu_{k+1}}{\mu_k}\mathcal{I}_k^{+}(e) \\
&= \frac{2\mu_{k+1}}{\mu_k}\mathcal{I}_k^{+} + \sum_{e\in\mathcal{E}_{k,k+1}^1} w^e\mu_k\mu_{k+1} \\
&\geq \frac{2\mu_{k+1}}{\mu_k}\mathcal{I}_k^{+} + w^{e_0}\mu_k\mu_{k+1},
\end{aligned}$$

where the last inequality follows from the fact that $\mathcal{E}_{k,k+1}^1$ is non-empty and $w^e \geq w^{e_0}$ for each $e \in \mathcal{E}_{k,k+1}^1$ (by the definition of e_0). This completes the proof. ■