Sequential Contests*

Shanglyu Deng† Qiang Fu‡ Zenan Wu§ Yuxuan Zhu¶

January, 2022

Abstract

This paper provides a general study of a sequential contest modeled as a multi-player incomplete-information all-pay auction. The contest consists of multiple periods. Players arrive and exert efforts sequentially to compete for a prize. They observe the efforts made by their earlier opponents, but not those of their contemporaneous and future rivals. We establish the existence and uniqueness of a symmetric Perfect Bayesian equilibrium (PBE) and fully characterize the equilibrium. Based on the equilibrium result, we show that players' ex ante expected efforts are, in general, nonmonotone with respect to their timing positions. However, a later mover always secures a larger ex ante expected payoff. Further, we endogenize the timing of moves and show that all players choose to move in the last period in the unique equilibrium that survives iterated elimination of strictly dominated strategies (IESDS).

Keywords: Sequential Contest; All-pay Auction; Later-mover Advantage; Endogenous Timing

JEL Classification Codes: C72, D43, D44, D82, L13.

---

*We thank Aislinn Bohren, Jidong Chen, Zhuoqiong Chen, Matthias Dahm, Hong Feng, Kai Konrad, Bin Liu, John Morgan, Roland Strausz, Qianfeng Tang, Tat-How Teh, Felix Várdy, Jun Yu, Huaxia Zeng, and seminar/conference participants at the Chinese University of Hong Kong, Shenzhen (CUHK-Shenzhen), Harbin Institute of Technology (Shenzhen), Renmin University of China, the Shanghai University of Finance and Economics (SUFE), Workshop on Industrial Organization and Competition Policy (UIBE), and the Seventh Annual Conference on Contests: Theory and Evidence for helpful discussions, suggestions, and comments. Fu thanks the Singapore Ministry of Education Tier-1 Academic Research Fund (R-313-000-139-115) for financial support. Wu thanks the National Natural Science Foundation of China (Nos. 71803003 and 72173002) and the seed fund of the School of Economics, Peking University, for financial support. Any errors are our own.

†Department of Economics, University of Maryland, 3114 Tydings Hall, 7343 Preinkert Dr., College Park, MD 20742. Email: sdeng1@umd.edu

‡Department of Strategy and Policy, National University of Singapore, 15 Kent Ridge Drive, Singapore, 119245. Email: bizfq@nus.edu.sg

§School of Economics, Peking University, Beijing, China, 100871. Email: zenan@pku.edu.cn

¶School of Economics, Peking University, Beijing, China, 100871. Email: zhuyuxuan@pku.edu.cn

1 Introduction

Many competitive activities resemble a contest, in which contenders strive to leapfrog while their efforts are nonrefundable regardless of win or loss. Such phenomena are widespread in socioeconomic contexts, ranging from electoral campaigns (Snyder, 1989), lobbying (Che and Gale, 1998; Baye, Kovenock, and De Vries, 1993), internal labor markets inside firms (Lazear and Rosen, 1981; Rosen, 1986; Green and Stokey, 1983), and sporting events (Brown, 2011), to R&D races (Loury, 1979; Lee and Wilde, 1980; Taylor, 1995; Fullerton and McAfee, 1999; Che and Gale, 2003).

Contest-like competitions in practice are often sequential inherently, in that contenders enter and act in succession. Political campaigns, for instance, take place in series (see, e.g., Morgan, 2003). Firms may enter a race for an innovative technology successively. Firms' actions are often subject to disclosure requirement or leaked to competitors. For instance, EU countries typically require mandatory disclosure of firms' R&D activities (La Rosa and Liberatore, 2014). In the United States, the Honest Leadership and Open Government Act of 2007 amended the Lobbying Disclosure Act of 1995, which elevated public disclosure requirements regarding lobbying activity and funding. On the leading crowdsourcing platform Taskcn, a participant is given access to earlier submissions (Liu, Yang, Adamic, and Chen, 2014).

Dynamic interactions arise in such scenarios. Later movers condition their actions on prior moves; while an earlier mover shapes their strategies in anticipation of future opponents' reactions. These complicate the strategical analysis of the contest game. The complexity can be further compounded when the contest allows for richer timing architectures: For instance, multiple players can enter and act in a single period simultaneously; they observe prior actions but not the contemporaneous, which embeds simultaneous competitions in a dynamic structure. A full-fledged analysis involves substantial analytical subtlety, which confines the majority of previous studies to limited settings, e.g., two-player two-period structure.[1,2]

Sequential moves have spawned two related classical questions in oligopoly theory (Amir and Stepanova, 2006). The first concerns players' payoff comparison with respect to their timing positions, i.e., the earlier- vs. later-mover advantages (see, e.g.,

[1] See, e.g., Dixit (1987), Baik and Shogren (1992), and Hoffmann and Rota-Graziosi (2012).

[2] Hinnosaar (2021) provides one remarkable exception.

Gal-Or, 1987; Dowrick, 1986; Dixit, 1987). The second views the timing architecture of an oligopoly as the endogenously determined outcome of players' strategic choices (see, e.g., Hamilton and Slutsky, 1990; Amir, 1995; Morgan, 2003), which addresses the classical Cournot/Stackelberg debate. The conventional wisdom obtained in the usual duopolistic settings, however, do not readily extend under more general sequential structures and deserve to be reexamined. Shinkai (2000), for instance, considers a three-firm three-period model. He shows that players' payoffs can be nonmonotone along the sequence, which precludes a convenient answer in general to the question regarding early- or later-mover advantage in oligopoly.

We consider a general sequential contest game, which imposes no restrictions on the number of players and accommodates a full spectrum of timing architectures. Analogous to standard static all-pay auction models—e.g., Moldovanu and Sela (2001) and Moldovanu, Sela, and Shi (2007)—ex ante symmetric players strive for a commonly valued prize, and the highest bidder wins; players' private types (abilities) are independently and identically distributed, with a higher ability yielding a lower marginal effort cost. The contest proceeds in multiple periods, while multiple players can be clustered in a single period; all the players within each period act simultaneously, while they observe earlier moves. A fully sequential contest and the standard simultaneous-move benchmark boil down to special cases of our model. The literature has yet to provide an equilibrium analysis of this game, and our paper fills the gap.[3] The equilibrium result further enables us to tackle the two aforementioned classical questions.

Next, we briefly discuss the limitations of a duopolistic setting and the nuance caused by a richer timing architecture, and then provide a snapshot of our results and implications.

Two-player Benchmark: Observations and Limitations We begin with a two-player sequential contest. Two players compete for a prize of unity value. Denote by b_1 and b_2, respectively, the first and second movers' efforts. Each bears an effort cost b_i/a_i, with $i \in \{1, 2\}$, where a_1 and a_2—the first and second movers' respective

[3]Segev and Sela (2014) and Jian, Li, and Liu (2017) allow for multiple players but assume a fully sequential structure. Hinnosaar (2021) provides a remarkable exception to the literature that allows for an unrestricted timing architecture but assumes a lottery contest, which differs from our setting.

abilities—are independently and identically distributed and privately known. The contest takes the form of an all-pay auction, so the player with a higher effort wins. Consider a simple tie-breaking rule such that the later mover wins in case of a tie.

The second mover, upon observing the first mover's effort b_1, either simply matches the earlier bid—i.e., $b_2 = b_1$—or stays inactive—i.e., $b_2 = 0$—if his ability a_2 falls below b_1, in which case matching b_1 incurs excessive cost. Obviously, in the equilibrium, the first mover exerts an ex ante higher expected effort than the second, because the latter player stays inactive with a positive probability and never strictly outbids the former. Furthermore, the first mover is ex ante less likely to win: He wins only if his late opponent's ability substantially falls below his own, such that the latter player stays inactive. It deserves to note that with sequential moves, the second mover can win even if his type is lower than that of the first—i.e., when $b_1 < a_1$ and $a_2 \in [b_1, a_1]$—which stands in contrast to a simultaneous contest. With a higher winning probability and a lower expected effort, the second mover receives a higher ex ante expected payoff.

It is unclear a priori whether this later-mover advantage extends to a more general setting. Imagine a simple case with three players and fully sequential moves. The second mover cannot simply match the earlier effort, which allows him to defeat the first mover but may not be optimal given the threat from the third. The optimal response depends on his expectation of the future competition. The second mover can be tempted to preemptively contribute the highest ex ante expected effort, which leads to a nonmonotone effort ranking along the sequence and nullifies the simple logic laid out above. It deserves to note that this rationale does not fully reveal the subtlety of a multi-player sequential contest. Recall that our model allows multiple players to be clustered in the same period, which further complicates bidding strategies, as a player faces both future and contemporaneous competitions. A comprehensive analysis is required to address these nuances.

Findings and Implications: Summary Our paper first conducts a comprehensive equilibrium analysis of the sequential contest game described above. To meet the analytical challenges posed by the dynamic interactions, we take advantage of the recursive property of the payoff structure and convert the game into one that resembles a simultaneous-move all-pay auction with an *endogenously* determined prize. The pseudo prize is shaped by players' ability distribution function and can expressed as a function of the maximum effort sunk by one's earlier opponents. Our model does not impose specific requirement on the curvatures of players' ability distribution. This may cause irregularity to their payoff functions and therefore discontinuity in their players' bidding strategies. Despite the nuance, we establish that there exists a unique symmetric Perfect Bayesian equilibrium (PBE) in the game and provide a complete equilibrium characterization. The equilibrium result enables three applications that shed light on the fundamentals of the sequential contest game.

We first explore players' effort ranking with respect to their timing positions and demonstrate that efforts are nonmonotone in general, which contradicts the convenient observations discussed above in two-player cases. We then investigate whether a player in a later timing position would receive a higher (lower) payoff than his earlier opponents. Despite the nonmonotone effort ranking, we establish that a payoff monotonicity arises: Regardless of the prevailing contest architecture, a player ends up with a higher ex ante expected payoff in a later timing position vis-à-vis an earlier one. Our result thus provides a formal argument for an unambiguous later-mover advantage in the context of multi-player contests.

Finally, we allow the players to simultaneously commit to the timing of their moves prior to the contest, which endogenizes the timing architecture of the contest. It deserves to note that despite the inherent overlap, the above-mentioned analysis—which establishes a later-mover advantage—does not address a player's timing choice. The later-mover advantage is obtained by comparing players' ex ante expected payoff across different periods under a given timing architecture. One's timing choice, however, affects the timing architecture of the contest; as a result, the analysis requires that we compare a player's equilibrium expected payoffs across different timing architectures. We formally verify that all players choose the last period for their moves, which constitutes the unique equilibrium that survives iterated elimination of strictly dominated strategies. A fully simultaneous contest arises when each player makes autonomous timing choices.

Link to the Literature This paper belongs to the small but burgeoning literature on sequential contests. Dixit (1987), Baik and Shogren (1992), Morgan (2003), and Hoffmann and Rota-Graziosi (2012) all consider complete-information Tullock contests in which two players move sequentially. Morgan and Várdy (2007) adopt a similar framework but assume that the follower has to bear a small cost to observe the leader's effort. Glazer and Hassin (2000) allow for three-period sequential plays. The analysis of multi-player sequential contests involves substantial technical difficulty because standard backward induction comes to no avail. Kahana and Klunover (2018) apply an "inverted best response" approach to a fully sequential lottery contest with multiple symmetric players. Hinnosaar (2021) allows for a general setup which imposes no restrictions on the prevailing timing architecture. He remarkably generalizes and formalizes Dixit's thesis that earlier players exert strictly higher efforts and are rewarded with strictly higher payoffs, which results from the strategic substitutability of efforts in a symmetric sequential lottery contest.

Our paper examines a drastically contrasting game theoretical context. All-pay auctions does not generate a continuous and well-behaved best-response correspondence as a lottery or Tullock contest. Our results diverge from that of Hinnosaar (2021): We demonstrate nonmonotonicity in equilibrium efforts and establish a later-mover advantage.

Our paper is more closely related to Segev and Sela (2014) and Jian, Li, and Liu (2017) in terms of the model setups. Both studies consider incomplete-information all-pay auctions. Segev and Sela (2014) examine fully-sequential all-pay auctions with multiple ex ante heterogeneous players and investigate how the expected highest effort depends on the number of players and ability distributions. Jian, Li, and Liu (2017) assume symmetric players and compare players' winning probabilities with respect to the order of moves. Our setting also assume symmetric players, but imposes no restrictions on timing architectures and allows for a broader class of ability distribution functions.[4] Konrad and Leininger (2007) consider two-stage multi-player complete-information all-pay auctions. It is shown that as in simultaneous-move contests, only the player with the lowest cost ends up with a positive expected payoff, while the payoff depends on his own timing position vis-à-vis those of the others.

This paper contributes to the extensive literature on players' payoff comparison with respect to their timing positions in sequential-move games—such as Gal-Or (1985, 1987), Dixit (1987), Dowrick (1986), Daughety (1990), Deneckere and

[4]Segev and Sela (2014) consider a concave distribution function, while Jian, Li, and Liu (2017) assume that the distribution function takes a power functional form.

Kovenock (1992), Amir and Grilo (1999), Van Damme and Hurkens (1999, 2004), Amir and Stepanova (2006), and von Stengel (2010), among many others—in various contexts, ranging from quantity/capacity- to price-setting competitions.[5] As stated above, this strand of literature typically focuses on duopolistic rivalry. Shinkai (2000) extends the framework to a three-firm three-period setting and illuminates the nuance caused by the more extensive sequence. To the best of our knowledge, our paper and Hinnosaar (2021) provide the few exceptions in the literature that examine earlier-/later-mover advantage under an unrestricted timing architecture.[6]

Our analysis adds to the literature on endogenous timing in oligopoly, such as Hamilton and Slutsky (1990), Mailath (1993), Amir (1995), and Amir and Stepanova (2006). A handful of studies explore this issue in contest settings, including Baik and Shogren (1992), Leininger (1993), and Morgan (2003). All these studies consider two-player models. Konrad and Leininger (2007) allow for multiple contestants, but impose a two-period structure.

The rest of the article is organized as follows. Section 2 sets up the model. Section 3 characterizes the equilibrium. Section 4 provides three applications of the equilibrium results. Section 5 presents two extensions that demonstrate the robustness of our main results. Proofs of the results are relegated to the Appendix. An Online Appendix provides analyses and discussions omitted from the main text and collects additional proofs.

2 The Model

A contest involves $N \geq 2$ ex-ante identical risk-neutral players, indexed by $i \in \mathcal{N} \equiv \{1, \dots, N\}$. The players arrive sequentially and each exert effort upon arrival to compete for a prize of a unity value. The contest proceeds in $T \geq 1$ period(s), and the players are, accordingly, partitioned into T groups. Denote by $\mathcal{N}_t$ the set of players in period t, and let $n_t := |\mathcal{N}_t| \geq 1$ indicate the number of players in $\mathcal{N}_t$. A player observes the efforts sunk by his earlier opponents but not those in

[5] Kempf and Rota-Graziosi (2010) consider a setting in which two jurisdictions set tax rates and endogenize leadership in tax competition.

[6] Jian, Li, and Liu (2017) establish a later-mover advantage in terms of players' equilibrium winning probabilities in a sequential all-pay auction.

contemporaneous and future periods. The architecture of the contest is fully described by a vector $\boldsymbol{n} := (n_1, \dots, n_T)$, with $N = \sum_{t=1}^{T} n_t$. The contest is fully sequential with $\boldsymbol{n} = (1, \dots, 1)$, while it degenerates to a fully simultaneous one with $\boldsymbol{n} = (N)$.

A player i, when exerting an effort (or bid) $b^i \geq 0$, incurs a cost $c(b^i) = b^i/a^i$, where $a^i > 0$ measures one's ability and is privately known.[7] Abilities are drawn independently from an interval $(0, 1]$ according to a common distribution function $F(\cdot)$. We assume that $F(\cdot)$ admits a positive and smooth density $f(\cdot) \equiv F'(\cdot)$ and is piecewise analytic.

The competition is modeled as an all-pay auction. The player with the highest effort wins. Specifically, a player $i \in \mathcal{N}_t$, when exerting an effort $b^i \geq 0$, is the *sole winner* if and only if (i) his effort is greater than or equal to those in earlier periods—i.e., $b^i \geq b^j$ for $j \in \cup_{k=1}^{t-1} \mathcal{N}_k$—and (ii) his effort is strictly larger than those in contemporaneous and future periods—i.e., $b^i > b^j$ for $j \in \cup_{k=t}^{T} \mathcal{N}_k \setminus \{i\}$. In the event that (i) multiple players in period t place the same highest bid and (ii) no future players match that, the prize is randomly distributed among them. To put it more formally, fixing a set of effort entries $\boldsymbol{b} \equiv (b^1, \dots, b^N)$, contestant i's winning probability is

$$p^i(\boldsymbol{b}) := \begin{cases} 1, & \text{if } b^i \geq \max_{j \in \cup_{k=1}^{t-1} \mathcal{N}_k} \{b^j\} \text{ and } b^i > \max_{j \in \cup_{k=t}^{T} \mathcal{N}_k \setminus \{i\}} \{b^j\}, \\ 1/m, & \begin{array}{l} \text{if } b^i \geq \max_{j \in \cup_{k=1}^{t-1} \mathcal{N}_k} \{b^j\},\ b^i > \max_{j \in \cup_{k=t+1}^{T} \mathcal{N}_k} \{b^j\}, \\ \text{and } b^j \text{ is among the } m \text{ highest of } \{b^j\}_{j \in \mathcal{N}_t} \text{ with a tie,} \end{array} \\ 0, & \text{if } b^i < \max_{j \in \cup_{k=1}^{t} \mathcal{N}_k \setminus \{i\}} \{b^j\} \text{ or } b^i \leq \max_{j \in \cup_{k=t+1}^{T} \mathcal{N}_k} \{b^j\}, \end{cases} \tag{1}$$

and his ex post expected payoff, for a given ability level a^i, is

$$p^i(\boldsymbol{b}) - b^i/a^i, \text{for all } i \in \mathcal{N}. \tag{2}$$

We consider the solution concept of perfect Bayesian equilibrium (PBE) for the sequential contest game throughout the paper except for the case in Section 5.1. We focus on the symmetric equilibrium in which all the players in the same period adopt

[7] We follow the tradition in the contest literature and accommodate player heterogeneity in their cost functions (e.g., Moldovanu and Sela, 2001, 2006; Moldovanu, Sela, and Shi, 2007; Brown and Minor, 2014). It is noteworthy that the model is isomorphic to an alternative setting in which effort is interpreted as bid in the auction literature: Players value prize differently but bear the same effort costs. All our results remain qualitatively unchanged under this model specification.

the same bidding strategy.

Before we characterize the equilibrium of the game, the following remark elaborates on the tie-breaking rule adopted in (1).

Remark 1 ***(Tie-breaking Rule)*** *Cautious readers may have noticed that the tie-breaking rule assumed in* (1) *is literally asymmetric, in that a later-mover can gain a lead by simply matching previous efforts. This tie-breaking rule is common in the literature, as it ensures the existence of a PBE.*[8] *A player's best response otherwise might not be well-defined.*[9] *We will demonstrate that our results largely remain intact under a symmetric tie-breaking rule.*

3 Equilibrium Analysis

In this section, we first lay out the fundamentals of the analysis, and then formally characterize the equilibrium.

3.1 Preliminaries of Equilibrium Analysis

Fix an effort profile $(b^j)_{j\in\cup_{k=1}^{t-1}\mathcal{N}_k}$, define $\gamma_t := \max_{j\in\cup_{k=1}^{t-1}\mathcal{N}_k}\{b^j\}$ for $t \in \{2, \ldots, T\}$, and let $\gamma_1 \equiv 0$. In words, γ_t is the maximum effort in the contest prior to period t. A symmetric PBE is denoted by $\{b_t^*(a;\gamma_t)\}_{t=1}^T$, where a is a player's ability and $b_t^*(a;\gamma_t)$ is the equilibrium bidding strategy for a player in period t.[10]

Let us introduce several notations before we proceed. Fixing a contest architecture $\boldsymbol{n} \equiv (n_1, \ldots, n_T)$, define a sequence of functions $\{H_t(b), a_t^*(\gamma), \tilde{\pi}_t(b,a)\}_{t=1}^T$, recursively, as follows:

[8] See, e.g., Andreoni, Che, and Kim (2007), Simon and Zame (1990), and Maskin and Riley (2000).

[9] Similarly, consider an asymmetric Bertrand duopoly game. A pure-strategy Nash equilibrium exists if ties are broken in favor of the lower-cost firm, whereas a pure-strategy equilibrium does not exist with an "equal sharing" tie-breaking rule.

[10] In principle, a period-t player's information set is summarized by the bid profile $(b^j)_{j\in\cup_{k=1}^{t-1}\mathcal{N}_k}$ from all previous periods rather than the maximum bid $\gamma_t \equiv \max_{j\in\cup_{k=1}^{t-1}\mathcal{N}_k}\{b^j\}$ and we need to specify a belief system when defining a PBE. However, in our setting, a player's payoff depends only on his own ability and whether his bid is the highest among all contestants; there is no need for the player to infer his rivals' types from the bidding profile. Therefore, a period-t player's bidding behaviors are the same across all information sets that lead to the same maximum bid γ_t, and it is unnecessary to specify a belief system.

$$H_T(b) \equiv 1, H_{t-1}(b) := H_t(b)F^{n_t}\left(a_t^*(b)\right), \forall b \in [0,1], \tag{3}$$

$$a_t^*(\gamma) := \max\left\{0 < a \leq 1 : \tilde{\pi}_t(b,a) \leq 0, \forall b \in [\gamma, 1]\right\}, \tag{4}$$

$$\tilde{\pi}_t(b,a) := H_t(b)F^{n_t-1}(a) - b/a. \tag{5}$$

It can be verified from the above definition that $a_t^*(\gamma)$ and $H_t(b)$ are well-defined and satisfy the following properties:

(a) $a_t^*(\gamma)$ is continuous, piecewise differentiable, and weakly increasing on $[0,1]$, satisfying $a_t^*(\gamma) \geq \gamma$ for all $t \in \mathcal{T}$; and

(b) $H_t(b)$ is continuous, piecewise differentiable, and strictly increasing on $[0,1]$, with $H_t(0) = 0$ and $H_t(1) = 1$ for all $t \in \mathcal{T} \setminus \{T\}$.[11]

The sequence of functions $\{H_t(b), a_t^*(\gamma), \tilde{\pi}_t(b,a)\}_{t=1}^T$ is key to equilibrium characterization and will be intuitively interpreted as our analysis unfolds. We hereby present four lemmas that pave the way for our equilibrium result. Lemma 1 delineates useful properties of players' bidding strategy in a hypothetical symmetric PBE.

Lemma 1 ***(Properties of Equilibrium Bidding Strategy)*** *Consider a sequential contest $\boldsymbol{n} \equiv (n_1, \ldots, n_T)$, and suppose that a symmetric PBE exists. A period-t player's equilibrium bidding strategy $b_t^*(a;\gamma_t)$ satisfies the following properties:*

(i) $b_t^*(a;\gamma_t)$ is increasing in a on $(0,1)$;

(ii) $b_t^*(a;\gamma_t) = 0$ for $a \leq a_t^*(\gamma_t)$ and $b_t^*(a;\gamma_t) \geq \gamma_t$ for $a > a_t^*(\gamma_t)$;

(iii) $b_t^*(a;\gamma_t)$ strictly increases with a on $\left(a_t^*(\gamma_t), 1\right)$ if $n_t \geq 2$.

Lemma 1(i) is intuitive: A stronger player tends to bid more aggressively. Lemma 1(ii) reveals the nature of $a_t^*(\cdot)$: $a_t^*(\gamma_t)$ is the threshold ability level above (below) which a period-t player would stay active (inactive) in equilibrium. Recall that $a_t^*(\cdot)$ increases with its arguments and γ_t is the maximum effort prior to period-t, which implies that higher earlier effort elevates the threshold for active bidding, thereby discouraging future competition. By Lemma 1(iii), when a period t involves two or more players,

[11] See the proof of Lemma 7 in the appendix for more details.

one's effort strictly increases with his ability provided that he is willing to place a positive bid, i.e., $a > a_t^*(\gamma_t)$. Note that the strict monotonicity does not necessarily hold in the case with $n_t = 1$. To see this, recall that in a two-player sequential contest $\boldsymbol{n} = (1,1)$, the later mover simply matches the first mover's effort γ_2 irrespective of his own ability, provided that it exceeds $a_2^*(\gamma_2) = \gamma_2$, i.e., $b_2^*(a;\gamma_2) = \gamma_2$ for $a > a_2^*(\gamma_2)$.

Let $b \geq 0$ be the realized highest effort by the end of a period $t \in T$, which leads to an eventual win if and only if it exceeds all subsequent efforts from period $t+1$. Lemma 1(ii) allows us to derive the equilibrium probability of this event. By Lemma 1, b ends up as the eventual winning effort if and only if all subsequent players stay inactive—i.e., every period-ℓ player's ability falls below the threshold $a_\ell^*(b)$, $\forall \ell \in \{t+1, \ldots, T\}$—which occurs with a probability of $\Pi_{\ell=t+1}^{T} F^{n_\ell}\left(a_\ell^*(b)\right)$: Otherwise, at least one player in later periods would stay active and exert an effort above b. Notably, this probability can be expressed recursively, which boils down to the function $H_t(b)$ in (3). We formally state this fact in the following lemma.

Lemma 2 ***(Equilibrium Winning Probability of a Provisional Winner)*** *Consider a sequential contest $\boldsymbol{n} \equiv (n_1, \ldots, n_T)$ and suppose that a symmetric PBE exists. Let $b \geq 0$ be the realized highest effort by the end of period $t \in \mathcal{T}$. Then $H_t(b)$ gives the probability of the effort b's exceeding all subsequent efforts in equilibrium.*

Lemma 2 enables us to exploit the recursive nature of this sequential contest game, which simplifies the equilibrium analysis. Consider a period-t player with ability $a > 0$. In a symmetric PBE, by exerting an arbitrary effort $b \geq \gamma_t$, he earns an expected payoff[12]

$$\pi_t(b, a; \gamma_t) := H_t(b) F^{n_t - 1}\left((b_t^*)^{-1}(b;\gamma_t)\right) - b/a. \tag{6}$$

He wins with a probability $H_t(b)F^{n_t-1}\left((b_t^*)^{-1}(b;\gamma_t)\right)$: As stated above, the effort b allows him to beat future opponents with a probability $H_t(b)$, and prevail over his contemporaneous competitors with a probability $F^{n_t-1}\left((b_t^*)^{-1}(b;\gamma_t)\right)$.

The strategic interactions between a period-t player and his future opponents are encapsulated in the provisional winning probability function $H_t(\cdot)$: His equilibrium

[12] The inverse function $(b_t^*)^{-1}(b;\gamma_t)$ is well-defined by Lemma 1(iii) for $b \geq \gamma_t$ and $n_t \geq 2$. Note that there is no need to specify $(b_t^*)^{-1}(b;\gamma_t)$ for the case of $n_t = 1$ given that $F^{n_t-1}(a) = 1$ for all $a \in (0,1)$.

bidding strategy can be solved for as if he competed in a static contest for a prize of a value $H_t(b)$, which technically dissolves the *dynamic* linkages between contestants across different periods. Despite the analogy, the pseudo prize value, $H_t(b)$, endogenously depends on the player's own effort b, so the equilibrium bidding strategy fundamentally differs from that in a standard static contest.

Next, we narrow down the set of possible equilibrium efforts. Consider a period-t player with ability $a > 0$ who faces contemporaneous competition, i.e., $n_t \geq 2$. Recall the function $\tilde{\pi}_t(b,a)$ in (5):

$$\tilde{\pi}_t(b,a) \equiv H_t(b)F^{n_t-1}(a) - b/a.$$

Fixing $\gamma_t \geq 0$, we define

$$S_t(a;\gamma_t) := \left\{ b \in [\gamma_t, 1] : \tilde{\pi}_t(b,a) > \tilde{\pi}_t(b',a), \forall b' \in (b,1] \right\}.$$

The set $S_t(a;\gamma_t)$ is type-dependent and shrinks as ability a increases, i.e., $S_t(a';\gamma_t) \subseteq S_t(a;\gamma_t)$ for $a < a'$. Further, recall the piecewise analyticity of the ability distribution $F(\cdot)$. This implies that $H_t(b)$ and, therefore, $\tilde{\pi}_t(b,a)$, are continuous and piecewise analytic with respect to b on $[\gamma_t, 1]$; $S_t(a;\gamma_t)$, in turn, can be expressed as the union of finitely many disjoint intervals. For ease of exposition, we denote by $m_t(a;\gamma_t) \in \mathbb{N}_+$ the number of disjoint intervals included in $S_t(a;\gamma_t)$. Then $S_t(a;\gamma_t)$ can be written as

$$S_t(a;\gamma_t) = \left[s_t^1(a;\gamma_t), e_t^1(a;\gamma_t)\right) \bigcup \left[s_t^2(a;\gamma_t), e_t^2(a;\gamma_t)\right) \bigcup \cdots \bigcup \left[s_t^{m_t(a;\gamma_t)}(a;\gamma_t), e_t^{m_t(a;\gamma_t)}(a;\gamma_t)\right],$$

with $e_t^m(a;\gamma_t) < s_t^{m+1}(a;\gamma_t)$ for $1 \leq m \leq m_t(a;\gamma_t) - 1$ and $e_t^{m_t(a;\gamma_t)}(a;\gamma_t) \equiv 1$. Figure 1 graphically illustrates the set $S_t(a;\gamma_t)$, in which case $m_t(a;\gamma_t) = 2$.

Now we are ready to show that a period-t player's equilibrium effort must be contained within the above-defined set, provided that it exceeds γ_t. To put this formally, $b_t^*(a,\gamma_t) \in S_t(a;\gamma_t)$ if $b_t^*(a,\gamma_t) \geq \gamma_t$ is continuous in the neighborhood of a. To see this, suppose, to the contrary, that $b_t^*(a,\gamma_t) \notin S_t(a;\gamma_t)$. By definition, there exists some effort $b' > b_t^*(a,\gamma_t)$ such that $\tilde{\pi}_t(b',a) \geq \tilde{\pi}_t(b_t^*(a,\gamma_t),a)$. We claim that the player's payoff would strictly exceed the value of the constructed function $\tilde{\pi}_t(b',a)$

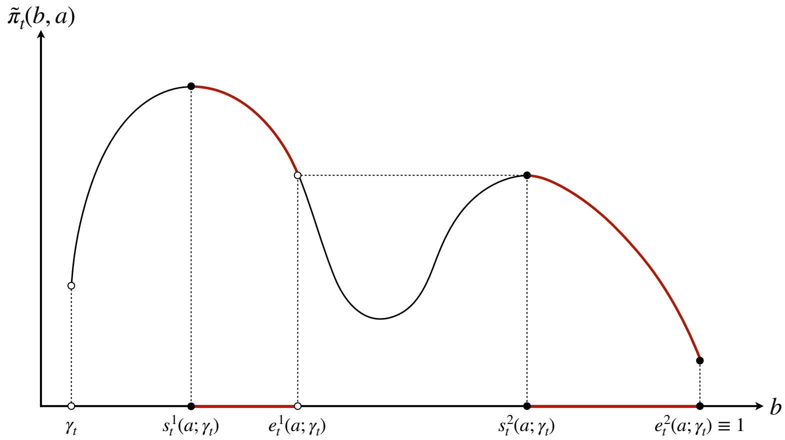


Figure 1: Illustration of $S_t(a; \gamma_t)$.

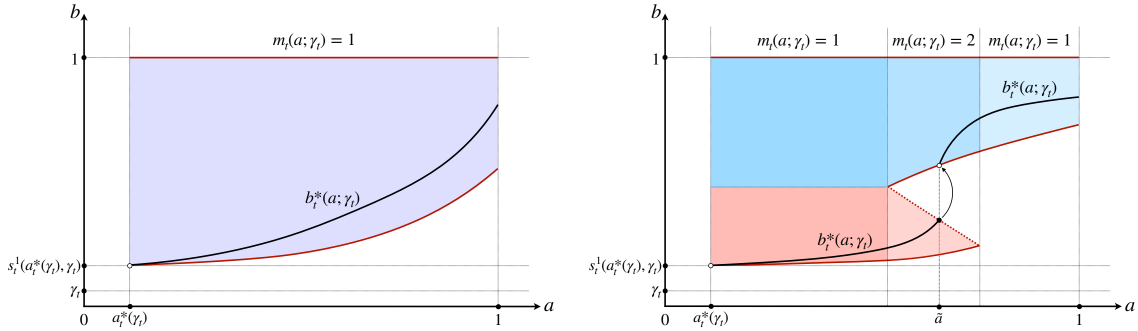
when he deviates from $b_t^*(a, \gamma_t)$ to b'. This is caused by the fact that the higher effort b' increases not only the probability of outbidding his future opponents but also that of beating the contemporaneous. More specifically, let a' be the maximum ability such that $b_t^*(a, \gamma_t) \leq b'$. Because the equilibrium bidding strategy $b_t^*(a, \gamma_t)$ is strictly increasing and continuous around a, we can conclude that $a' > a$. In a symmetric PBE, the player's actual expected payoff from the deviation ends up as $H_t(b')F^{n_t-1}(a') - b'/a$: In other words, he behaves as if he has an ability a', which allows him to defeat his contemporaneous opponents with a probability $F^{n_t-1}(a')$. This payoff strictly exceeds $\tilde{\pi}_t(b', a) \equiv H_t(b')F^{n_t-1}(a) - b'/a$ and thus overshadows the equilibrium payoff $\tilde{\pi}_t(b_t^*(a, \gamma_t), a)$. Contradiction ensues.

The next lemma demonstrates that the smallest element in $S_t(a; \gamma_t)$ is indeed the bid a player of the threshold ability $a = a_t^*(\gamma_t)$ tends to place in equilibrium.

Lemma 3 ***(Equilibrium Bid at $\mathbf{a_t^*(\gamma_t)}$)*** *Consider a sequential contest $\boldsymbol{n} \equiv (n_1, \ldots, n_T)$ and suppose that a symmetric PBE exists. If $n_t \geq 2$ and $a_t^*(\gamma_t) < 1$, then $\lim_{a \searrow a_t^*(\gamma_t)} b_t^*(a; \gamma_t) = s_t^1(a_t^*(\gamma_t); \gamma_t)$.*

For notational convenience, we use $b_t^*(a - 0; \gamma_t)$ and $b_t^*(a + 0; \gamma_t)$ to denote the left and right limits of $b_t^*(a; \gamma_t)$, respectively. The following lemma can be obtained.

Lemma 4 ***(Potential Discontinuity of Players' Bidding Strategy)*** *Consider a sequential contest $\boldsymbol{n} \equiv (n_1, \ldots, n_T)$ and suppose that a symmetric PBE exists. Fix*



(a) Continuous Equilibrium Bidding Strategy (b) Discontinuous Equilibrium Bidding Strategy

Figure 2: Illustration of Equilibrium Bidding Strategy $b_t^*(a;\gamma_t)$.

a period t with $n_t \geq 2$ and a player's ability $\tilde{a} \in (a_t^*(\gamma_t), 1]$. If $b_t^*(\tilde{a}-0;\gamma_t) = e_t^m(\tilde{a};\gamma_t)$ for some $1 \leq m \leq m_t(\tilde{a};\gamma_t) - 1$, then $b_t^*(\tilde{a}+0;\gamma_t) = s_t^{m+1}(\tilde{a};\gamma_t)$.

Lemma 4, as well as Lemma 3, suggests the possibility of discontinuous equilibrium bidding strategy. By Lemma 1, a period-t player, for a given γ_t, would increase his effort as his ability ascends. When the effort is in the interior of the set $S_t(a;\gamma_t)$, the player's bidding strategy would gradually increase with a for $a > a_t^*(\gamma_t)$, as Figure 2(a) illustrates. Recall that the set of eligible efforts $S_t(a;\gamma_t)$ shrinks as a ascends (see Figure 2). When the player's effort reaches the end of some interval in the set $S_t(a;\gamma_t)$—i.e. $e_t^m(a;\gamma_t)$—he would refrain from exerting an effort in the "undesirable" region $\big(e_t^m(a;\gamma_t), s_t^{m+1}(a;\gamma_t)\big)$; his effort jumps directly to the lower bound of the next adjacent interval in $S_t(a;\gamma_t)$, i.e., $s_t^{m+1}(a;\gamma_t)$. This scenario is depicted in Figure 2(b).

To understand why the boundary of the set $S_t(a;\gamma_t)$ can be played in the equilibrium, it is useful to further inspect the constructed function (5) and the equilibrium payoff function (6). Consider a symmetric PBE. When a player's effort increases, he ends up with a higher probability of outperforming his contemporaneous opponents: Such an *equilibrium effect* is nevertheless omitted in the expression of (5). Suppose that all other players' bidding strategy contains a jump from $b^\sharp$ to $b^{\sharp\sharp}$ when one's ability increases. All efforts between $b^\sharp$ and $b^{\sharp\sharp}$ would yield the same probability of winning the contemporaneous competition. As a result, the aforementioned equilibrium effect dissolves around the jump.

The jump predicted in Lemma 4 and Figure 2(b) is impossible in a Bayesian

Nash equilibrium of a *static* all-pay auction. This discontinuity, if it exists, largely stems from the dynamic interaction in the game, which is captured by the provisional winning probability function $H_t(\cdot)$ in the interim expected payoff (6).

3.2 Equilibrium Result: Existence, Uniqueness, and Characterization

We are ready to verify the existence and uniqueness of a symmetric PBE in the sequential contest game under an arbitrary contest architecture $\boldsymbol{n} \equiv (n_1, \ldots, n_T)$ and fully characterize it. Let $h_t(b) := H_t'(b)$.

Theorem 1 ***(Equilibrium of Sequential Contests)*** *Consider a sequential contest $\boldsymbol{n} \equiv (n_1, \ldots, n_T)$. There exists a unique symmetric PBE $\{b_t^*(a;\gamma_t)\}_{t=1}^T$ of the contest game, which is fully characterized as follows:*[13]

(i) If $n_t = 1$, then

$$b_t^*(a;\gamma_t) \begin{cases} = 0, & \text{if } a \leq a_t^*(\gamma_t), \\ = \gamma_t, & \text{if } a_t^*(\gamma_t) < a \leq a_t^{**}(\gamma_t), \\ \in \arg\max_{b>\gamma_t}[H_t(b) - b/a], & \text{if } a > a_t^{**}(\gamma_t), \end{cases} \tag{7}$$

where $a_t^*(\gamma_t)$ is defined in (4) and can be simplified as $a_t^*(\gamma_t) = \min_{b \geq \gamma_t} b/H_t(b)$, and $a_t^{**}(\gamma_t) := \sup_{a_t^*(\gamma_t) \leq a \leq 1} \big\{a : H_t(\gamma_t) - \gamma_t/a > H_t(b') - b'/a, \forall b' \in (\gamma_t, 1]\big\}$.

(ii) If $n_t \geq 2$, then $b_t^*(a;\gamma_t) = 0$ for $a \leq a_t^*(\gamma_t)$. For $a > a_t^*(\gamma_t)$, $b_t^*(a;\gamma_t)$ increases continuously and is governed by the following differential equation:

$$(n_t-1)aF^{n_t-2}(a)f(a)H_t\big(b_t^*(a;\gamma_t)\big)+aF^{n_t-1}(a)h_t\big(b_t^*(a;\gamma_t)\big)(b_t^*)'(a;\gamma_t)-(b_t^*)'(a;\gamma_t) = 0, \tag{8}$$

with the initial condition $b_t^*\big(a_t^*(\gamma_t) + 0;\gamma_t\big) = s_t^1\big(a_t^*(\gamma_t);\gamma_t\big)$. When $b_t^*(\tilde{a};\gamma_t) = e_t^m(\tilde{a};\gamma_t)$ for some $\tilde{a} \in (0,1)$ and $1 \leq m \leq m_t(\tilde{a};\gamma_t) - 1$, $b_t^*(a;\gamma_t)$ jumps to

[13] The symmetric PBE is unique in the sense that if there exist two symmetric PBE of the contest game—denoted by $\{b_t^*(a;\gamma_t)\}_{t=1}^T$ and $\{b_t^{**}(a;\gamma_t)\}_{t=1}^T$—then for all $t \in \mathcal{T}$ and $\gamma_t \geq 0$, the collection of ability a such that $b_t^*(a;\gamma_t) = b_t^{**}(a;\gamma_t)$ has F-measure one.

$s_t^{m+1}(\tilde{a};\gamma_t)$ at $a = \tilde{a}$ and then increases continuously from $\tilde{a}$ again according to (8), with the initial condition $b_t^*(\tilde{a}+0;\gamma_t) = s_t^{m+1}(\tilde{a};\gamma_t)$.

Theorem 1 establishes the existence and uniqueness of a symmetric PBE in sequential contests. A player's equilibrium bidding strategy depends on the number of contemporaneous opponents. Theorem 1(i) considers a scenario in which a single player arrives in a period, while Theorem 1(ii) addresses the case in which multiple players are clustered in one set $\mathcal{N}_t$. It is straightforward to observe that a player would remain inactive if he is of low ability—i.e., $a \leq a_t^*(\gamma_t)$—in either scenario, as predicted in Lemma 1(ii). The predictions diverge between the two scenarios when the player's ability is sufficiently high. With $n_t = 1$, the player matches the highest prior bid γ_t when his ability remains in an intermediate range—i.e., $a_t^*(\gamma_t) < a < a_t^{**}(\gamma_t)$, while he strictly outbids γ_t if his ability exceeds the cutoff $a_t^{**}(\gamma_t)$. In contrast, with $n_t \geq 2$, he strictly outbids γ_t whenever his ability exceeds $a_t^*(\gamma_t)$: Contemporaneous competition compels him to step up effort to avoid a tie.

A closer look at Lemma 1 and Theorem 1 allows us to identify the set of players who constantly stay inactive, i.e., exerting zero effort irrespective of their own types and previous efforts. Recall from Lemma 1(ii) that a period-t player would be completely discouraged if his ability falls below $a^*(\gamma_t)$. Obviously, he would do so if $a^*(\gamma_t) = 1$ for all $\gamma_t \in [0,1]$, which is equivalent to $\tilde{\pi}_t(b,1) \leq 0$ for all $b \in [0,1]$ by (4). This condition, together with (5) and $F(1) = 1$, implies $H_t(b) \leq b$ for all $b \in [0,1]$. Let $\mathcal{T}_0 := \{t \in \mathcal{T} : H_t(b) \leq b, \forall b \in [0,1]\}$. It can be verified that $t' \in \mathcal{T}_0$ if $t \in \mathcal{T}_0$ and $t' < t$. Define $t_0 := \max \mathcal{T}_0$; it is obvious to infer $0 \leq t_0 \leq T-1$.[14] The following result naturally ensues.

Proposition 1 ***(Players Who Always Remain Inactive)*** *Consider a sequential contest $\boldsymbol{n} \equiv (n_1, \ldots, n_T)$. In the unique symmetric PBE $\{b_t^*(a;\gamma_t)\}_{t=1}^T$ of the contest game, all players in periods 1 through t_0 choose to stay inactive regardless of his ability and the previous maximum bid, i.e., $b_t^*(a;\gamma_t) = 0$ for all $a \in (0,1]$, $\gamma_t \in [0,1]$, and $t \in \mathcal{T}_0$.*

Proposition 1 states that players that arrive in early periods—i.e., $t \leq t_0$—always stay inactive, regardless of their own types. These players obviously receive zero

[14] In the case that $\mathcal{T}_0$ is an empty set, we let $t_0 = 0$.

expected payoff in equilibrium, which alludes to a disadvantage of being earlier movers in a contest. The players who arrive subsequently, in contrast, exert positive efforts with positive probabilities.

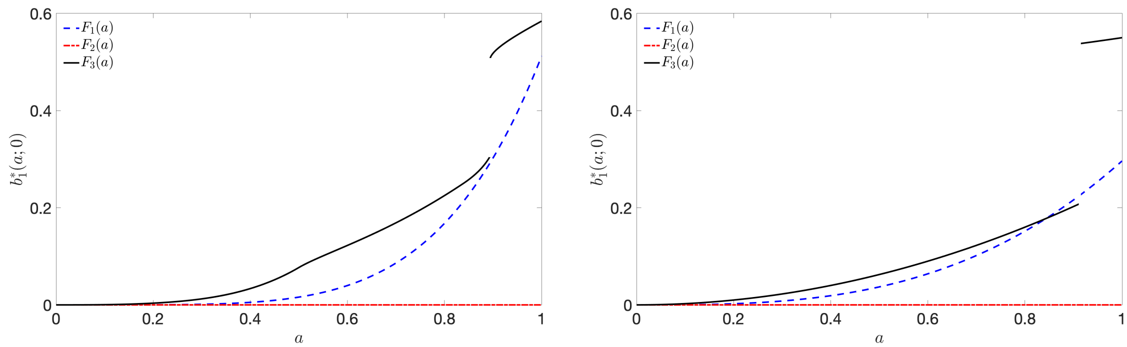
We illustrate our equilibrium results in more specific settings. First, let us consider a case with a concave ability distribution (e.g., Segev and Sela, 2014). Assume that $F(\cdot)$ is continuous, twice differentiable, strictly concave, and satisfies $\lim_{a \searrow 0}[f(a)a] = 0$, which is assumed in Segev and Sela (2014) for an analysis of a fully sequential all-pay auction. Recall Proposition 1, which states that early players—i.e., those who arrive in from periods 1 to t_0—stay inactive regardless, with $t_0 := \max \mathcal{T}_0$ and $\mathcal{T}_0 := \{t \in \mathcal{T} : H_t(b) \leq b, \forall b \in [0, 1]\}$. It is straightforward to verify that $H_t(b) > b$ for all $b \in (0, 1)$, which, in turn, implies $t_0 = 0$. That is, all players exert positive efforts in equilibrium with positive probabilities.

Further, consider the case with convex ability distribution. Let $F(\cdot)$ be continuous, twice differentiable, and weakly convex. A (weakly) convex ability distribution implies $H_t(b) \leq b$ for all $b \in [0, 1]$ and $t \in \{1, \ldots, T-1\}$, which further leads to $t_0 = T - 1$. The equilibrium is straightforward: All the players who arrive prior to period T would stay inactive in equilibrium regardless of their own types and period-T players behave as if they participate in a simultaneous all-pay auction with n_T players, where standard result applies.

The result of Theorem 1 can readily be adapted to derive the equilibrium in these cases. Further, recall that Lemma 4, as well as Lemma 3, alludes to the possibility of discontinuous equilibrium bidding strategies. However, discontinuity arises in neither of these two cases laid out above—i.e., concave or convex ability distributions—which are discussed in more detail in Online Appendix B. Next, we demonstrate that discontinuity, indeed, may emerge under irregular distribution—e.g., one to be concave in some regions and convex in the others.

3.3 Discussion: Discontinuity in Equilibrium Strategies

Consider the following three ability distributions: (i) $F_1(a) = a^{2/3}$; (ii) $F_2(a) = a^2$; and (iii)



(a) Contest $(1, 1)$

(b) Contest $(2, 1)$

Figure 3: First Mover's Bidding Strategy in Two-Period Sequential Contests under Different Ability Distributions.

$$F_3(a) = \begin{cases} \sqrt{a}, & a < \frac{1}{4}, \\ \frac{(a+0.75)^2}{2}, & \frac{1}{4} \leq a \leq \frac{1}{2}, \\ 0.3837\sqrt{a - 0.4764} + 0.7224, & a > \frac{1}{2}. \end{cases}$$

Note that $F_3(\cdot)$ is convex in a on $[\frac{1}{4}, \frac{1}{2}]$ and concave on $(0, \frac{1}{4})$ and $(\frac{1}{2}, 1]$. Consider a simple two-player sequential contest $(n_1, n_2) = (1, 1)$. Figure 3(a) illustrates the first mover's equilibrium bidding strategy under each distribution. A jump in the bidding function with respect to the first mover's ability, a, arises under $F_3(\cdot)$. Further, the discontinuity persists when an additional player is added into the first period, which yields contemporaneous competition ($n_1 = 2$), as is evidenced by Figure 3(b).

4 Applications

In this part, we apply our equilibrium results to further delve into the fundamentals of this sequential contest game. First, we demonstrate the complexity involved in ranking players' ex ante expected efforts with respect to their timing positions, which nullifies the insights obtained in two-player settings. Second, despite the nuances, we establish that players' ex ante expected payoffs remain monotone in the general setting. Finally, we endogenize players' moving order in the contest.

4.1 Ambiguous Effort Ranking

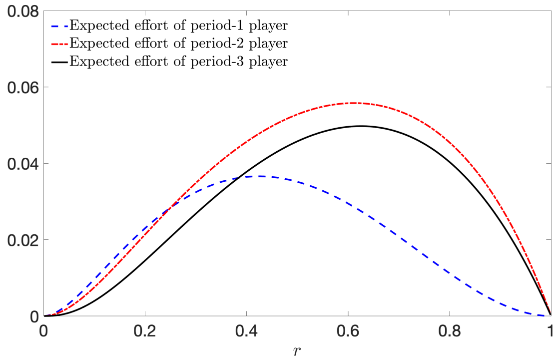
Although Theorem 1 fully characterizes the equilibrium, a closed-form solution for the equilibrium bidding strategy is unavailable in general. For now, we assume a distribution function of the form $F(a) = a^r$—with $r \in (0, 1)$—and a fully sequential architecture with $\boldsymbol{n} = (1, \ldots, 1)$. This case enables a handy solution, which suffices to demonstrate the nuance caused by a multi-player setting. Note that a larger r implies a more *favorable* distribution, as it causes the probability mass to concentrate more on the upper end of the support and competent players are more likely to realize. Let b_t^* be a period-t player's bid in equilibrium and $\mathbb{E}[b_t^*]$ be its expectation.

Proposition 2 ***(Ranking Players' Efforts in Fully Sequential Contests)*** *Suppose $F(a) = a^r$, with $r \in (0, 1)$. Consider an N-player fully sequential contest, with $\boldsymbol{n} = (1, \ldots, 1)$ and $T = N$. The following statements hold:*

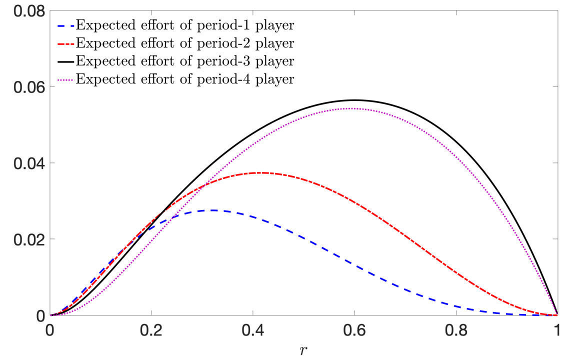
(i) If $N = 2$, then $\mathbb{E}[b_1^*] > \mathbb{E}[b_2^*]$ for all $r \in (0, 1)$.

(ii) If $N \geq 3$, then $\mathbb{E}[b_1^*] > \mathbb{E}[b_2^*] > \cdots > \mathbb{E}[b_T^*]$ when r is sufficiently close to 0, and $\mathbb{E}[b_{T-1}^*] > \mathbb{E}[b_T^*] > \mathbb{E}[b_{T-2}^*] > \cdots > \mathbb{E}[b_1^*]$ when r is sufficiently close to 1.

Proposition 2(i) reiterates the observation laid out in the Introduction: The first mover is expected to exert more effort than the second in a two-player sequential contest. Proposition 2(ii) nevertheless demonstrates that the monotonicity may not persist when more than two players are involved. Specifically, when r is close to 1, the *penultimate* mover exerts the highest expected effort. To understand the logic, consider a three-player fully sequential contest $(1, 1, 1)$ and focus on the second mover. Fix an arbitrary period-1 effort b_1^*, the second mover has no incentive to exert an effort strictly higher than b_1^* in the absence of a third mover. However, anticipating a third player in the final period, matching b_1^* is no longer necessarily optimal; by Theorem 1(i), he may exert a strictly higher effort than b_1^* once his ability a_2 surpasses the cutoff $a_2^{**}(b_1)$ to avoid being overtaken by the third player. It is thus straightforward to infer that the case of $\mathbb{E}[b_2^*] > \mathbb{E}[b_1^*]$ is more likely as r ascends: A high-ability third player is more likely to emerge, which compels him to step up his effort. It is also intuitive to observe $\mathbb{E}[b_2^*] > \mathbb{E}[b_3^*]$. The last player behaves essentially in the same way



(a) Contest $(1,1,1)$: $T = N = 3$



(b) Contest $(1,1,1,1)$: $T = N = 4$

Figure 4: Players' Expected Efforts in Fully Sequential All-pay Auctions.

as the second mover in the two-player case: He either matches $\gamma_3 = \max\{b_1^*, b_2^*\}$ or stays inactive, since no future competition is ahead.

Figure 4 depicts players' individual efforts for all values of $r \in (0,1)$, assuming $N = 3$ or 4. By Figure 4(a), three patterns arise as r ascends: (i) $\mathbb{E}[b_1^*] > \mathbb{E}[b_2^*] > \mathbb{E}[b_3^*]$; (ii) $\mathbb{E}[b_2^*] > \mathbb{E}[b_1^*] > \mathbb{E}[b_3^*]$; and (iii) $\mathbb{E}[b_2^*] > \mathbb{E}[b_3^*] > \mathbb{E}[b_1^*]$. The comparison is further complicated when the number of players increases from three to four, as Figure 4(b) illustrates. Six patterns emerge as r increases: (i) $\mathbb{E}[b_1^*] > \mathbb{E}[b_2^*] > \mathbb{E}[b_3^*] > \mathbb{E}[b_4^*]$; (ii) $\mathbb{E}[b_2^*] > \mathbb{E}[b_1^*] > \mathbb{E}[b_3^*] > \mathbb{E}[b_4^*]$; (iii) $\mathbb{E}[b_2^*] > \mathbb{E}[b_3^*] > \mathbb{E}[b_1^*] > \mathbb{E}[b_4^*]$; (iv) $\mathbb{E}[b_3^*] > \mathbb{E}[b_2^*] > \mathbb{E}[b_1^*] > \mathbb{E}[b_4^*]$; (v) $\mathbb{E}[b_3^*] > \mathbb{E}[b_2^*] > \mathbb{E}[b_4^*] > \mathbb{E}[b_1^*]$; and (vi) $\mathbb{E}[b_3^*] > \mathbb{E}[b_4^*] > \mathbb{E}[b_2^*] > \mathbb{E}[b_1^*]$.

It is noteworthy that effort ranking across periods would be substantially complicated in a partially sequential all-pay auction, in which case multiple players may act simultaneously in certain periods.[15]

4.2 Monotone Payoff Ranking

We now formally address the following research question: Holding fixed the contest architecture, does a player benefit from being an earlier/later mover? An answer to this question requires that we compare players' expected payoffs with respect to their timing positions. Despite the complexity in effort ranking shown in Proposition 2, we verify that players' equilibrium expected payoffs can be ranked monotonically.

[15]Numerical results for four-player three-period sequential contests–i.e., $(N,T) = (4,3)$—are available from the authors upon request.

Let Π_t^* denote a period-t player's equilibrium expected payoff, with $t \in \mathcal{T}$, in a sequential contest $\boldsymbol{n} \equiv (n_1, \dots, n_T)$. Recall that $\mathcal{T}_0 \equiv \{t \in \mathcal{T} : H_t(b) \leq b, \forall b \in [0,1]\}$ indicates the set of the periods in which players always stay inactive in equilibrium, with $t_0 \equiv \max \mathcal{T}_0$. The following result can be obtained.

Theorem 2 ***(Later-mover Advantage in Sequential Contests)*** *Consider a sequential contest $\boldsymbol{n} \equiv (n_1, \dots, n_T)$. A player's expected payoff is higher than those of all earlier movers in the unique symmetric PBE. To put this formally, $0 = \Pi_1^* = \cdots = \Pi_{t_0}^* < \Pi_{t_0+1}^* < \cdots < \Pi_T^*$.*

Recall that $t_0 = 0$ under a concave ability distribution and $t_0 = T - 1$ under a convex distribution. The following result can immediately be obtained.

Corollary 1 ***(Later-mover Advantage with Concave/Convex Ability Distributions)*** *Consider a sequential contest $\boldsymbol{n} \equiv (n_1, \dots, n_T)$. The following statements hold in the unique symmetric PBE:*

(i) Suppose that $F(\cdot)$ is continuous, twice differentiable, strictly concave, and satisfies $\lim_{a \searrow 0}[f(a)a] = 0$. Then $0 < \Pi_1^* < \cdots < \Pi_T^*$.

(ii) Suppose that $F(\cdot)$ is continuous, twice differentiable, and weakly convex. Then $\Pi_1^* = \cdots = \Pi_{T-1}^* = 0 < \Pi_T^*$.

Theorem 2 and Corollary 1 formally establish later-mover advantage in a multi-player sequential all-pay auction. We sketch the proof as follows. For ease of exposition, let us consider a fully sequential contest, with $N = T$. Recall that the profile of equilibrium bidding strategies is denoted by $\boldsymbol{b}^* := \left\{b_1^*(a;\gamma_1), \dots, b_T^*(a;\gamma_T)\right\}$. Fix an arbitrary period $\tau \in \{t_0+1, \dots, T-1\}$, we conduct the following thought experiment. Let us modify the period-$(\tau+1)$ player's bidding strategy from $b_{\tau+1}^*(a;\gamma_{\tau+1})$ to

$$b_{\tau+1}^\dagger(a;\gamma_{\tau+1}) := b_\tau^*(a;\gamma_\tau).$$

In other words, he hypothetically ignores the period-τ player's effort and replicates the latter's equilibrium strategy (*not his effort*). Denote players' expected payoffs under the constructed strategy profile $\boldsymbol{b}^\dagger := \left\{b_1^*(a;\gamma_1), \dots, b_\tau^*(a;\gamma_\tau), b_{\tau+1}^\dagger(a;\gamma_{\tau+1}),\right.$ $\left.b_{\tau+2}^*(a;\gamma_{\tau+2}), \dots, b_T^*(a;\gamma_T)\right\}$ by $(\Pi_1^\dagger, \dots, \Pi_T^\dagger)$.

The key is to show that the period-τ player would be strictly better off with the period-$(\tau+1)$ player's hypothetical deviation, i.e., $\Pi^*_\tau < \Pi^\dagger_\tau$. The intuition is as follows. A later mover, ceteris paribus, tends to be more aggressive in competition than an earlier mover: The former needs to beat a smaller number of future opponents for a win than the latter, which encourages the later mover. Thus, when the period-$(\tau+1)$ player deviates and replicates his immediate predecessor's strategy, he would be less likely to outperform the latter. This obviously benefits the period-τ player.

To fix ideas, consider a period-τ player, with ability $a_\tau > a^{**}_\tau(\gamma_\tau)$, for a given γ_τ. By Theorem 1, he would exert an effort strictly above γ_τ. When the period-$(\tau+1)$ player mimics the period-τ player, the former can defeat the latter if and only if the period-$(\tau+1)$ player is of a higher type, which occurs with probability $1-F(a_\tau)$. Under the equilibrium strategy profile, in contrast, the period-$(\tau+1)$ player outperforms the period-τ player as long as the former chooses to stay active: He does so whenever his ability exceeds the threshold $a^*_{\tau+1}(\gamma_{\tau+1}) = a^*_{\tau+1}(b^*_\tau(a_\tau;\gamma_\tau))$, which occurs with a probability of $1-F(a^*_{\tau+1}(b^*_\tau(a_\tau;\gamma_\tau)))$. We formally show in the Appendix that $a_\tau > a^*_{\tau+1}(b^*_\tau(a_\tau;\gamma_\tau))$:[16] In other words, the period-$(\tau+1)$ player behaves less aggressively when he mimics his immediate predecessor.

To complete the proof, first note that $\Pi^\dagger_{\tau+1} \le \Pi^*_{\tau+1}$ by the definition of PBE. We further have $\Pi^\dagger_\tau \le \Pi^\dagger_{\tau+1}$ by the construction of $b^\dagger_{\tau+1} = b^*_\tau$. Combining these inequalities yields $\Pi^*_\tau < \Pi^\dagger_\tau \le \Pi^\dagger_{\tau+1} \le \Pi^*_{\tau+1}$, which concludes that a period-$(\tau+1)$ player receives a higher equilibrium payoff than a period-τ player.

As previously mentioned, the tie-breaking rule literally "favors" later movers. It, however, deserves to note that the later-mover advantage is not an artifact of the tie-breaking rule. In Section 5.1 and Online Appendix A, we present an extension of a symmetric tie-breaking rule and demonstrates the robustness of our result.

4.3 Endogenous Timing

Our equilibrium results enable us to explore how the architecture of the contest game could endogenously arise. Let the contest be preceded by a timing-choice stage, in which players simultaneously commit to the timing of their moves. Each player

[16] We show in the proof of Theorem 2 that $a_\tau > a^*_{\tau+1}(b^*_\tau(a_\tau;\gamma_\tau))$ holds for an arbitrary contest architecture.

picks one from $L \geq 2$ available periods, denoted by $\mathcal{L} := \{1, \ldots, L\}$, before he learns his realized type and acts accordingly. Before the contest begins, the architecture $\tilde{\boldsymbol{n}}$ is announced publicly, and each player learns his own ability privately. The sequential contest takes place as described in Section 2 thereafter, and Theorem 1 fully characterizes the unique PBE of the contest subgame.[17]

It is noteworthy that the later-mover advantage established in Theorem 2 does not imply choosing a late period is a dominant strategy for each player. To be more specific, the later-mover advantage is obtained by comparing different players' expected payoffs with respect to their timing positions under a predetermined contest architecture. With endogenous timing of moves, however, a player's autonomous timing choice would reshape the resultant contest architecture and affect all players' equilibrium payoffs. Understanding a player's timing choice requires that we compare a given player's equilibrium payoffs across *different* contest architectures. The subsequent analysis takes up this challenge.

The analysis begins with players' equilibrium winning probabilities. Fix an arbitrary contest architecture $\boldsymbol{n} \equiv (n_1, \ldots, n_T)$, with $n_t \geq 1$ for $t \in \{1, \ldots, T\}$. Consider a period-t player of ability $a \in (0,1)$ and denote by $WP_t^*(a; \boldsymbol{n})$ his expected equilibrium winning probability in the unique symmetric PBE. The following lemma can be obtained.

Lemma 5 *Consider two arbitrary contest architectures* $\boldsymbol{n}' \equiv (n'_1, \ldots, n'_{T'})$*—with* $n'_t \geq 1$, $t \in \{1, \ldots, T'\}$, $T' \geq 2$, *and* $\sum_{t=1}^{T'} n'_t = N$*—and* $\boldsymbol{n}'' \equiv (n''_1, \ldots, n''_{T''})$*—with* $n''_t \geq 1$, $t \in \{1, \ldots, T''\}$, $T'' \geq 2$, *and* $\sum_{t=1}^{T''} n''_t = N$. *For almost every* $a \in (0,1)$, *we have*

$$\max\left\{WP_1^*(a; \boldsymbol{n}'), WP_1^*(a; \boldsymbol{n}'')\right\} < F^{N-1}(a) < \min\left\{WP_{T'}^*(a; \boldsymbol{n}'), WP_{T''}^*(a; \boldsymbol{n}'')\right\}.$$

That is, for almost every a, a player is more likely to win when acting in the last period of the contest than being one of the first movers, regardless of the prevailing contest architecture. The comparison is bridged through $F^{N-1}(a)$, which is a player's equilibrium winning probability in a simultaneous contest.

[17] Theorem 1 is established under the assumption that each period possesses at least one player. With endogenous timing, this assumption may not be satisfied due to the possibility that no players choose to move in a certain period. In such a scenario, we can simply remove these periods and relabel the rest to invoke Theorem 1.

This inequality paves the way for a comparison of equilibrium payoffs. We invoke the standard payoff-equivalence argument for direct mechanisms. A period-t player's equilibrium payoff in a sequential contest $\boldsymbol{n} \equiv (n_1, \ldots, n_T)$—which we denote by $\Pi_t^*(\boldsymbol{n})$—can be pinned down by his equilibrium expected winning probability as follows:

$$\Pi_t^*(\boldsymbol{n}) = \mathbb{E}\left[\frac{1}{a}\int_0^a WP_t^*(x;\boldsymbol{n})dx\right] = \int_0^1 \int_0^a \frac{WP_t^*(x;\boldsymbol{n})}{a} dxdF(a).$$

We further define $\Pi^{SIM} := \int_0^1 \int_0^a \frac{1}{a} F^{N-1}(a)dxdF(a)$, which is one's expected payoff in a simultaneous contest. Lemma 5 can then be translated into a comparison of equilibrium payoffs:

$$\max\left\{\Pi_1^*(\boldsymbol{n}'), \Pi_1^*(\boldsymbol{n}'')\right\} < \Pi^{SIM} < \min\left\{\Pi_{T'}^*(\boldsymbol{n}'), \Pi_{T''}^*(\boldsymbol{n}'')\right\}. \tag{9}$$

By this inequality, we are ready to explore players' incentives in their timing choices. The following result ensues.

Lemma 6 ***(Strictly Dominated Strategy with Endogenous Moving Order)*** *For every player, choosing to move in period* 1 *is strictly dominated by choosing to move in period L.*

Lemma 6 implies that the equilibrium in the timing-choice stage is solvable by iterated elimination of strictly dominated strategies (IESDS).

Theorem 3 ***(Unique Equilibrium with Endogenous Moving Order)*** *All players choosing to move in the last period constitutes a Nash equilibrium of the first-stage game that uniquely survives IESDS.*

When players are allowed to pick the timing of their moves, all players will choose the last period and a simultaneous contest endogenously emerges. Again, the result is not an artifact of our tie-breaking rule that literally favors a later mover. The robustness of our results will be discussed below.

5 Extensions and Robustness

In this section, we discuss briefly two extensions to our baseline model, which demonstrate the robustness and versatility of our main results.

5.1 Symmetric Tie-breaking Rule

As discussed above, our current tie-breaking rule appears to favor a later mover. We now consider a symmetric tie-breaking rule. Specifically, a contestant i's winning probability for a given effort profile $\boldsymbol{b} \equiv (b^1, \ldots, b^N)$ is

$$p^{i,s}(\boldsymbol{b}) := \begin{cases} 1, & \text{if } b^i > \max_{\mathcal{N}\setminus\{i\}}\{b^j\}, \\ 1/m, & \text{if } b^i \text{ is among the } m \text{ highest of } \{b^j\}_{j\in\mathcal{N}} \text{ with a tie}, \\ 0, & \text{if } b^i < \max_{\mathcal{N}\setminus\{i\}}\{b^j\}, \end{cases} \tag{10}$$

By (10), when multiple players exert the same highest effort, they win the contest with equal probabilities regardless of their arrival time. It deserves to highlight that the unique symmetric PBE characterized in Theorem 1 is entirely independent of the prevailing tie-breaking rule if each period of the contest involves at least two players—i.e., $n_t \geq 2$ for all $t \in \mathcal{T}$. A PBE, however, may fail to exist if there exists some $t \in \mathcal{T}$ with $n_t = 1$, which calls for an alternative equilibrium notion.

We adopt the notion of ϵ-equilibrium firstly proposed by Radner (1980).[18,19] Intuitively, it requires that a player deviate only when this deviation leads to a substantially higher expected payoff; and the parameter $\epsilon > 0$ captures how much of payoff improvement is required to trigger a deviation. For brevity, we do not present the details, which are laid out in the Online Appendix A. Fixing $\epsilon > 0$, we show that there exists an ϵ-equilibrium in which players' strategies do not depart from the counterpart in the unique symmetric PBE as derived in Theorem 1. This equilibrium result enables us to reaffirm a later-mover advantage given an arbitrary contest

[18] For applications of ϵ-optimality see, e.g., Baye and Morgan (2004), Milgrom (2010), and Martin and Schlag (2020).

[19] Mailath, Postlewaite, and Samuelson (2005) introduce the notion of contemporaneous perfect ϵ-equilibrium in dynamic games with complete information, which is stronger than Radner's equilibrium notion; Jackson, Rodriguez-Barraquer, and Tan (2012) define interim and ex-ante ϵ-equilibrium in static games with incomplete information. Given that the contest we consider is a dynamic game with incomplete information, their definitions can naturally be combined and extended to define an ϵ-equilibrium in our setting.

architecture, as well as our prediction in Section 4.3 regarding endogenous timing in the game, under the symmetric tie-breaking rule.

5.2 Hybrid Payment Rule for Losers

Our main results do not rely on the all-pay feature. Specifically, we allow each loser to bear only a portion of his effort cost. The associated payment rule is specified as follows: The winner in the contest is obliged to pay the full cost of his own effort, while a loser pays $\theta \in [0,1]$ of that.[20,21] To put this formally, fixing a contestant i's ability a^i and the effort profile $\boldsymbol{b} \equiv (b^1, \ldots, b^N)$, his ex post payoff is

$$p^i(\boldsymbol{b})(1 - b^i/a^i) - [1 - p^i(\boldsymbol{b})]\theta b^i/a^i, \text{for all } i \in \mathcal{N}.$$

The above expression degenerates to (2) in the baseline setting as $\theta = 1$, and the contest game turns into a sequential *first-price auction* with $\theta = 0$. A $\theta \in (0,1)$ depicts a hybrid payment rule that involves both winner-pay and all-pay elements.

Fixing a contest architecture $\boldsymbol{n} \equiv (n_1, \ldots, n_T)$ and $\theta \in [0,1]$, a sequence of functions $\{H_t(b;\theta), a_t^*(\gamma;\theta), \tilde{\pi}_t(b,a;\theta)\}_{t=1}^T$ in parallel with (3), (4), and (5) can be defined recursively as follows:[22]

$$H_T(b;\theta) \equiv 1, H_{t-1}(b;\theta) := H_t(b;\theta)F^{n_t}\left(a_t^*(b;\theta)\right), \forall b \in [0,1], \tag{11}$$

$$a_t^*(\gamma;\theta) := \max\{0 < a \leq 1 : \tilde{\pi}_t(b,a;\theta) \leq 0, \forall b \geq \gamma\}, \tag{12}$$

$$\tilde{\pi}_t(b,a;\theta) := H_t(b;\theta)F^{n_t-1}(a)\left[1 - (1-\theta)b/a\right] - \theta b/a. \tag{13}$$

With slight abuse of notation, let $\mathcal{T}_0(\theta) := \{t \in \mathcal{T} : H_t(b;\theta) \leq \frac{\theta b}{1-b+\theta b}, \forall b \in [0,1]\}$ and define $t_0(\theta) := \max \mathcal{T}_0(\theta)$. Again, we can obtain $0 \leq t_0(\theta) \leq T-1$. The analysis in Section 3 can be adapted to characterize a unique symmetric PBE as in Theorem 1. A similar analysis would reinstate the main results of our baseline

[20] See Amann and Leininger (1996) and Baye, Kovenock, and De Vries (2005, 2012) for similar parameterization.

[21] Note that a bid $b \in (0, \gamma_t)$ always leads to a loss and is suboptimal to a period-t player for $\theta > 0$. In contrast, when $\theta = 0$, bidding $b \in (0, \gamma_t)$ is strategically equivalent to bidding zero because a bid does not incur a cost to a loser. In this case, we impose the restriction that period-t players bid 0 or weakly above γ_t without any loss of generality when characterizing the symmetric PBE.

[22] We add θ into $\{H_t(b), a_t^*(\gamma), \tilde{\pi}_t(b,a)\}_{t=1}^T$ to highlight the fact that the defined sequence of functions depends on θ.

model—i.e., Theorems 2 and 3—under this hybrid payment rule. The details are available from the authors upon request.

6 Concluding Remarks

In this paper, we conduct a general analysis of an incomplete-information sequential contest in the form of (first-price) all-pay auctions. Our model allows for a flexible architecture, such that multiple players can be clustered in a single period: They move simultaneously within the period, while observing earlier efforts and anticipating future competitions. Our analysis fully characterizes the unique symmetric equilibrium under a general ability distribution, which adds to the contest literature, as a general analysis of sequential contests remains scarce.

Based on our equilibrium analysis, we formally establish a later-mover advantage, in that one secures a higher ex ante expected payoff when he is assigned to a later timing position vis-à-vis an earlier one, despite that players' ex ante expected efforts can be nonmonotone. We further allow players to choose the timing of their moves in a pre-contest stage. The unique equilibrium that survives iterated elimination of strictly dominated strategies requires that all players choose the last period.

Large room for extensions remains. For instance, a model of multiple prizes with sequential moves deserves serious scholarly effort. A second-price contest—which depicts a "war of attrition"(e.g., Krishna and Morgan, 1997; Bulow and Klemperer (1999); Hafer, 2006; Bergemann, Brooks, and Morris, 2019)—also deserves serious research effort under a sequential timing architecture, and will be attempted in the future.

References

Amann, E., and W. Leininger (1996): "Asymmetric all-pay auctions with incomplete information: The two-player case," *Games and Economic Behavior*, 14(1), 1–18.

Amir, R. (1995): "Endogenous timing in two-player games: a counterexample," *Games and Economic Behavior*, 9(2), 234–237.

Amir, R., and I. Grilo (1999): "Stackelberg versus Cournot equilibrium," *Games and Economic Behavior*, 26(1), 1–21.

Amir, R., and A. Stepanova (2006): "Second-mover advantage and price leadership in Bertrand duopoly," *Games and Economic Behavior*, 55(1), 1–20.

Andreoni, J., Y.-K. Che, and J. Kim (2007): "Asymmetric information about rivals' types in standard auctions: An experiment," *Games and Economic Behavior*, 59(2), 240–259.

Baik, K. H., and J. F. Shogren (1992): "Strategic behavior in contests: Comment," *American Economic Review*, 82(1), 359–362.

Baye, M. R., D. Kovenock, and C. G. De Vries (1993): "Rigging the lobbying process: An application of the all-pay auction," *American Economic Review*, 83(1), 289–294.

——— (2005): "Comparative analysis of litigation systems: An auction-theoretic approach," *Economic Journal*, 115(505), 583–601.

——— (2012): "Contests with rank-order spillovers," *Economic Theory*, 51, 315–350.

Baye, M. R., and J. Morgan (2004): "Price dispersion in the lab and on the internet: Theory and evidence," *RAND Journal of Economics*, 35(3), 449–466.

Bergemann, D., B. Brooks, and S. Morris (2019): "Revenue guarantee equivalence," *American Economic Review*, 109(5), 1911–29.

Brown, J. (2011): "Quitters never win: The (adverse) incentive effects of competing with superstars," *Journal of Political Economy*, 119(5), 982–1013.

Brown, J., and D. B. Minor (2014): "Selecting the best? Spillover and shadows in elimination tournaments," *Management Science*, 60(12), 3087–3102.

Bulow, J., and P. Klemperer (1999): "The generalized war of attrition," *American Economic Review*, 89(1), 175–189.

Che, Y.-K., and I. L. Gale (1998): "Caps on political lobbying," *American Economic Review*, 88(3), 643–651.

——— (2003): "Optimal design of research contests," *American Economic Review*, 93(3), 646–671.

Daughety, A. F. (1990): "Beneficial concentration," *American Economic Review*, 80(5), 1231–1237.

Deneckere, R. J., and D. Kovenock (1992): “Price leadership,” *Review of Economic Studies*, 59(1), 143–162.

Dixit, A. (1987): “Strategic behavior in contests,” *American Economic Review*, 77(5), 891–898.

Dowrick, S. (1986): “von Stackelberg and Cournot duopoly: Choosing roles,” *RAND Journal of Economics*, 17(2), 251–260.

Fullerton, R. L., and R. P. McAfee (1999): “Auctioning entry into tournaments,” *Journal of Political Economy*, 107(3), 573–605.

Gal-Or, E. (1985): “First mover and second mover advantages,” *International Economic Review*, 26(3), 649–653.

——— (1987): “First mover disadvantages with private information,” *Review of Economic Studies*, 54(2), 279–292.

Glazer, A., and R. Hassin (2000): “Sequential rent seeking,” *Public Choice*, 102(3-4), 219–228.

Green, J. R., and N. L. Stokey (1983): “A comparison of tournaments and contracts,” *Journal of Political Economy*, 91(3), 349–364.

Hafer, C. (2006): “On the origins of property rights: Conflict and productionin the state of nature,” *Review of Economic Studies*, 73(1), 119–143.

Hamilton, J. H., and S. M. Slutsky (1990): “Endogenous timing in duopoly games: Stackelberg or Cournot equilibria,” *Games and Economic Behavior*, 2(1), 29–46.

Hinnosaar, T. (2021): “Optimal sequential contests,” *Working Paper*.

Hoffmann, M., and G. Rota-Graziosi (2012): “Endogenous timing in general rent-seeking and conflict models,” *Games and Economic Behavior*, 75(1), 168–184.

Jackson, M. O., T. Rodriguez-Barraquer, and X. Tan (2012): “Epsilon-equilibria of perturbed games,” *Games and Economic Behavior*, 75(1), 198–216.

Jian, L., Z. Li, and T. X. Liu (2017): “Simultaneous versus sequential all-pay auctions: An experimental study,” *Experimental Economics*, 20(3), 648–669.

Kahana, N., and D. Klunover (2018): “Sequential lottery contests with multiple participants,” *Economics Letters*, 163, 126–129.

Kempf, H., and G. Rota-Graziosi (2010): "Endogenizing leadership in tax competition," *Journal of Public Economics*, 94(9-10), 768–776.

Konrad, K. A., and W. Leininger (2007): "The generalized Stackelberg equilibrium of the all-pay auction with complete information," *Review of Economic Design*, 11(2), 165–174.

Krishna, V., and J. Morgan (1997): "An analysis of the war of attrition and the all-pay auction," *Journal of Economic Theory*, 72(2), 343–362.

La Rosa, F., and G. Liberatore (2014): "Biopharmaceutical and chemical firms R&D disclosure, and cost of equity: The impact of the regulatory regime," *European Management Journal*, 32(5), 806–820.

Lazear, E. P., and S. Rosen (1981): "Rank-order tournaments as optimum labor contracts," *Journal of Political Economy*, 89(5), 841–864.

Lee, T., and L. L. Wilde (1980): "Market structure and innovation: A reformulation," *Quarterly Journal of Economics*, 94(2), 429–436.

Leininger, W. (1993): "More efficient rent-seeking: A Münchhausen solution," *Public Choice*, 75(1), 43–62.

Liu, T. X., J. Yang, L. A. Adamic, and Y. Chen (2014): "Crowdsourcing with all-pay auctions: A field experiment on taskcn," *Management Science*, 60(8), 2020–2037.

Loury, G. C. (1979): "Market structure and innovation," *Quarterly Journal of Economics*, 93(3), 395–410.

Mailath, G. J. (1993): "Endogenous sequencing of firm decisions," *Journal of Economic Theory*, 59(1), 169–182.

Mailath, G. J., A. Postlewaite, and L. Samuelson (2005): "Contemporaneous perfect epsilon-equilibria," *Games and Economic Behavior*, 53(1), 126–140.

Martin, S., and K. H. Schlag (2020): "Split it up to create incentives: Investment, public goods and crossing the river," *Journal of Economic Theory*, 189, 105092.

Maskin, E., and J. Riley (2000): "Equilibrium in sealed high bid auctions," *Review of Economic Studies*, 67(3), 439–454.

Milgrom, P. (2010): "Simplified mechanisms with an application to sponsored-search auctions," *Games and Economic Behavior*, 70(1), 62–70.

MOLDOVANU, B., AND A. SELA (2001): "The optimal allocation of prizes in contests," *American Economic Review*, 91(3), 542–558.

——— (2006): "Contest architecture," *Journal of Economic Theory*, 126(1), 70–96.

MOLDOVANU, B., A. SELA, AND X. SHI (2007): "Contests for status," *Journal of Political Economy*, 115(2), 338–363.

MORGAN, J. (2003): "Sequential contests," *Public Choice*, 116(1-2), 1–18.

MORGAN, J., AND F. VÁRDY (2007): "The value of commitment in contests and tournaments when observation is costly," *Games and Economic Behavior*, 60(2), 326–338.

RADNER, R. (1980): "Collusive behavior in noncooperative epsilon-equilibria of oligopolies with long but finite lives," *Journal of Economic Theory*, 22(2), 136–154.

ROSEN, S. (1986): "Prizes and incentives in elimination tournaments," *American Economic Review*, 76(4), 701–715.

SEGEV, E., AND A. SELA (2014): "Multi-stage sequential all-pay auctions," *European Economic Review*, 70, 371–382.

SHINKAI, T. (2000): "Second mover disadvantages in a three-player Stackelberg game with private information," *Journal of Economic Theory*, 90(2), 293–304.

SIMON, L. K., AND W. R. ZAME (1990): "Discontinuous games and endogenous sharing rules," *Econometrica*, 58(4), 861–872.

SNYDER, J. M. (1989): "Election goals and the allocation of campaign resources," *Econometrica*, 57(3), 637–660.

TAYLOR, C. R. (1995): "Digging for golden carrots: An analysis of research tournaments," *American Economic Review*, 85(4), 872–890.

VAN DAMME, E., AND S. HURKENS (1999): "Endogenous stackelberg leadership," *Games and Economic Behavior*, 28(1), 105–129.

——— (2004): "Endogenous price leadership," *Games and Economic Behavior*, 47(2), 404–420.

VON STENGEL, B. (2010): "Follower payoffs in symmetric duopoly games," *Games and Economic Behavior*, 69(2), 512–516.

Appendix: Proofs

Proofs of Lemmas 1 and 2. It is useful to prove several intermediate results.

Lemma 7 *The following statements hold:*

(i) $a_t^*(\gamma)$ *is continuous, piecewise differentiable, and weakly increasing on* $[0,1]$*, satisfying* $a_t^*(\gamma) \geq \gamma$ *for all* $t \in \mathcal{T}$*; and*

(ii) $H_t(b)$ *is continuous, piecewise differentiable, and strictly increasing on* $[0,1]$*, with* $H_t(0) = 0$ *and* $H_t(1) = 1$ *for all* $t \in \mathcal{T} \setminus \{T\}$*.*

Proof. We prove the lemma by induction. Note that piecewise analyticity implies piecewise differentiability. Therefore, to show that $a_t^*(\gamma)$ and $H_t(b)$ are piecewise differentiable, it suffices to show that they are piecewise analytic. Denote by $G_n(\cdot)$ the inverse function of $aF^{n-1}(a)$ for an arbitrary positive integer $n \in \mathbb{N}_+$. It can be verified that $G_n(\cdot)$ is strictly increasing, piecewise analytic, and differentiable on $[0,1]$, with $G_n(0) = 0$ and $G_n(1) = 1$.

Base case: By definition, $H_T(b) = 1$. Therefore, $\tilde{\pi}_T(b,a) \equiv H_T(b)F^{n_T-1}(a) - b/a = F^{n_T-1}(a) - b/a$. These facts, together with (4), imply that

$$a_T^*(\gamma) := \max\left\{0 < a \leq 1 : \tilde{\pi}_T(b,a) \leq 0, \forall b \in [\gamma, 1]\right\} = G_{n_T}(\gamma)$$

and $H_{T-1}(b) = F^{n_T}\big(G_{n_T}(b)\big)$. It is straightforward to verify that $a_T^*(\gamma)$ satisfies part (i) of the lemma and $H_{T-1}(b)$ satisfies part (ii).

Inductive step: Suppose that $H_t(b)$ satisfies part (ii) of the lemma for some $t \leq T-1$. It suffices to show that $a_t^*(\gamma)$ satisfies part (i) of the lemma and $H_{t-1}(b)$ satisfies part (ii).

Fixing $b \in (0,1]$, $\tilde{\pi}_t(b,a)$ strictly increases with $a \in (0,1)$. Define $\breve{a}_t(b)$ as follows:

$$\breve{a}_t(b) := \begin{cases} G^{n_t}\left(\min\left\{\frac{1}{H_t'(0)}, 1\right\}\right), & \text{if } b = 0, \\ 1, & \text{if } b \in (0,1] \text{ and } \tilde{\pi}_t(b,1) < 0, \\ \text{the unique solution to } \tilde{\pi}_t(b,a) = 0, & \text{otherwise.} \end{cases}$$

It can be verified that $a_t^*(\gamma) = \min_{b \geq \gamma} \breve{a}_t(b)$ and $\breve{a}_t(b)$ is continuous on $[0, 1]$. This in turn implies that $a_t^*(\gamma)$ is continuous, piecewise analytic, and weakly increasing on $[0, 1]$. Further, for $b \geq \gamma$, we have

$$\tilde{\pi}_t(b, \gamma) = F^{n_t - 1}(\gamma) H_t(b) - b/\gamma \leq 0,$$

which indicates that $\gamma \in \{0 < a \leq 1 : \tilde{\pi}_t(b, a) \leq 0, \forall b \geq \gamma\}$ and thus $a_t^*(\gamma) \geq \gamma$. To summarize, $a_t^*(\gamma)$ satisfies part (i) of the lemma.

Because $H_t(b)$ satisfies part (ii) of the lemma by assumption and $a_t^*(\gamma)$ satisfies part (i), we can conclude that $H_{t-1}(b) = H_t(b) F^{n_t}\big(a_t^*(b)\big)$ satisfies part (ii). This completes the inductive step.

Conclusion: By the principle of induction, $a_t^*(\gamma)$ satisfies part (i) of Lemma 7 for all $t \in \mathcal{T}$ and $H_t(b)$ satisfies part (ii) for all $t \in \mathcal{T} \setminus \{T\}$. This concludes the proof. ■

Lemma 8 *The following statements hold for all $t \in \mathcal{T}$.*

(i) If $a_t^*(\gamma) < a < 1$, then there exists $b \in [\gamma, 1]$ such that $\tilde{\pi}_t(b, a) > 0$.

(ii) If $0 < a < a_t^*(\gamma)$, then $\tilde{\pi}_t(b, a) < 0$ for all $b \in [\gamma, 1] \setminus \{0\}$.

Proof. Part (i) of the lemma is obvious and it remains to prove part (ii). Fix $0 < a < a_t^*(\gamma)$. Suppose, to the contrary, that $\tilde{\pi}_t(b_0, a) \geq 0$ for some $b_0 \in [\gamma, 1] \setminus \{0\}$. It follows immediately that $a \geq b_0$. Further, we have that $\tilde{\pi}_t(b_0, a_t^*(\gamma)) > \tilde{\pi}_t(b_0, a) \geq 0$, which contradicts with the fact that $\tilde{\pi}_t(b_0, a_t^*(\gamma)) \leq 0$ for all $b \in [\gamma, 1]$. This completes the proof. ■

Now we can prove Lemmas 1 and 2 by induction.

Base case: Consider the last period, i.e., $t = T$. It is evident that the realized highest effort by the end of period T wins the contest with certainty. By definition, $H_T(b) = 1$. Therefore, Lemma 2 holds for $t = T$ and it remains to show that Lemma 1 holds for the last period. We consider the following two cases:

(a) Suppose $n_T = 1$. Then the optimal bidding strategy of the unique period-T player is to bid γ_T if $a > \gamma_T$ and bid 0 otherwise. Therefore, Lemma 1(i) and (ii) hold.

(b) Suppose $n_T \geq 2$. We first show that $b_T^*(a;\gamma_T)$ is increasing in a. Suppose, to the contrary, that there exists an ability pair (a', a''), with $0 < a' < a'' < 1$, such that $b'' := b_T^*(a'';\gamma_T) < b' := b_T^*(a';\gamma_T)$. Denote the equilibrium winning probability of bidding b by $\mathcal{WP}_T^*(b)$. It is obvious that $\mathcal{WP}_T^*(b'') < \mathcal{WP}_T^*(b')$; otherwise, a type-$a'$ player has a strict incentive to bid b''. Moreover, from players' incentive compatibility constraints, we have that

$$\mathcal{WP}_T^*(b')a' - b' \geq \mathcal{WP}_T^*(b'')a' - b'', \text{ and } \mathcal{WP}_T^*(b'')a'' - b'' \geq \mathcal{WP}_T^*(b')a'' - b',$$

which is equivalent to

$$a'\left[\mathcal{WP}_T^*(b') - \mathcal{WP}_T^*(b'')\right] \geq b' - b'', \text{ and } a''\left[\mathcal{WP}_T^*(b') - \mathcal{WP}_T^*(b'')\right] \leq b' - b''.$$

Combining the above inequalities yields

$$(a' - a'') \times \left[\mathcal{WP}_T^*(b') - \mathcal{WP}_T^*(b'')\right] \geq 0,$$

which is a contradiction given that $\mathcal{WP}_T^*(b'') < \mathcal{WP}_T^*(b')$ and the postulated $a' < a''$.

Let $\bar{a}_T := \inf\{a : b_T^*(a;\gamma_T) > 0\}$. We first show that $b_T^*(a;\gamma_T)$ strictly increases with a for $a > \bar{a}_T$. Suppose, to the contrary, that $\bar{a}_T < a' < a'' < 1$ and $b' := b_T^*(a';\gamma_T) = b'' := b_T^*(a'';\gamma_T)$. It follows immediately that $b_T^*(a;\gamma_T) = b'$ for $a \in [a', a'']$. Then a type-a' player has an incentive to deviate from bidding b'. Specifically, he can raise his effort by an infinitesimal amount to substantially increase his winning probability, which leads to an increase in his interim expected payoff. A contradiction.

Further, note that $b_T^*(a;\gamma_T) \geq \gamma_T$ for $a > \bar{a}_T$ and $b_T^*(a;\gamma_T) = 0$ for $a \leq \bar{a}_T$, and it thus remains to prove that $\bar{a}_T = a_T^*(\gamma_T) \equiv \max\left\{0 < a \leq 1 : \tilde{\pi}_T(b,a) \leq 0, \forall b \in [\gamma_T, 1]\right\}$, where $\tilde{\pi}_T(b,a) \equiv H_T(b)F^{n_T-1}(a) - b/a$. We consider the following two cases:

(i) Suppose that $\bar{a}_T < a_T^*(\gamma_T)$. Consider a type-$a'$ player, with $\bar{a}_T < a' < a_T^*(\gamma_T)$. Recall that $b_T^*(a;\gamma_T)$ strictly increases with a for $a > \bar{a}_T$. Therefore, we have $b_T^*(a';\gamma_T) > 0$. His equilibrium expected payoff is

$$F^{n_T-1}(a') - \frac{b_T^*(a';\gamma_T)}{a'} = \tilde{\pi}_T\left(b_T^*(a';\gamma_T), a'\right) < 0,$$

where the strict inequality follows from $b_T^*(a';\gamma_T) \neq 0$, $b_T^*(a';\gamma_T) \geq \gamma_T$, and Lemma 8(ii). However, he can secure a nonnegative expected payoff by bidding zero. A contradiction.

(ii) Suppose that $\bar{a}_T > a_T^*(\gamma_T)$. Fix $a' \in \left(a_T^*(\gamma_T), \bar{a}_T\right)$. It follows immediately from $a' < \bar{a}_T$ that $b_T^*(a';\gamma_T) = 0$. Note that bidding zero must generate zero expected payoff to a type-a' player. Otherwise, we must have $\gamma_T = 0$; together with $\bar{a}_T > 0$, we can conclude that a player whose type falls below $\bar{a}_T$ can strictly increase his expected payoff by exerting an infinitesimal amount of effort. A contradiction.

By Lemma 8(i), there exists some $b' \in [\gamma_T, 1]$ such that $\tilde{\pi}_T(b', a') > 0$. Then type-$a'$ player's expected payoff of bidding b' is bounded from below by

$$F^{n_T-1}(a') - \frac{b'}{a'} = \tilde{\pi}_T(b', a') > 0.$$

Therefore, a type-a' player has a strict incentive to deviate from exerting zero effort, which is a contradiction.

Inductive step: Suppose that the equilibrium bidding strategy $b_t^*(a;\gamma_t)$ satisfies the properties stated in Lemma 1 and $H_t(b)$ gives the probability of the effort b's exceeding all subsequent efforts in equilibrium, as predicted in Lemma 2, for some $t \leq T$. We show that the same holds for period $t-1$.

Suppose that the realized highest effort by the end of period $t-1$ is b. Then the probability of the effort b's exceeding all subsequent efforts in equilibrium is $H_t(b)F^{n_t}\left(a_t^*(b)\right)$, which is exactly $H_{t-1}(b)$ from (3).

For the case of $n_{t-1} = 1$, note that the problem of the only period-$(t-1)$ player with ability a is $\max_{b\in\{0\}\cup[\gamma_{t-1},1]}[H_{t-1}(b) - b/a]$. It is then straightforward to verify that (i) $b_{t-1}^*(a;\gamma_{t-1})$ is increasing in a on $(0,1)$; and (ii) $b_{t-1}^*(a;\gamma_{t-1}) = 0$ for $a \leq a_{t-1}^*(\gamma_{t-1})$ and $b_t^*(a;\gamma_{t-1}) \geq \gamma_{t-1}$ for $a > a_{t-1}^*(\gamma_{t-1})$. For the case of $n_{t-1} \geq 2$, by the same argument as in the base case, we can show that $b_{t-1}^*(a, \gamma_{t-1})$ satisfies all properties stated in Lemma 1. This completes the inductive step.

Conclusion: By the principle of induction, $b_t^*(a;\gamma_t)$ satisfies all properties stated in Lemma 1 for all $t \in \mathcal{T}$. Moreover, $H_t(b)$ gives the probability of the effort b's exceeding all subsequent efforts in equilibrium for all $t \in \mathcal{T}$, as predicted in Lemma 2. This concludes the proof. ∎

Proof of Lemma 3. It is useful to prove the following intermediate result.

Lemma 9 *Suppose that $a_t^*(\gamma_t) < 1$. Then $\tilde{\pi}_t\big(s_t^1(a_t^*(\gamma_t);\gamma_t), a_t^*(\gamma_t)\big) = 0$.*

Proof. Evidently, $s_t^1(a_t^*(\gamma_t);\gamma_t) \geq \gamma_t$; together with the definition of $a_t^*(\gamma_t)$, we can obtain $\tilde{\pi}_t\big(s_t^1(a_t^*(\gamma_t);\gamma_t), a_t^*(\gamma_t)\big) \leq 0$. Suppose, to the contrary, that $\tilde{\pi}_t\big(s_t^1(a_t^*(\gamma_t);\gamma_t), a_t^*(\gamma_t)\big) \neq 0$. Then we must have

$$\tilde{\pi}_t\big(s_t^1(a_t^*(\gamma_t);\gamma_t), a_t^*(\gamma_t)\big) < 0.$$

The above inequality, together with the fact that $s_t^1\big(a_t^*(\gamma_t);\gamma_t\big) \in S_t(a;\gamma_t)$, implies that

$$\tilde{\pi}_t\big(b', a_t^*(\gamma_t)\big) < \tilde{\pi}_t\big(s_t^1(a_t^*(\gamma_t);\gamma_t), a_t^*(\gamma_t)\big) < 0, \text{ for all } s_t^1\big(a_t^*(\gamma_t);\gamma_t\big) < b' \leq 1. \tag{14}$$

Next, note that by definition, $s_t^1\big(a_t^*(\gamma_t);\gamma_t\big)$ is the smallest element in the set $S_t(a;\gamma_t)$. Therefore, we have that

$$\tilde{\pi}_t\big(b', a_t^*(\gamma_t)\big) \leq \tilde{\pi}_t\big(s_t^1(a_t^*(\gamma_t);\gamma_t), a_t^*(\gamma_t)\big) < 0, \text{ for all } \gamma_t \leq b' \leq s_t^1\big(a_t^*(\gamma_t);\gamma_t\big). \tag{15}$$

Combining (14) and (15), $\tilde{\pi}_t(b', a_t^*(\gamma_t) + \epsilon) < 0$ for all $b' \in [\gamma_t, 1]$ for sufficiently small $\epsilon > 0$, which contradicts the definition of $a_t^*(\gamma_t)$ and concludes the proof. ∎

Now we can prove Lemma 3. Suppose, to the contrary, that $n_t \geq 2$, $a_t^*(\gamma_t) < 1$, and $\lim_{a \searrow a_t^*(\gamma_t)} b_t^*(a;\gamma_t) \neq s_t^1(a_t^*(\gamma_t);\gamma_t)$. We consider the following two cases:

(a) Suppose that $\lim_{a \searrow a_t^*(\gamma_t)} b_t^*(a;\gamma_t) < s_t^1(a_t^*(\gamma_t);\gamma_t)$. Then for sufficiently small $\epsilon > 0$, we have $b_t^*(a;\gamma_t) < s_t^1\big(a_t^*(\gamma_t);\gamma_t\big)$ for all $a < a_t^*(\gamma_t) + \epsilon$. Consider a type-$a_t^*(\gamma_t)$ player. His expected payoff of bidding $s_t^1\big(a_t^*(\gamma_t);\gamma_t\big)$ is at least

$$H_t\left(s_t^1(a_t^*(\gamma_t);\gamma_t)\right) F^{n_t-1}\left(a_t^*(\gamma_t) + \epsilon\right) - \frac{s_t^1(a_t^*(\gamma_t);\gamma_t)}{a_t^*(\gamma_t)}$$

$$
\begin{aligned}
&> H_t\left(s_t^1(a_t^*(\gamma_t);\gamma_t)\right) F^{n_t-1}\left(a_t^*(\gamma_t)\right) - \frac{s_t^1(a_t^*(\gamma_t);\gamma_t)}{a_t^*(\gamma_t)} \\
&= \tilde{\pi}_t\left(s_t^1\left(a_t^*(\gamma_t);\gamma_t\right), a_t^*(\gamma_t)\right) = 0,
\end{aligned}
$$

where the equality follows from Lemma 9. Meanwhile, it follows from Lemma 1(ii) that a type-$a_t^*(\gamma_t)$ player would bid 0 and thus earns zero expected payoff in equilibrium. A contradiction.

(b) Suppose that $\lim_{a\searrow a_t^*(\gamma_t)} b_t^*(a;\gamma_t) > s_t^1(a_t^*(\gamma_t);\gamma_t)$. Consider a player whose type is $a' = a_t^*(\gamma_t) + \epsilon$ for sufficiently small $\epsilon > 0$. His expected payoff of bidding $b' = b_t^*(a';\gamma_t)$ is $\tilde{\pi}_t(b', a')$. It follows from the postulated $\lim_{a\searrow a_t^*(\gamma_t)} b_t^*(a;\gamma_t) > s_t^1(a_t^*(\gamma_t);\gamma_t)$ and the definition of $S_t(a;\gamma)$ that

$$
\tilde{\pi}_t\left(\lim_{a\searrow a_t^*(\gamma_t)} b_t^*(a;\gamma_t), a_t^*(\gamma_t)\right) < \tilde{\pi}_t\left(s_t^1\left(a_t^*(\gamma_t);\gamma_t\right), a_t^*(\gamma_t)\right) = 0,
$$

where the equality again follows from Lemma 9. By continuity, $\tilde{\pi}_t(b', a') < 0$ for sufficiently small $\epsilon > 0$. Therefore, a type-a' player can secure a strictly higher expected payoff by exerting zero effort, which is a contradiction. This concludes the proof.

■

Proof of Lemma 4. Fix some type $\tilde{a} \in (a_t^*(\gamma_t), 1]$ such that $b_t^*(\tilde{a}-0;\gamma_t) = e_t^m(\tilde{a};\gamma_t)$ for some $1 \le m \le m_t(\tilde{a};\gamma_t) - 1$. Then we have that (see Figure 1)

$$
\tilde{\pi}_t\left(e_t^m(\tilde{a};\gamma_t), \tilde{a}\right) = \tilde{\pi}_t\left(s_t^{m+1}(\tilde{a};\gamma_t), \tilde{a}\right). \tag{16}
$$

Suppose, to the contrary, that $b_t^*(\tilde{a}+0;\gamma_t) \neq s_t^{m+1}(\tilde{a};\gamma_t)$. We consider the following two cases:

(a) Suppose that $b_t^*(\tilde{a}+0;\gamma_t) < s_t^{m+1}(\tilde{a};\gamma_t)$. Then there exists $\epsilon > 0$ such that $b_t^*(a;\gamma_t) < s_t^{m+1}(\tilde{a};\gamma_t)$ for all $a < \tilde{a}+\epsilon$. Consider a player whose ability is $\tilde{a}-\epsilon'$ for sufficiently small $\epsilon' > 0$. His expected payoff of bidding $s_t^{m+1}(\tilde{a};\gamma_t)$ is no less than

$$
F^{n_t-1}(\tilde{a}+\epsilon)H_t\left(s_t^{m+1}(\tilde{a};\gamma_t)\right) - \frac{s_t^{m+1}(\tilde{a};\gamma_t)}{\tilde{a}-\epsilon'} > F^{n_t-1}(\tilde{a}+\epsilon)H_t\left(s_t^{m+1}(\tilde{a};\gamma_t)\right) - \frac{s_t^{m+1}(\tilde{a};\gamma_t)}{\tilde{a}}
$$

$$
\begin{aligned}
&>\tilde{\pi}_t\left(s_t^{m+1}(\tilde{a};\gamma_t),\tilde{a}\right)\\
&=\tilde{\pi}_t\left(e_t^{m}(\tilde{a};\gamma_t),\tilde{a}\right),
\end{aligned}
$$

where the second equality follows from (16). Note that $b_t^*(\tilde{a}-0;\gamma_t)=e_t^m(\tilde{a};\gamma_t)$ and thus $\tilde{\pi}_t(e_t^m(\tilde{a};\gamma_t),\tilde{a})$ is the limit of player's equilibrium expected payoff as ϵ' approaches 0. Therefore, the player can obtain a strictly higher payoff by bidding $s_t^{m+1}(\tilde{a};\gamma_t)$, which is a contradiction.

(b) Suppose that $b_t^*(\tilde{a}+0;\gamma_t)>s_t^{m+1}(\tilde{a};\gamma_t)$. Consider a player whose type is $\tilde{a}+\epsilon'$ for sufficiently small $\epsilon'>0$. Note that his equilibrium expected payoff of bidding $b_t^*(\tilde{a}+\epsilon';\gamma_t)$ can then be bounded from above by

$$
\tilde{\pi}_t\left(b_t^*(\tilde{a}+\epsilon';\gamma_t),\tilde{a}+\epsilon'\right)<\tilde{\pi}_t\left(s_t^{m+1}(\tilde{a};\gamma_t),\tilde{a}\right)=\tilde{\pi}_t\left(e_t^{m}(\tilde{a};\gamma_t),\tilde{a}\right),
$$

where the inequality follows from the definition of $S_t(a;\gamma)$ and the equality from (16). Meanwhile, his expected payoff of bidding $b_t^*(\tilde{a}-0;\gamma_t)$ is

$$
F^{n_t-1}(\tilde{a})H_t\left(e_t^m(\tilde{a};\gamma_t)\right)-\frac{e_t^m(\tilde{a};\gamma_t)}{\tilde{a}+\epsilon'}\geq\tilde{\pi}_t\left(e_t^{m}(\tilde{a};\gamma_t),\tilde{a}\right).
$$

Therefore, the player has a strict incentive to deviate from his equilibrium bid $b_t^*(\tilde{a}+\epsilon';\gamma_t)$, which is a contradiction.

■

Proof of Theorem 1. We consider the following two cases:

(a) Suppose $n_t=1$. It is evident that $b_t^*(a;\gamma_t)=0$ for $a\leq a_t^*(\gamma_t)$ and $b_t^*(a;\gamma_t)$ solves

$$
\max_{b\geq\gamma_t}\left[H_t(b)-b/a\right]
$$

for $a>a_t^*(\gamma_t)$. Further, let $a_t^{**}(\gamma_t):=\sup_{a_t^*(\gamma_t)\leq a\leq 1}\left\{a:H_t(\gamma_t)-\gamma_t/a>H_t(b')-b'/a,\forall b'\in(\gamma_t,1]\right\}$. It follows immediately that

$$
\frac{b'-\gamma_t}{a}>\frac{b'-\gamma_t}{a_t^{**}(\gamma_t)}\geq H_t(b')-H_t(\gamma_t),\text{ for all } a_t^*(\gamma_t)\leq a<a_t^{**}(\gamma_t)\text{ and } b'\in(\gamma_t,1],
$$

which in turn implies that $b_t^*(a;\gamma_t)=\gamma_t$ for when the player's ability a lies between $a^*(\gamma_t)$ and $a^{**}(\gamma_t)$. To summarize, period-t player's equilibrium bidding strategy for the case of $n_t = 1$ is characterized by (7) in part (i) of the theorem.

(b) Suppose that $n_t \geq 2$. For $a \leq a_t^*(\gamma_t)$, it follows immediately from Lemma 1(ii) that $b_t^*(a;\gamma_t) = 0$. For $a > a_t^*(\gamma_t)$, we have that

$$b_t^*(a;\gamma_t) \in \arg\max_{b>\gamma_t} \left[H_t(b) F^{n_t-1}\left((b_t^*)^{-1}(b;\gamma_t)\right) - b/a \right].$$

This implies that

$$a \in \arg\max_{\check{a}>a_t^*(\gamma_t)} \check{\pi}_t(\check{a}, a;\gamma_t) := H_t\left(b_t^*(\check{a};\gamma_t)\right) F^{n_t-1}(\check{a}) - b_t^*(\check{a};\gamma_t)/a.$$

Suppose that $b_t^*(a;\gamma_t)$ is continuous in some interval $\mathcal{U}_{\tilde{a}} = (\tilde{a}, \tilde{a}+\epsilon)$. In the equilibrium, the following first-order condition should be satisfied:

$$\left.\frac{\partial \check{\pi}_t(\check{a}, a;\gamma_t)}{\partial \check{a}}\right|_{\check{a}=a} = 0, \text{ for } a \in \mathcal{U}_{\tilde{a}},$$

which is equivalent to

$$(n_t - 1) H_t\left(b_t^*(a;\gamma_t)\right) F^{n_t-2}(a) f(a) + (b_t^*)'(a;\gamma_t) \times \left.\frac{\partial \tilde{\pi}_t(b,a)}{\partial b}\right|_{b=b_t^*(a;\gamma_t)} = 0, \text{ for } a \in \mathcal{U}_{\tilde{a}}, \tag{17}$$

and can be further simplified as (8) in the text. Condition (17), together with Lemma 1(ii), Lemma 3, and Lemma 4, indicates that the equilibrium bidding strategy $b_t^*(a;\gamma_t)$, if a PBE exists, is fully characterized as in Theorem 1(ii).

It remains to verify that $b_t^*(a;\gamma_t)$ as described in Theorem 1(ii) indeed constitutes a PBE of the contest game. We first verify the monotonicity of $b_t^*(a;\gamma_t)$. Evidently, the first term on the left-hand side of (17) always remains positive, indicating that $(b_t^*)'(a;\gamma_t) \neq 0$. Moreover, suppose that there exists $\tilde{a} \geq a_t^*(\gamma_t)$ such that $b_t^*(\tilde{a}+0;\gamma_t) = s_t^m(\tilde{a};\gamma_t)$ for some $1 \leq m \leq m_t(\tilde{a};\gamma_t)$. From the definition of $s_t^m(\tilde{a};\gamma_t)$, for any sufficiently small $\epsilon > 0$, we have that

$$\left.\frac{\partial \tilde{\pi}_t\left(b, \tilde{a}\right)}{\partial b}\right|_{b=s_t^m(\tilde{a};\gamma_t)+\epsilon} < 0.$$

Therefore, $(b_t^*)'(a;\gamma_t) > 0$ at $a = \tilde{a} + \epsilon$; otherwise, (17) cannot be satisfied. We can thus conclude from these facts that $b_t^*(a;\gamma_t)$ strictly increases with a whenever $b_t^*(a;\gamma_t)$ is continuous and is governed by (17). It remains to verify the monotonicity of $b_t^*(a;\gamma_t)$ at discontinuity points. Suppose that there exists $\tilde{a}$ such that $b_t^*(\tilde{a}-0;\gamma_t) = e_t^m(\tilde{a};\gamma_t)$ for some $1 \leq m \leq m_t(\tilde{a};\gamma_t)-1$. By Lemma 4, we have $b_t^*(\tilde{a}-0;\gamma_t) = e_t^m(\tilde{a};\gamma_t) < s_t^{m+1}(\tilde{a};\gamma_t) = b_t^*(\tilde{a}+0;\gamma_t)$.

Simple algebra would verify that

$$\frac{\partial^2 \breve{\pi}_t(\breve{a}, a;\gamma_t)}{\partial \breve{a} \partial a} > 0,$$

which implies that $\frac{\partial \breve{\pi}_t(\breve{a},a;\gamma_t)}{\partial \breve{a}}$ is increasing in a. Therefore, we have

$$\frac{\partial \breve{\pi}_t(\breve{a}, a;\gamma_t)}{\partial \breve{a}} > \left.\frac{\partial \breve{\pi}_t(\breve{a}, a;\gamma_t)}{\partial \breve{a}}\right|_{\breve{a}=a} = 0, \text{ for } \breve{a} < a,$$

and

$$\frac{\partial \breve{\pi}_t(\breve{a}, a;\gamma_t)}{\partial \breve{a}} < \left.\frac{\partial \breve{\pi}_t(\breve{a}, a;\gamma_t)}{\partial \breve{a}}\right|_{\breve{a}=a} = 0, \text{ for } \breve{a} > a.$$

That is, the necessary first-order condition $\left.\frac{\partial \breve{\pi}_t(\breve{a},a;\gamma_t)}{\partial \breve{a}}\right|_{\breve{a}=a} = 0$ is also a sufficient condition for global maximizer. This concludes the proof.

■

Proof of Proposition 1. See main text. ■

Proof of Proposition 2. The analysis for the case of $N = 2$ is laid out in the main text and it suffices to consider the case of $N \geq 3$. Assuming $F(a) = a^r$, with $r \in (0,1)$, and a fully sequential contest architecture, we can derive players' equilibrium expected efforts as follows. The detailed derivation is provided in Online Appendix C.

For $r \searrow 0$, we have

$$\mathbb{E}[b_t^*] = (T-t)r^2 + o(r^2),$$

from which we can conclude that $\mathbb{E}[b_1^*] > \mathbb{E}[b_2^*] > \cdots > \mathbb{E}[b_T^*]$ when r is sufficiently close to 0.

For $r \nearrow 1$, we have

$$\mathbb{E}[b_t^*] = \begin{cases} \frac{1}{e}(1-r)^{T-t} + o((1-r)^{T-t}), & \text{if } t < T, \\ \frac{2e-1}{2e^2}(1-r) + o(1-r), & \text{if } t = T, \end{cases}$$

from which we can conclude that $\mathbb{E}[b_{T-1}^*] > \mathbb{E}[b_T^*] > \mathbb{E}[b_{T-2}^*] > \cdots > \mathbb{E}[b_1^*]$ when r is sufficiently close to 1. This concludes the proof. ■

Proof of Theorem 2. Recall the unique symmetric PBE is denoted by $\boldsymbol{b}^* := \{b_1^*(a;\gamma_1), \ldots, b_T^*(a;\gamma_T)\}$. Fix an arbitrary period $\tau \in \{t_0+1, \ldots, T-1\}$ and a player i in period $\tau+1$, i.e., $i \in \mathcal{N}_{\tau+1}$. We conduct the following thought experiment. Holding fixed all other players' strategies—including those in period $\tau+1$, if any—we modify player i's equilibrium bidding strategy from $b_{\tau+1}^*(a;\gamma_{\tau+1})$ to

$$b_{\tau+1}^{i\dagger}(a;\gamma_{\tau+1}) := b_\tau^*(a;\gamma_\tau).$$

For ease of exposition, denote the constructed profile of bidding strategies by $\boldsymbol{b}^\dagger$. Further, denote player i's expected payoff and a period-τ player's under $\boldsymbol{b}^\dagger$ by $\Pi_{\tau+1}^{i\dagger}$ and $\Pi_\tau^\dagger$, respectively.

We first show that $\Pi_\tau^* < \Pi_\tau^\dagger$. Consider an indicative period-τ player j, $j \in \mathcal{N}_\tau$, whose ability we denote by a^j. Denote his interim expected payoff under $\boldsymbol{b}^*$ and that under $\boldsymbol{b}^\dagger$ by $\pi_\tau^*(a^j;\gamma_\tau)$ and $\pi_\tau^\dagger(a^j;\gamma_\tau)$, respectively.

If $b^j := b_\tau^*(a^j;\gamma_\tau) = 0$, then the period-$\tau$ player loses under both $\boldsymbol{b}^*$ and $\boldsymbol{b}^\dagger$, indicating $\pi_\tau^*(a^j;\gamma_\tau) = \pi_\tau^\dagger(a^j;\gamma_\tau) = 0$. If $b^j > 0$, then we have $b^j \geq \gamma_\tau$ and

$$\begin{aligned} \pi_\tau^*(a^j;\gamma_\tau) &= F^{n_\tau - 1}(a^j) H_\tau(b^j) - b^j/a^j \\ &= F^{n_\tau - 1}(a^j) H_{\tau+1}(b^j) F^{n_{\tau+1}}\left(a_{\tau+1}^*(b^j)\right) - b^j/a^j, \end{aligned}$$

where the second equality follows from (3). Similarly, we have

$$\pi_\tau^\dagger(a^j;\gamma_\tau) = F(\acute{a}^j) F^{n_\tau - 1}(a^j) H_{\tau+1}(b^j) F^{n_{\tau+1}-1}\left(a_{\tau+1}^*(b^j)\right) - b^j/a^j,$$

where $\acute{a}^j$ is defined as

$$\acute{a}^j := \begin{cases} a^*_\tau(\gamma_\tau), & \text{if } n_\tau = 1 \text{ and } a^*_\tau(\gamma_\tau) < a^j \leq a^{**}_\tau(\gamma_\tau), \\ a^j, & \text{otherwise,} \end{cases}$$

and satisfies $b^*_\tau(\acute{a}^j; \gamma_\tau) = b^*_\tau(a^j; \gamma_\tau) > 0$. It is straightforward to verify that

$$\pi^*_\tau(a^j; \gamma_\tau) < \pi^\dagger_\tau(a^j; \gamma_\tau) \iff F\left(a^*_{\tau+1}(b^j)\right) < F(\acute{a}^j) \iff a^*_{\tau+1}(b^j) < \acute{a}^j. \tag{18}$$

It follows immediately from $b^*_\tau(\acute{a}^j; \gamma_\tau) = b^*_\tau(a^j; \gamma_\tau) > 0$ that $\pi^*_\tau(\acute{a}^j; \gamma_\tau) > 0$, from which we can conclude

$$H_{\tau+1}(b^j) F^{n_{\tau+1}-1}\left(a^*_{\tau+1}(b^j)\right) - b^j/\acute{a}^j > 0. \tag{19}$$

Further, it follows from the definition of $a^*_{\tau+1}(\cdot)$ [see Equation (4)] that

$$\tilde{\pi}_{\tau+1}\left(b^j, a^*_{\tau+1}(b^j)\right) = H_{\tau+1}(b^j) F^{n_{\tau+1}-1}\left(a^*_{\tau+1}(b^j)\right) - b^j/a^*_{\tau+1}(b^j) \leq 0. \tag{20}$$

Combining (19) and (20) yields

$$a^*_{\tau+1}(b^j) \leq \frac{b^j}{H_{\tau+1}(b^j) F^{n_{\tau+1}-1}\left(a^*_{\tau+1}(b^j)\right)} < \acute{a}^j.$$

The above condition, together with (18), implies that $\pi^*_\tau(a^j; \gamma_\tau) < \pi^\dagger_\tau(a^j; \gamma_\tau)$ and

$$\Pi^*_\tau = \mathbb{E}\left[\pi^*_\tau(a^j; \gamma_\tau)\right] < \mathbb{E}\left[\pi^\dagger_\tau(a^j; \gamma_\tau)\right] = \Pi^\dagger_\tau, \tag{21}$$

where the expectation is taken with respect to both a^j and γ_τ.

To complete the proof, first note that $\Pi^*_{\tau+1} \geq \Pi^{i\dagger}_{\tau+1}$ by the definition of PBE. Moreover, it follows immediately from the construction $b^{i\dagger}_{\tau+1}(a; \gamma_{\tau+1}) := b^*_\tau(a; \gamma_\tau)$ that $\Pi^{i\dagger}_{\tau+1} \geq \Pi^\dagger_\tau$. These inequalities, together with (21), imply that $\Pi^*_{\tau+1} \geq \Pi^{i\dagger}_{\tau+1} \geq \Pi^\dagger_\tau > \Pi^*_\tau$. ■

Proof of Lemma 5. Fixing an arbitrary architecture $\boldsymbol{n} \equiv (n_1, \ldots, n_T)$, with $n_t \geq 1$ for all $t \in \{1, \ldots, T\}$ and $T \geq 2$, it suffices to show that

$$WP^*_1(a; \boldsymbol{n}) < F^{N-1}(a) < WP^*_T(a; \boldsymbol{n}), \text{ for almost every } a \in (0, 1).$$

We first prove that $WP_1^*(a;\boldsymbol{n}) < F^{N-1}(a)$ for all $a \in (0,1)$. Consider a representative period-1 player $i \in \mathcal{N}_1$. Recall $\gamma_1 \equiv 0$. The inequality obviously holds if $b^i := b_1^*(a^i;\gamma_1) = 0$, and it remains to consider the case where $b^i > 0$. Player i's expected equilibrium payoff is

$$\pi_1^*(a^i;\boldsymbol{n}) := WP_1^*(a^i;\boldsymbol{n}) - \frac{b^i}{a^i} > 0. \tag{22}$$

Fixing $\ell \in \{2,\ldots,T\}$, we have that

$$WP_1^*(a^i;\boldsymbol{n}) = F^{n_1-1}(a^i)\prod_{t=2}^{\ell} F^{n_t}\left(a_t^*(b^i)\right) H_\ell(b^i) \le F^{n_\ell-1}\left(a_\ell^*(b^i)\right) H_\ell(b^i), \tag{23}$$

where the equality follows from Lemma 1 and Lemma 2. Combining (22) and (23) yields that

$$F^{n_\ell-1}\left(a_\ell^*(b^i)\right) H_\ell(b^i) - \frac{b^i}{a^i} > 0. \tag{24}$$

From (3), (4), and (5), we have that $\tilde{\pi}_\ell\left(b^i, a_\ell^*(b^i)\right) \le 0$, which is equivalent to

$$F^{n_\ell-1}\left(a_\ell^*(b^i)\right) H_\ell(b^i) - \frac{b^i}{a_\ell^*(b^i)} \le 0. \tag{25}$$

Comparing (24) with (25) yields that $a^i > a_\ell^*(b^i)$ for all $\ell \in \{2,\ldots,T\}$, which in turn implies that

$$WP_1^*(a^i;\boldsymbol{n}) = F^{n_1-1}(a^i)\prod_{\ell=2}^{T} F^{n_\ell}\left(a_\ell^*(b^i)\right) < F^{n_1-1}(a^i)\prod_{\ell=2}^{T} F^{n_\ell}(a^i) = F^{N-1}(a^i).$$

Next, we prove that $F^{N-1}(a) < WP_T^*(a;\boldsymbol{n})$ for almost every $a \in (0,1)$. Fix $a \in (0,1)$, $\gamma \in [0,1]$, and (t,ℓ), with $1 \le t < \ell \le T$. Following a similar argument as in the previous analysis, we can show that if $b_t^*(a;\gamma) > 0$, then

$$a > a_\ell^*\left(b_t^*(a;\gamma)\right). \tag{26}$$

Consider a representative period-T player, $j \in \mathcal{N}_T$, with ability $a^j \in (0,1)$. By (26), we have $a^j > a_T^*\left(b_1^*(a^j;0)\right)$. Note that $a_T^*\left(b_1^*(a;0)\right)$ weakly increases with a and thus is continuous almost everywhere. We can focus on the case where $a_T^*\left(b_1^*(a;0)\right)$ is continuous at $a = a^j$. Therefore, there exists $\epsilon > 0$ such that $a^j > a_T^* \left(b_1^*(a^j + \epsilon; 0) \right)$. Let $\underline{a} := a^j + \epsilon$. It follows immediately that

$$\underline{a} > a^j > a_T^* \left(b_1^*(\underline{a}; 0) \right). \tag{27}$$

It is useful to prove the following intermediate result.

Lemma 10 *Fix an arbitrary architecture $\boldsymbol{n} \equiv (n_1, \ldots, n_T)$—with $n_t \geq 1$ for all $t \in \{1, \ldots, T\}$ and $T \geq 2$—and consider an indicative period-T player $j \in \mathcal{N}_T$. He wins the contest in the unique PBE if $\underline{a} > a^{j'}$ for all $j' \in \mathcal{N}_1$ and $a^j > a^{j'}$ for all $j' \in \mathcal{N} \setminus \left(\{j\} \cup \mathcal{N}_1 \right)$.*

Proof. Fix an ability profile $\boldsymbol{a} := (a^1, \ldots, a^N)$ such that $\underline{a} > a^{j'}$ for all $j' \in \mathcal{N}_1$ and $a^j > a^{j'}$ for all $j' \in \mathcal{N} \setminus \left(\{j\} \cup \mathcal{N}_1 \right)$. Let $\imath$ denote the index of the provisional winner by the end of period $T-1$ given that all players use the equilibrium strategy and $\mathfrak{t}$ the period he moves. Then $\gamma_T = b_{\mathfrak{t}}^*(a^\imath; \gamma_{\mathfrak{t}})$. Evidently, the lemma holds if $b_{\mathfrak{t}}^*(a^\imath; \gamma_{\mathfrak{t}}) = 0$ and it remains to consider the situation where $b_{\mathfrak{t}}^*(a^\imath; \gamma_{\mathfrak{t}}) > 0$. We consider the following two cases:

(a) Suppose $\mathfrak{t} \geq 2$. Then we have

$$a^j > a^\imath > a_T^* \left(b_{\mathfrak{t}}^*(a^\imath; \gamma_{\mathfrak{t}}) \right) = a_T^* \left(\gamma_T \right),$$

where the first inequality follows from the postulated $\imath \notin \mathcal{N}_1$ and the second inequality from (26).

(b) Suppose $\mathfrak{t} = 1$. Then we have

$$a^j > a_T^* \left(b_1^*(\underline{a}; 0) \right) \geq a_T^* \left(b_1^*(a^\imath; 0) \right) = a_T^* \left(\gamma_T \right),$$

where the first inequality follows from (27).

To summarize, if $b_{\mathfrak{t}}^*(a^\imath; \gamma_{\mathfrak{t}}) > 0$, then $a^j > a_T^* \left(\gamma_T \right)$, indicating that player j places a positive amount of bid in equilibrium. Therefore, he outbids all players up to period $T - 1$. Next, note that $a^j > a^{j'}$ for all $j' \in \mathcal{N}_T$ by assumption; together with Lemma 1(iii), player j outbids all of his contemporaneous rivals in period T and wins the contest. ■

By Lemma 10, player j's expected winning probability, $WP_T^*(a^j;\boldsymbol{n})$, can be bounded from below by

$$WP_T^*(a^j;\boldsymbol{n}) \geq F^{n_1}(\underline{a})F^{\left(\sum_{t=2}^{T} n_t\right)-1}(a^j) > F^{N-1}(a^j).$$

This concludes the proof. ■

Proof of Lemma 6. Fix an indicative player $i \in \mathcal{N}$ and consider the following two cases:

(a) All other players choose to move in the last period. If player i chooses to move in period 1, the resultant contest architecture is $\hat{\boldsymbol{n}} = (1, N-1)$ and his equilibrium payoff in this subgame is $\Pi_1^*(\hat{\boldsymbol{n}})$. If player i chooses to move in the last period, all players move simultaneously in the second-stage game and his equilibrium payoff amounts to Π^{SIM}. By (9), we have $\Pi_1^*(\hat{\boldsymbol{n}}) < \Pi^{SIM}$.

(b) At least one player chooses not to move in the last period. Denote the resultant contest architecture when player i chooses to move in period 1 and that when he chooses to move in period L by $\hat{\boldsymbol{n}}'$ and $\hat{\boldsymbol{n}}''$, respectively. Note that $\hat{\boldsymbol{n}}'$ degenerates to a simultaneous contest if all other players choose to move in period 1 and a sequential one otherwise. By (9), we have $\Pi_1^*(\hat{\boldsymbol{n}}') \leq \Pi^{SIM}$. Next, note that $\hat{\boldsymbol{n}}''$ is a sequential contest. Denote the number of periods with at least one player by $\hat{T}''$. Clearly, player i's equilibrium payoff under $\hat{\boldsymbol{n}}''$ is $\Pi_{\hat{T}''}^*(\hat{\boldsymbol{n}}'')$. Again, we can obtain $\Pi^{SIM} < \Pi_{\hat{T}''}^*(\hat{\boldsymbol{n}}'')$ from (9). Therefore, we have $\Pi_1^*(\hat{\boldsymbol{n}}') \leq \Pi^{SIM} < \Pi_{\hat{T}''}^*(\hat{\boldsymbol{n}}'')$.

To summarize, moving in period L yields a strictly higher payoff to player i than moving in period 1. This concludes the proof. ■

Proof of Theorem 3. The theorem follows immediately from Lemma 6. ■

Sequential Contests

ONLINE APPENDIX

(Not Intended for Publication)

Shanglyu Deng* Qiang Fu† Zenan Wu‡ Yuxuan Zhu§

In this Online Appendix, we collect the analyses and discussions omitted from the main text that complement our baseline results. Online Appendix A presents the detailed analysis of an extension to our baseline model—i.e., the case with symmetric tie-breaking rule—which demonstrates the robustness of our main results. Online Appendix B presents the analysis under concave/convex ability distributions omitted in the main body of the text. Online Appendix C provides detailed derivation of players' expected efforts in the proof of Proposition 2.[1]

A Symmetric Tie-breaking Rule

Recall the symmetric tie-breaking rule (10) described in Section 5.1. Fix a player $i \in \mathcal{N}_t$ and an effort profile $(b^j)_{j\in\cup_{k=1}^{t-1}\mathcal{N}_k}$. With a symmetric tie-breaking rule, the player's strategy depends not only on his ability and the highest prior effort $\gamma_t \equiv \max_{j\in\cup_{k=1}^{t-1}\mathcal{N}_k}\{b^j\}$, but also on the number of players who chose γ_t, which we denote by z_t. For notational convenience, denote, respectively, by $b^{i,s}(a;\gamma_t,z_t)$ a player i's strategy, with $i \in \mathcal{N}_t$, and $\pi^{i,s}(b,a;\gamma_t,z_t)$ his interim expected payoff when choosing effort b for a given ability a and his opponents' strategy profile $\left\{b^{j,s}(a;\gamma_t,z_t)\right\}_{j\in\mathcal{N}_t,t\in\mathcal{T},j\neq i}$.

*Department of Economics, University of Maryland, 3114 Tydings Hall, 7343 Preinkert Dr., College Park, MD 20742. Email: sdeng1@umd.edu

†Department of Strategy and Policy, National University of Singapore, 15 Kent Ridge Drive, Singapore, 119245. Email: bizfq@nus.edu.sg

‡School of Economics, Peking University, Beijing, China, 100871. Email: zenan@pku.edu.cn

§School of Economics, Peking University, Beijing, China, 100871. Email: zhuyuxuan@pku.edu.cn

[1]This note is not self-contained; it is the online appendix of the paper "Sequential Contests."

Definition 1 ***(ϵ-equilibrium)*** *Fix $\epsilon > 0$. A strategy profile $\boldsymbol{b}^s \equiv \left\{b^{i,s}(a;\gamma_t, z_t)\right\}_{i\in\mathcal{N}_t, t\in\mathcal{T}}$ constitutes an ϵ-equilibrium of a sequential contest $\boldsymbol{n} \equiv (n_1, \ldots, n_T)$ under a symmetric tie-breaking rule* (10)*, if for all $i \in \mathcal{N}_t$, with $t \in \mathcal{T}$, and any history associated with $\gamma_t \in [0,1]$ and $z_t \geq 1$,*

$$\pi^{i,s}(b^{i,s}\left(a;\gamma_t, z_t), a; \gamma_t, z_t\right) \geq \pi^{i,s}\left(b', a; \gamma_t, z_t\right) - \epsilon,$$

for all $a \in (0,1]$ and $b' \geq 0$.

Definition 1 requires that given any history summarized by (γ_t, z_t), a player's equilibrium strategy be an ϵ-best response after his ability realizes.

Proposition A1 ***(ϵ-equilibrium in Sequential Contests with a Symmetric Tie-Breaking Rule)*** *Consider a sequential contest $\boldsymbol{n} \equiv (n_1, \ldots, n_T)$. Fix an arbitrary $\epsilon > 0$ and let*

$$b^{i,s}(a;\gamma_t, z_t) := \begin{cases} b_t^*(a;\gamma_t) + a\epsilon \mathbb{1}_{\{a_t^*(\gamma_t) < a \leq a_t^{**}(\gamma_t)\}}, & \text{if } n_t = 1, \\ b_t^*(a;\gamma_t), & \text{if } n_t \geq 2, \end{cases} \tag{A1}$$

for all $i \in \mathcal{N}_t$, with $t \in \mathcal{T}$. The strategy profile $\boldsymbol{b}^s \equiv \left\{b^{i,s}(a;\gamma_t, z_t)\right\}_{i\in\mathcal{N}_t, t\in\mathcal{T}}$ constitutes a symmetric ϵ-equilibrium of the contest game under a symmetric tie-breaking rule as specified in (10).

Proof. Consider the strategy profile $\boldsymbol{b}^s \equiv \left\{b^{i,s}(a;\gamma_t, z_t)\right\}_{i\in\mathcal{N}_t, t\in\mathcal{T}}$ as defined in (A1). Fix $t \in \mathcal{T}$ and consider an indicative player $i \in \mathcal{N}_t$. Note that a tie occurs with zero probability under $\boldsymbol{b}^s$. We consider the following two cases:

(a) Suppose that $n_t = 1$. Then player i's interim expected payoff of bidding b^i under (10), given that all other players bid according to $\boldsymbol{b}^s$, is

$$\pi^{i,s}(b^i, a^i; \gamma_t, z_t) = \begin{cases} -b^i/a^i, & \text{if } b^i < \gamma_t, \\ \frac{1}{z_t+1}\Pr\left(a^l \leq a_k^*(b^i), \forall k \in \{t+1, \ldots, T\}, \forall l \in \mathcal{N}_k\right) - b^i/a^i, & \text{if } b^i = \gamma_t, \\ \Pr\left(a^l \leq a_k^*(b^i), \forall k \in \{t+1, \ldots, T\}, \forall l \in \mathcal{N}_k\right) - b^i/a^i, & \text{if } b^i > \gamma_t. \end{cases}$$

With slight abuse of notation, denote player i's interim expected payoff under (1) and $\boldsymbol{b}^* \equiv \left\{b_t^*(a;\gamma_t)\right\}_{t\in\mathcal{T}}$ by $\pi^i(b^i,a^i;\gamma_t)$. Note that if $b^i \neq \gamma_t$, $\pi^{i,s}(b^i,a^i;\gamma_t,z_t) = \pi^i(b^i,a^i;\gamma_t)$; if $b^i=\gamma_t$, $\pi^{i,s}(b^i,a^i;\gamma_t,z_t) \le \pi^i(b^i,a^i;\gamma_t)$.

We consider the following two subcases:

(i) Suppose that $b_t^*(a^i;\gamma_t)\neq\gamma_t$, which occurs when $a^i \le a_t^*(\gamma_t)$ or $a^i > a_t^{**}(\gamma_t)$. By (A1), we have $b^{i,s}(a^i;\gamma_t,z_t) = b_t^*(a^i;\gamma_t)$. Note that $b_t^*(a^i;\gamma_t)$ solves $\max_{b^i\ge 0}\pi^i(b^i,a^i;\gamma_t)$ in the unique symmetric PBE under (1). Moreover, it can be verified that $\pi^{i,s}(b^i,a^i;\gamma_t,z_t) = \pi^i(b^i,a^i;\gamma_t)$ except for the case of $b^i=\gamma_t$. Therefore, $b_t^*(a^i;\gamma_t)$ also solves $\max_{b^i\ge 0}\pi^{i,s}(b^i,a^i;\gamma_t,z_t)$ and is thus optimal for player i.

(ii) Suppose that $b_t^*(a^i;\gamma_t)=\gamma_t$, which occurs when $a_t^*(\gamma_t) < a^i \le a_t^{**}(\gamma_t)$. It follows from (A1) that $b^{i,s}(a^i;\gamma_t,z_t) = b_t^*(a^i;\gamma_t)+a^i\epsilon = \gamma_t + a^i\epsilon$. Note that for all $b^i\ge 0$, we have that

$$\pi^{i,s}(b^i,a^i;\gamma_t,z_t) \le \pi^i(b^i,a^i;\gamma_t) \le \pi^i(b_t^*(a^i;\gamma_t),a^i;\gamma_t) = \pi^i(\gamma_t,a^i;\gamma_t).$$

That is, player i's interim expected payoff is bounded from above by $\pi^i(\gamma_t,a^i;\gamma_t)$. Moreover, bidding $\gamma_t+a^i\epsilon$ would result in an expected payoff of $\pi^i(\gamma_t+a^i\epsilon,a^i;\gamma_t)$, which is no less than $\pi^i(\gamma_t,a^i;\gamma_t)-\epsilon$. Therefore, bidding $b^{i,s}(a;\gamma_t,z_t)$ is ϵ-optimal for player i.

(b) Suppose that $n_t \ge 2$. Then

$$\pi^{i,s}(b^i,a^i;\gamma_t,z_t) = \begin{cases} -b^i/a^i, & \text{if } b^i<\gamma_t,\\ \frac{1}{z_t+1}\Pr\begin{pmatrix} a^l \le a_k^*(b^i), \forall k\in\{t,\dots,T\}, \\ \forall l \in \mathcal{N}_k\setminus\{i\}\end{pmatrix} - b^i/a^i, & \text{if } b^i=\gamma_t,\\ \Pr\begin{pmatrix} a^j \le (b_t^*)^{-1}(b^i;\gamma_t), \forall j\in\mathcal{N}_t\setminus\{i\}, \text{ and} \\ a^l \le a_k^*(b^i), \forall k\in\{t+1,\dots,T\}, \forall l\in\mathcal{N}_k\end{pmatrix} - b^i/a^i, & \text{if } b^i>\gamma_t.\end{cases}$$

Again, it can be verified that $\pi^{i,s}(b^i,a^i;\gamma_t,z_t) = \pi^i(b^i,a^i;\gamma_t)$ if $b_i\neq\gamma_t$ and $\pi^{i,s}(b^i,a^i;\gamma_t,z_t) \le \pi^i(b^i,a^i;\gamma_t)$ if $b^i=\gamma_t$. By (A1), we have $b^{i,s}(a^i;\gamma_t,z_t) =$

$b_t^*(a^i;\gamma_t)$. Moreover, it follows from Lemma 1(ii) and (iii) that $b_t^*(a^i;\gamma_t) = 0$ for $a^i \leq a_t^*(\gamma_t)$ and $b_t^*(a^i;\gamma_t) > \gamma_t$ for $a^i > a_t^*(\gamma_t)$. Therefore, $b_t^*(a^i;\gamma_t) \neq \gamma_t$ for all $a^i \in (0,1)$. Further, for all $a^i \in (0,1)$ and $b^i \geq 0$, we have that

$$\pi^{i,s}\left(b^i, a^i; \gamma_t, z_t\right) \leq \pi^i\left(b^i, a^i; \gamma_t\right) \leq \pi^i\left(b_t^*(a^i;\gamma_t), a^i; \gamma_t\right) = \pi^{i,s}\left(b^{i,s}(a^i;\gamma_t, z_t), a^i; \gamma_t, z_t\right).$$

Therefore, $b^{i,s}(a^i;\gamma_t,z_t)$ solves $\max_{b^i\geq 0} \pi^{i,s}(b^i, a^i;\gamma_t,z_t)$ and is optimal for player i.

To summarize, $\pi^{i,s}(b^{i,s}(a;\gamma_t,z_t), a;\gamma_t,z_t) \geq \pi^{i,s}(b', a;\gamma_t,z_t) - \epsilon$ for all $a \in (0,1]$ and $b' \geq 0$, and thus $\boldsymbol{b}^s \equiv \left(b^{i,s}(a;\gamma_t,z_t)\right)_{i\in\mathcal{N}_t, t\in\mathcal{T}}$ constitutes a symmetric ϵ-equilibrium of the contest game under (10). This concludes the proof. ■

The equilibrium construction in Proposition A1 is intuitive. For periods that have multiple players—i.e., $n_t \geq 2$—the alternative tie-breaking rule does not play a critical role because players, by Theorem 1(ii), would never match the highest prior effort. As a result, players' strategies do not depart from the counterpart in the PBE. For periods that have a unique player—i.e., $n_t = 1$—we slightly increase a player's effort whenever he wants to match the highest effort from previous periods under (1) to ensure that he can beat all his predecessors and will not share the prize with them.

As discussed in Remark 1, the rule (1) literally "favors" later movers, which may cast doubt on the robustness of our result, as this sparks the concern that the later-mover advantage established in Theorem 2 is an artifact of the asymmetric tie-breaking rule. By Proposition A1, the ϵ-equilibrium we construct under (10) largely reinstates the prediction of Theorem 1 for sufficiently small ϵ. As a result, all our results obtained under the original tie-breaking rule remain qualitatively intact. Let $\Pi_t^{*,s}$ denote a period-t player's equilibrium expected payoff in the symmetric ϵ-equilibrium described in Proposition A1, with $t \in \mathcal{T}$. The following can formally be stated.

Proposition A2 ***(Later-mover Advantage with a Symmetric Tie-Breaking Rule)*** *Consider a sequential contest* $\boldsymbol{n} \equiv (n_1, \dots, n_T)$ *under a symmetric tie-breaking rule as specified in* (10). *For sufficiently small* $\epsilon > 0$, *a player's expected payoff is higher than those of all earlier movers in the symmetric* ϵ*-equilibrium* $\boldsymbol{b}^s \equiv$

$\left\{b^{i,s}(a;\gamma_t,z_t)\right\}_{i\in\mathcal{N}_t,t\in\mathcal{T}}$, *where* $b^{i,s}(a;\gamma_t,z_t)$ *is defined in* (A1)*. Formally,* $0=\Pi_1^{*,s}=\cdots=\Pi_{t_0}^{*,s}<\Pi_{t_0+1}^{*,s}<\cdots<\Pi_T^{*,s}$.

The following result in parallel with Theorem 3 can be established.

Proposition A3 ***(Unique Equilibrium with Endogenous Moving Order and a Symmetric Tie-breaking rule)*** *Suppose that each player picks one from* $L\geq 2$ *available periods before he learns his realized type and acts accordingly. Further, fix sufficiently small* $\epsilon>0$ *and an arbitrary contest architecture, players adopt the equilibrium strategies as described in* (A1)*. Then all players choosing to move in the last period constitutes a Nash equilibrium of the first-stage game that uniquely survives IESDS.*

B Analysis under Concave/Convex Ability Distributions

In this section, we adapt our results of Theorem 1 and Proposition 1 to derive the respective equilibria under concave and convex ability distributions. We begin with the case of a concave ability distribution.

Assumption A1 *The ability distribution function* $F(\cdot)$ *is continuous, twice differentiable, strictly concave, and satisfies* $\lim_{a\searrow 0}[f(a)a]=0$*.*

Recall that we establish $t_0=0$ under Assumption A1 in Section 3: That is, all players will exert positive efforts with positive probabilities. The following result—in parallel with Theorem 1—can be obtained.

Proposition A4 ***(Concave Ability Distribution and Equilibrium Bidding Strategy)*** *Suppose that Assumption A1 is satisfied and consider a sequential contest* $\boldsymbol{n}\equiv(n_1,\ldots,n_T)$*. The unique symmetric PBE of the contest game,* $\{b_t^*(a;\gamma_t)\}_{t=1}^T$*, is characterized as follows:*

(i) If $n_t = 1$, then

$$b_t^*(a;\gamma_t) = \begin{cases} 0, & \text{if } a \le a_t^*(\gamma_t), \\ \gamma_t, & \text{if } a_t^*(\gamma_t) < a \le a_t^{**}(\gamma_t), \\ h_t^{-1}\left(\frac{1}{a}\right), & \text{if } a > a_t^{**}(\gamma_t), \end{cases}$$

where $a_t^*(\gamma_t) = \gamma_t / H_t(\gamma_t)$, and $a_t^{**}(\gamma_t) = 1/h_t(\gamma_t)$.

(ii) If $n_t \ge 2$, then for $a \le a_t^*(\gamma_t)$, $b_t^*(a;\gamma_t) = 0$; for $a > a_t^*(\gamma_t)$, $b_t^*(a;\gamma_t)$ solves the differential equation (8) with the initial condition $b_t^*(a_t^*(\gamma_t)+0;\gamma_t) = \gamma_t$.

Proof. It is useful to prove the following intermediate result.

Lemma A1 *Suppose that Assumption A1 is satisfied. Then $H_t(b)$ is continuous, twice differentiable, weakly increasing, strictly concave on $[0,1]$, and satisfies $\lim_{b\searrow 0}[bh_t(b)] = 0$ for all $t \le T-1$.*

Proof. We prove the lemma by induction.

Base case: Consider the penultimate period, i.e., $t = T-1$. Recall from the proof of Lemma 7 that $G_n(\cdot)$ is defined as the inverse function of $aF^{n-1}(a)$ for an arbitrary positive integer $n \in \mathbb{N}_+$. It follows from (4), (5), and Assumption A1 that $a_T^*(\gamma) = G_{n_T}(\gamma)$; together with (3), we have $H_{T-1}(b) = F^{n_{T-1}}\left(G_{n_{T-1}}(b)\right)$. Evidently, $H_{T-1}(b)$ is continuous, twice differentiable, and weakly increasing, and it remains to show that $H_{T-1}(b)$ is strictly concave on $[0,1]$ and satisfies $\lim_{b\searrow 0}[bh_{T-1}(b)] = 0$.

For notational convenience, define $\widehat{G}_t(b) := F^{n_t}\left(G_{n_t}(b)\right)$, $\forall t \in \{2,\ldots,T\}$. It follows from Assumption A1 that $\widehat{G}_t(b)$ is continuous, twice differentiable, and weakly increasing. We first show that $\widehat{G}_t(b)$ is strictly concave. Carrying out the algebra, we have that

$$\widehat{G}_t'(b) = \frac{n_t F\left(G_{n_t}(b)\right) f\left(G_{n_t}(b)\right)}{F\left(G_{n_t}(b)\right) + (n_t-1)G_{n_t}(b) f\left(G_{n_t}(b)\right)}.$$

Because $G_{n_t}(b)$ is strictly increasing in b, it suffices to show that for all $x \in (0,1)$,

$$\frac{d}{dx}\frac{F(x)f(x)}{F(x)+(n_t-1)xf(x)} < 0 \iff \frac{d}{dx}\left[\frac{1}{f(x)} + (n_t-1)\frac{x}{F(x)}\right] > 0.$$

The strict concavity of $F(x)$ implies that both $\frac{1}{f(x)}$ and $\frac{x}{F(x)}$ are strictly increasing in x. Therefore, $\widehat{G}_t(b)$ is strictly concave in b.

Next, we show that $\lim_{b\searrow 0}[b\widehat{G}_t'(b)] = 0$. The analysis is straightforward for $n_t = 1$, and it suffices to consider the case of $n_t \geq 2$. Carrying out the algebra, we have that

$$\lim_{b\searrow 0} \frac{b}{G_{n_t}(b)} = \lim_{b\searrow 0} \left[F^{n_t-1}\left(G_{n_t}(b)\right)\right] = 0,$$

and

$$\lim_{b\searrow 0} \left[G_{n_t}(b)\widehat{G}_t'(b)\right] = \lim_{b\searrow 0} \frac{n_t bF(b)f(b)}{F(b) + (n_t-1)bf(b)} = \lim_{b\searrow 0} \frac{n_t}{\frac{1}{bf(b)} + (n_t-1)\frac{1}{F(b)}} = 0.$$

Therefore,

$$\lim_{b\searrow 0}[b\widehat{G}_t'(b)] = \lim_{b\searrow 0} \frac{b}{G_{n_t}(b)} \times \lim_{b\searrow 0}[G_{n_t}(b)\widehat{G}_t'(b)] = 0.$$

Note that $H_{T-1}(b) = \widehat{G}_{T-1}(b)$. The above analyses indicate that $H_{T-1}(b) = \widehat{G}_{T-1}(b)$ is strictly concave and $\lim_{b\searrow 0}[bh_{T-1}(b)] = \lim_{b\searrow 0}[b\widehat{G}_{T-1}'(b)] = 0$.

Inductive step: Suppose that $H_t(b)$ is continuous, twice differentiable, weakly increasing, strictly concave on $[0,1]$, and satisfies $\lim_{b\searrow 0}[bh_t(b)] = 0$ for some $t \leq T-1$. Next, we show that $H_{t-1}(b)$ has the same properties. Before we proceed, note that $H_t(0) = 0$ for all $t \leq T-1$.

It is straightforward to verify that $H_{t-1}(b)$ is continuous and twice differentiable from its definition. Further, it can be verified from the concavity of $H_t(b)$ that $b/H_t(b)$ is strictly increasing in b. Because $H_t(b)$, $\widehat{G}_t(b)$, and $b/H_t(b)$ are all increasing, $H_{t-1}(b)$ is an increasing function.

Next, we prove the strict concavity of $H_{t-1}(b)$. Carrying out the algebra, we can obtain that

$$h_{t-1}(b) = h_t(b)\left[\widehat{G}_t\left(\frac{b}{H_t(b)}\right) - \widehat{G}_t'\left(\frac{b}{H_t(b)}\right)\frac{b}{H_t(b)}\right] + \widehat{G}_t'\left(\frac{b}{H_t(b)}\right),$$

and

$$h'_{t-1}(b) = \underbrace{h'_t(b)}_{<0} \underbrace{\left[\widehat{G}_t\left(\frac{b}{H_t(b)}\right) - \widehat{G}'_t\left(\frac{b}{H_t(b)}\right)\frac{b}{H_t(b)}\right]}_{>0} + \underbrace{\widehat{G}''_t\left(\frac{b}{H_t(b)}\right)}_{<0} \underbrace{\left[\frac{H_t(b) - bh_t(b)}{H_t(b)}\right]}_{>0} \underbrace{\left[\frac{b}{H_t(b)}\right]'}_{>0}.$$

From the previous analysis, $\widehat{G}_t(b)$ is strictly concave and $\lim_{b\searrow 0}[b\widehat{G}'_t(b)] = 0$, which implies that $\widehat{G}''_t\left(\frac{b}{H_t(b)}\right) < 0$ and

$$\widehat{G}_t\left(\frac{b}{H_t(b)}\right) - \widehat{G}'_t\left(\frac{b}{H_t(b)}\right)\frac{b}{H_t(b)} > 0, \forall b \in (0, 1).$$

The monotonicity of $b/H_t(b)$ implies that $\frac{H_t(b)-bh_t(b)}{H_t(b)} > 0$ and $\left[\frac{b}{H_t(b)}\right]' > 0$. Further, the strict concavity of $H_t(b) < 0$ implies that $h'_t(b) < 0$. Therefore, $h'_{t-1}(b) < 0$ and thus $H_{t-1}(b)$ is strictly concave.

Finally, we have

$$\lim_{b\searrow 0}\left[bh_{t-1}(b)\right] = \lim_{b\searrow 0}\left\{bh_t(b)\left[\widehat{G}_t\left(\frac{b}{H_t(b)}\right) - \widehat{G}'_t\left(\frac{b}{H_t(b)}\right)\frac{b}{H_t(b)}\right] + H_t(b)\frac{b}{H_t(b)}\widehat{G}'_t\left(\frac{b}{H_t(b)}\right)\right\}. \tag{A2}$$

Note that

$$0 \le \widehat{G}_t\left(\frac{b}{H_t(b)}\right) - \widehat{G}'_t\left(\frac{b}{H_t(b)}\right)\frac{b}{H_t(b)} \le \widehat{G}_t\left(\frac{b}{H_t(b)}\right) \le 1, \tag{A3}$$

$$0 \le \frac{b}{H_t(b)}\widehat{G}'_t\left(\frac{b}{H_t(b)}\right) \le \widehat{G}_t\left(\frac{b}{H_t(b)}\right) \le 1, \tag{A4}$$

Equations (A2) to (A4), together with $H_t(0) = 0$ and the postulated $\lim_{b\searrow 0}[bh_t(b)] = 0$, imply that $\lim_{b\searrow 0}[bh_{t-1}(b)] = 0$. This completes the inductive step.

Conclusion: By the principle of induction, $H_t(b)$ is continuous, twice differentiable, weakly increasing, strictly concave on $[0, 1]$, and satisfies $\lim_{b\searrow 0}[bh_t(b)] = 0$ for all $t \le T - 1$. This concludes the proof. ■

Now we can prove Proposition A4. By Lemma A1, $H_t(b)$ is concave. Therefore, $\tilde{\pi}_t(b, a)$ is concave in b and $S_t(a; \gamma)$ reduces to a single interval. The proposition then follows immediately from Theorem 1. ■

As mentioned above, there is no jump in a player's equilibrium bidding strategy once he decides to exert a positive effort: It is straightforward to see that the set $S_t(a;\gamma_t)$ degenerates to a single interval for all $a \in (a_t^*(\gamma_t), 1]$, as Figure 2(a) depicts. To understand the logic, let us consider a two-player sequential contest $(1, 1)$ and focus on the first mover. The second mover would simply match the first mover's effort as long as his ability is above the early effort and remain inactive otherwise. Thus, the first mover's expected payoff of placing a bid b is $F(b) - b/a$. A concave ability distribution ensures that his expected payoff function is concave in the effort, which implies that he will *gradually* increase his effort as his ability increases.

We then proceed to the case with convex ability distribution.

Assumption A2 *The ability distribution function $F(\cdot)$ is continuous, twice differentiable, and weakly convex.*

The following result is obtained, which characterizes the symmetric PBE in the sequential contest.

Proposition A5 ***(Convex Ability Distribution and Equilibrium Bidding Strategy)*** *Suppose that Assumption A2 is satisfied and consider a sequential contest $\boldsymbol{n} \equiv (n_1, \ldots, n_T)$. In the unique symmetric PBE, $b_t^*(a;\gamma_t) = 0$ for all $a \in (0, 1]$, $\gamma_t \in [0, 1]$ and $t \in \{1, \ldots, T-1\}$, and*

$$b_T^*(a;\gamma_T) = \begin{cases} 0, & \text{if } a \leq G_{n_T}(\gamma_T), \\ \gamma_T + \int_{G_{n_T}(\gamma_T)}^{a} (n_T - 1)sf(s)F^{n_T-2}(s)ds, & \text{if } a > G_{n_T}(\gamma_T). \end{cases}$$

As stated in Section 3.2, all players who arrive prior to period T stay inactive, so the contest boils down to a simultaneous all-pay auction with n_T players, i.e., those who arrive in the last period. This observation stands in sharp contrast to that in the case with a concave ability distribution function, which attests to the nontrivial role played by the curvature of distribution function in shaping equilibrium behavior. We continue with the two-player sequential contest $(1, 1)$ to elaborate on the intuition. Recall that the first mover's expected payoff of placing a bid b is $F(b) - b/a$, which is convex with a convex ability distribution. Thus, the payoff can be maximized by either a zero or a sufficiently large effort. Further, a convex ability distribution implies more intense future competition, as high-ability players are likely to emerge in later periods. This discourages players' investment to avoid futile effort and the resultant loss. As a result, the players except those from the last period choose to drop out of the competition.

C Derivation of Players' Expected Efforts in the Proof of Proposition 2

We first derive $\mathbb{E}[b_t^*|\gamma_t]$. According to the equilibrium bidding strategy characterized in Proposition A4(i), we have that

$$\begin{aligned}
\mathbb{E}[b_t^*|\gamma_t] &= \gamma_t \left\{ F\left[\frac{1}{h_t(\gamma_t)}\right] - F\left[\frac{\gamma_t}{H_t(\gamma_t)}\right] \right\} + \int_{h_t(1/\gamma_t)}^{1} \frac{1}{h_t^{-1}(a)} dF(a) \\
&= \gamma_t \left\{ F\left[\frac{1}{h_t(\gamma_t)}\right] - F\left[\frac{\gamma_t}{H_t(\gamma_t)}\right] \right\} + \int_{\gamma_t}^{\bar{\gamma}_t} x dF\left[\frac{1}{h_t(x)}\right] \\
&= \begin{cases} \bar{\gamma}_t - \gamma_t F\left[\frac{\gamma_t}{H_t(\gamma_t)}\right] - \int_{\gamma_t}^{\bar{\gamma}_t} F\left[\frac{1}{h_t(x)}\right] dx, & \text{if } \gamma_t \leq \bar{\gamma}_t, \\ \gamma_t \left\{ 1 - F\left[\frac{\gamma_t}{H_t(\gamma_t)}\right] \right\}, & \text{if } \gamma_t > \bar{\gamma}_t, \end{cases}
\end{aligned} \tag{A5}$$

where $\bar{\gamma}_t := h_t^{-1}(1)$ for $t < T$ and $\bar{\gamma}_T := 0$. Given that $F(a) = a^r$, $r \in (0, 1)$, it can be verified that $H_t(b) = b^{1-(1-r)^{T-t}}$, which implies that

$$F\left[\frac{\gamma}{H_t(\gamma)}\right] = \gamma^{r(1-r)^{T-t}}, \tag{A6}$$

and

$$F\left[\frac{1}{h_t(\gamma)}\right] = \min\left\{1, \frac{\gamma^{r(1-r)^{T-t}}}{[1-(1-r)^{T-t}]^r}\right\}. \tag{A7}$$

Combining (A5), (A6), and (A7), for $\gamma \le \bar{\gamma}_t$, we have

$$\begin{aligned}\mathbb{E}[b_t^*|\gamma_t = \gamma] &= \bar{\gamma}_t - \gamma^{1+r(1-r)^{T-t}} - \frac{\bar{\gamma}_t^{1+r(1-r)^{T-t}} - \gamma^{1+r(1-r)^{T-t}}}{[1+r(1-r)^{T-t}][1-(1-r)^{T-t}]^r} \\ &= \frac{r(1-r)^{T-t}}{1+r(1-r)^{T-t}}\bar{\gamma}_t + \frac{1-[1+r(1-r)^{T-t}][1-(1-r)^{T-t}]^r}{[1+r(1-r)^{T-t}][1-(1-r)^{T-t}]^r}\gamma^{1+r(1-r)^{T-t}},\end{aligned}$$

where the second equality follows from $\frac{\bar{\gamma}_t^{r(1-r)^{T-t}}}{[1-(1-r)^{T-t}]^r} = 1$; for $\gamma > \bar{\gamma}_t$, we have

$$\mathbb{E}[b_t^*|\gamma_t = \gamma] = \gamma\left[1 - \gamma^{r(1-r)^{T-t}}\right].$$

Next, we derive the CDF of γ_t. Note that

$$\gamma_t = \max_{\tau<t}\left\{h_\tau^{-1}\left(\frac{1}{a_\tau}\right)\right\}, \text{ for } t \ge 2. \tag{A8}$$

The above expression can be obtained by induction. Evidently, (A8) holds for $t = 2$. Suppose that (A8) holds for $t = \ell \ge 2$. It suffices to show that (A8) holds for $t = \ell+1$. Consider the period-ℓ player. From Proposition A4(i), we have that

$$\gamma_{\ell+1} = \begin{cases}\gamma_\ell, & \text{if } a_\ell \le 1/h_\ell(\gamma_\ell), \\ h_\ell^{-1}\left(\frac{1}{a_\ell}\right), & \text{if } a_\ell > 1/h_\ell(\gamma_\ell).\end{cases}$$

Note that $h_\ell^{-1}\left(\frac{1}{a_\ell}\right)$ is increasing in a_ℓ. The above expression can be rewritten as

$$\gamma_{\ell+1} = \max\left\{\gamma_\ell, h_\ell^{-1}\left(\frac{1}{a_\ell}\right)\right\};$$

together with the postulated $\gamma_\ell = \max_{\tau<\ell}\left\{h_\tau^{-1}\left(\frac{1}{a_\tau}\right)\right\}$, we can obtain

$$\gamma_{\ell+1} = \max\left\{\max_{\tau<\ell}\left\{h_\tau^{-1}\left(\frac{1}{a_\tau}\right)\right\}, h_\ell^{-1}\left(\frac{1}{a_\ell}\right)\right\} = \max_{\tau<\ell+1}\left\{h_\tau^{-1}\left(\frac{1}{a_\tau}\right)\right\}.$$

By (A8), the CDF of γ_t, with $t \geq 2$, is given by

$$\Gamma_t(\gamma) := \Pi_{\tau<t} F\left[\frac{1}{h_\tau(\gamma)}\right].$$

Note that $\bar{\gamma}_{\tau+1} < \bar{\gamma}_\tau$ for $1 \leq \tau \leq T-2$. Therefore, for $t \geq 2$, we have that

$$\Gamma_t(\gamma) = \begin{cases} \Pi_{\tau<t} F\left[\frac{1}{h_\tau(\gamma)}\right] = \frac{\gamma^{r\sum_{\tau<t}(1-r)^{T-\tau}}}{\Pi_{\tau<t}[1-(1-r)^{T-\tau}]^r}, & \text{if } \gamma \leq \bar{\gamma}_t, \\ \Pi_{\tau\leq k} F\left[\frac{1}{h_\tau(\gamma)}\right] = \frac{\gamma^{r\sum_{\tau\leq k}(1-r)^{T-\tau}}}{\Pi_{\tau\leq k}[1-(1-r)^{T-\tau}]^r}, & \text{if } \bar{\gamma}_{k+1} < \gamma \leq \bar{\gamma}_k, \text{ for } 1 \leq k \leq t-1. \end{cases} \tag{A9}$$

Now we can calculate $\mathbb{E}[b_t^*]$. Note that

$$\mathbb{E}[b_t^*] = \int_0^{\bar{\gamma}_1} \mathbb{E}[b_t^*|\gamma] d\Gamma_t(\gamma) = \int_0^{\bar{\gamma}_t} \mathbb{E}[b_t^*|\gamma] d\Gamma_t(\gamma) + \int_{\bar{\gamma}_t}^{\bar{\gamma}_1} \mathbb{E}[b_t^*|\gamma] d\Gamma_t(\gamma). \tag{A10}$$

Combining (A5) and (A9), for $t \geq 2$, we can obtain that

$$\begin{aligned} \int_0^{\bar{\gamma}_t} \mathbb{E}[b_t^*|\gamma] d\Gamma_t(\gamma) &= \frac{r(1-r)^{T-t}}{1+r(1-r)^{T-t}} \frac{\bar{\gamma}_t^{1+r\sum_{\tau<t}(1-r)^{T-\tau}}}{\Pi_{\tau<t}[1-(1-r)^{T-\tau}]^r} \\ &\quad + \frac{r\sum_{\tau<t}(1-r)^{T-\tau}}{\Pi_{\tau<t}[1-(1-r)^{T-\tau}]^r} \left(\frac{1}{A_{T-t}} - 1\right) \int_0^{\bar{\gamma}_t} \gamma^{r\sum_{\tau\leq t}(1-r)^{T-\tau}} dx \\ &= \frac{r(1-r)^{T-t}}{A_{T-t}} \frac{\bar{\gamma}_t^{1+r\sum_{\tau\leq t}(1-r)^{T-\tau}}}{\Pi_{\tau<t}[1-(1-r)^{T-\tau}]^r} + \frac{r\sum_{\tau<t}(1-r)^{T-\tau}}{\Pi_{\tau<t}[1-(1-r)^{T-\tau}]^r} \left(\frac{1}{A_{T-t}} - 1\right) \frac{\bar{\gamma}_t^{1+r\sum_{\tau\leq t}(1-r)^{T-\tau}}}{1+r\sum_{\tau\leq t}(1-r)^{T-\tau}} \\ &= \frac{\bar{\gamma}_t^{1+r\sum_{\tau\leq t}(1-r)^{T-\tau}}}{A_{T-t}\Pi_{\tau<t}[1-(1-r)^{T-\tau}]^r} \left[r(1-r)^{T-t} + (1-A_{T-t}) \frac{r\sum_{\tau<t}(1-r)^{T-\tau}}{1+r\sum_{\tau\leq t}(1-r)^{T-\tau}} \right], \end{aligned} \tag{A11}$$

where $A_{T-t} := [1 + r(1-r)^{T-t}][1-(1-r)^{T-t}]^r$; and that

$$\int_{\bar{\gamma}_t}^{\bar{\gamma}_1} \mathbb{E}[b_t^*|\gamma] d\Gamma_t(\gamma) = \sum_{k=1}^{t-1} \int_{\bar{\gamma}_{k+1}}^{\bar{\gamma}_k} \gamma \left\{ 1 - F\left[\frac{\gamma}{H_t(\gamma)}\right] \right\} d\Pi_{\tau \le k} F\left[\frac{1}{h_\tau(\gamma)}\right]$$

$$= \sum_{k=1}^{t-1} \frac{r \sum_{\tau \le k} (1-r)^{T-\tau}}{\Pi_{\tau \le k}[1-(1-r)^{T-\tau}]^r} \left(\frac{\bar{\gamma}_k^{1+r\sum_{\tau \le k}(1-r)^{T-\tau}} - \bar{\gamma}_{k+1}^{1+r\sum_{\tau \le k}(1-r)^{T-\tau}}}{1 + r\sum_{\tau \le k}(1-r)^{T-\tau}} \right.$$

$$\left. - \frac{\bar{\gamma}_k^{1+r(1-r)^{T-t}+r\sum_{\tau \le k}(1-r)^{T-\tau}} - \bar{\gamma}_{k+1}^{1+r(1-r)^{T-t}+r\sum_{\tau \le k}(1-r)^{T-\tau}}}{1 + r(1-r)^{T-t} + r\sum_{\tau \le k}(1-r)^{T-\tau}} \right). \tag{A12}$$

Therefore, for $t \geq 2$, $\mathbb{E}[b_t^*]$ can be calculated through (A10), (A11), and (A12). For $t = 1$, $\mathbb{E}[b_t^*]$ can again be calculated through (A10), (A11), and (A12) by setting $\Pi_{\tau<1}[1-(1-r)^{T-\tau}]^r$ to 1 and $\sum_{\tau<t}(1-r)^{T-\tau}$ to 0.