Competitive Personalized Pricing with Multidimensional Characteristics[*]

Qiang Fu[†] Zenan Wu[‡] Yuxuan Zhu[§]

June 5, 2026

Abstract

Consumers differ in both their brand-dependent preferences (loyalty) and brand-independent preferences (choosiness). Firms produce horizontally differentiated products and, depending on data availability or competition policy, tailor their prices based on what they learn about consumer characteristics. Information along different dimensions of consumer characteristics yields contrasting implications for consumer welfare and industry profit. Either loyalty-based pricing—i.e., price discrimination based only on loyalty—or fully personalized pricing—i.e., price discrimination based on both loyalty and choosiness—maximizes consumer welfare, while partially personalized pricing based on choosiness always maximizes industry profit.

Keywords: multidimensional consumer characteristics, personalized pricing.

JEL Classification Codes: D43, D82, L13.

---

[*] We are grateful to Coeditor Nageeb Ali and four anonymous reviewers for extremely helpful and constructive comments and suggestions. We thank Gaurab Aryal, Deepal Basak, Michael Baye, Yongmin Chen, Zhijun Chen, Jiajia Cong, Rahul Deb, Alexandre de Cornière, Hanming Fang, Rick Harbaugh, Jin Li, Zhuoran Lu, Pietro Ortoleva, Marilyn Peace, Andrew Rhodes, Xianwen Shi, Yossi Spiegel, Greg Taylor, Allen Vong, Julian Wright, Jidong Zhou, Junjie Zhou, and various seminar and conference audiences for helpful discussions, suggestions, and comments. Wu thanks the National Natural Science Foundation of China (Nos. 72173002) for financial support. Any errors are our own.

[†] Department of Strategy and Policy, National University of Singapore, Email: bizfq@nus.edu.sg.

[‡] School of Economics, Peking University, Email: zenan@pku.edu.cn.

[§] School of Economics, Peking University, Email: zhuyuxuan@pku.edu.cn.

1 Introduction

The widespread adoption of commercial surveillance technology, data analytics, and AI-enabled algorithmic tools has granted firms unprecedented flexibility in inferring consumer characteristics and customizing prices with increasing granularity (Dubé and Misra, 2023; Spann et al., 2025). The U.S. Federal Trade Commission's (FTC) recent surveillance-pricing report documents the pervasive use of consumer information to set individualized prices (FTC, 2025).

However, personalized pricing is frequently perceived as unfair, opaque, or intrusive, and has sparked extensive controversy. For example, major ride-hailing platforms have reportedly experimented with individualized fares based on past ride history and in-app behavioral data, fueling concerns that algorithms may infer and exploit users' willingness to pay. Similarly, executives at Delta Air Lines publicly discussed developing AI systems capable of tailoring airfares to individual customers using login data and browsing behavior, which triggered widespread criticism and heightened anxiety about consumers' vulnerability to algorithmic pricing (Shepardson, 2025).

In response, various regulatory measures have been enacted to govern the processing of personal data and protect consumers' rights to transparency: most notably, the European Union's General Data Protection Regulation (GDPR), the California Privacy Rights Act (CPRA), and the recently enacted New York Algorithmic Pricing Disclosure Act. The FTC has also launched investigations into firms that use AI and data analytics to help clients implement personalized pricing, accusing them of "exploiting vast troves of personal information to charge people higher prices" (Siddiqui, 2024; FTC, 2024).

This public controversy and regulatory friction call for systematic research into the distributional effects of consumer privacy and price personalization, namely, how firms' access to consumer information influences economic efficiency, consumer welfare, and profitability (Acquisti et al., 2016; Baye and Sappington, 2020; Ali et al., 2023; Anderson et al., 2023; Rhodes and Zhou, 2024). These questions lie at the core of ongoing regulatory debates (OECD, 2018; Rott et al., 2022), and our paper aims to address them.

The scope of this paper goes beyond the conventional binary debate over whether the use of consumer data should be permitted for pricing. Consumer characteristics are multidimensional, and different data sources reveal specific facets of consumer preferences. Firms' access to data or their ability to use such data, which is often constrained by regulatory requirements, determines their ability to tailor prices along various dimensions of consumer heterogeneity.

We present a systematic taxonomy of pricing regimes based on distinct dimensions of consumer characteristics and compare their welfare and profit implications. We demonstrate that the welfare and profit implications of (partial or perfect) price personalization critically depend on the nature of the data available to firms and the inferences it supports. This allows us to identify which types of consumer data are more likely to enable consumer surplus extraction and therefore merit closer regulatory scrutiny (Tucker, 2024; Heidhues et al., 2026).

Specifically, a consumer may prefer one brand over another due to "horizontal" taste differences, whether aesthetic or functional. At the same time, consumers differ in their "vertical" willingness to pay for products that more closely match their preferences. Borrowing the terminology of Armstrong (2006), we call the former brand-specific preferences a consumer's *loyalty*, while the latter non-brand-specific preferences her *choosiness*.

Demographic or socioeconomic information, such as ZIP codes, gender, or financial status, can better proxy for choosiness, while conveying less about how a consumer ranks specific brands (loyalty). In contrast, behavioral data such as browsing history or search queries—which platforms like Google routinely collect and make available to sellers for targeting—are often more informative about loyalty but less revealing about choosiness. The FTC's surveillance-pricing study (FTC, 2025), for instance, explicitly distinguishes between these data categories, by contrasting direct behavioral metrics (e.g., browsing history, clicks, and mouse movements) with third-party demographic data (e.g., gender and financial indicators sourced from data brokers). The study separately identifies consumer "loyalty" and "willingness to pay" as distinct targeting criteria derived from these different sources. Shiller (2020) also shows that price discrimination based on demographic versus web-browsing data generates markedly different profit implications.

Data limitations and regulatory constraints may prevent firms from fully inferring consumer characteristics or tailoring prices to complete consumer profiles. Consequently, prices are often customized along specific dimensions rather than being fully personalized. For instance, a firm may purchase third-party demographic data from data brokers but lack sufficient proprietary observations of web interactions; this allows the firm to infer a consumer's willingness to pay for preferred brands, but leaves it blind to their brand-specific preferences. As a historical example, Orbitz customized prices according to users' device types; while a device type may effectively proxy for socioeconomic status, it does not reveal one's relative preferences across hotel brands. In contrast, a consumer's frequent searches for Nike shoes reveal a preference for Nike over Adidas products, but do not effectively predict the premium the consumer would pay for Nike over Adidas. Deriving a precise estimate requires socioeconomic proxies, which may be unavailable or legally restricted.

As documented by the recent FTC surveillance-pricing report (FTC, 2025), online retailers segment their customers by inferred fondness for certain brands via real-time web interactions, such as clicks and time spent watching content; these data map directly onto consumers' brand-specific preferences in our context. Furthermore, privacy frameworks such as the GDPR, CPRA, and New York's Algorithmic Pricing Disclosure Act restrict the use of certain categories of "sensitive" data, which often include the financial and socioeconomic proxies that inform consumer choosiness. Concurrently, authorities such as the FTC and the UK's Competition and Markets Authority (CMA) have increasingly scrutinized and audited algorithmic pricing systems, and require firms to disclose and justify the data inputs used for price recommendations. These restrictions therefore push firms' pricing strategies toward one dimension of consumer characteristics while preventing them from conditioning prices on the other. Our model explicitly accommodates these partially personalized pricing regimes, which may arise from existing or hypothetical regulatory oversight.[1]

Consequently, four pricing regimes emerge as possible. Under *uniform pricing*—in which price discrimination is banned or firms observe neither consumers' loyalty nor choosiness—firms set a single price for all consumers. Under *fully personalized pricing*, firms perfectly observe both dimensions for each consumer and tailor prices accordingly. Under *loyalty-based pricing* (observing loyalty but not choosiness) or *choosiness-based pricing* (observing choosiness but not loyalty), firms rely on partial preference information. We compare equilibrium outcomes across these four regimes in terms of consumer welfare and industry profit.

Under full market coverage, fully personalized pricing intensifies market competition and improves consumer welfare relative to uniform pricing: Once consumer preferences are observed, each consumer becomes individually contestable, which sharpens competitive pressure. This affirms the insight of Thisse and Vives (1988) and Rhodes and Zhou (2024), established in settings with unidimensional consumer characteristics. However, partially personalized pricing may either outperform fully personalized pricing in terms of consumer welfare or underperform relative to uniform pricing, depending on which dimension of consumer characteristics—loyalty or choosiness—firms can infer and use to customize their prices.

Our analysis primarily focuses on the comparison between loyalty-based and fully personalized pricing, because consumer welfare is maximized under one of these two regimes. If firms observe loyalty but not choosiness, consumer welfare can exceed that under fully personalized pricing. Firms' inability to observe choosiness triggers a subtle marginal–inframarginal

---

[1] We acknowledge that, in practice, a data source that effectively predicts one dimension of consumer characteristics can often reveal some information about the other. However, different data streams typically differ substantially in the precision with which they support inferences about each dimension. Our model captures the idealized case in which this asymmetry in informational content is sufficiently stark. Furthermore, our robustness analysis in Section 5.2.1 demonstrates that the key welfare rankings remain intact when the two dimensions are not perfectly independent.

trade-off: A firm refrains from setting a high price—even for consumers who strongly prefer its product—because some of these consumers may not be sufficiently choosy to justify a high premium for the better fit. This uncertainty places downward pressure on prices, since the firm seeks to avoid alienating less choosy consumers. Put differently, when choosiness is unknown, less choosy consumers generate a positive externality that benefits their more choosy counterparts. The comparison between loyalty-based and fully personalized pricing ultimately depends on the distribution of consumer types. We provide plausible conditions under which either regime maximizes aggregate consumer welfare.

In contrast, if firms observe choosiness but not loyalty (i.e., under choosiness-based pricing), expected consumer welfare falls below that under uniform pricing.[2] Further, equilibrium industry profits can be unambiguously ranked: Choosiness-based pricing always maximizes industry profit, whereas loyalty-based pricing minimizes it. These observations reveal the qualitatively contrasting roles played by different data streams: Observing choosiness facilitates surplus extraction by eliminating the marginal–inframarginal trade-off, whereas observing loyalty enhances contestability and intensifies competition.

These findings yield intuitive policy implications and provide plausible theoretical rationales for evolving contemporary regulatory principles. Recent frameworks increasingly differentiate between distinct categories of personal information, and tighten restrictions on "sensitive" inputs used for pricing. For example, New York's recently enacted Algorithmic Pricing Disclosure Act focuses on transparency regarding the use of protected characteristics and wealth inferences in generating personalized prices. Meanwhile, authorities such as the FTC and CMA have moved toward auditing pricing algorithms by scrutinizing the "ingredients" underlying price recommendations in order to limit consumer exploitation.

More generally, our analysis provides a normative benchmark for evaluating data privacy and algorithmic pricing regulations, such as the debate regarding personal data trading in Europe (Heidhues et al., 2026). A robust policy analysis must move beyond the binary question of whether personalization and consumer data sharing should be allowed. Instead, it must account for how specific types of consumer information enable firms to shape their pricing strategies and redistribute surplus across market participants.

Link to the Literature A vast amount of scholarly effort has been devoted to the study of competitive price discrimination. One strand of the literature models personalized pricing in the form of imperfect price discrimination, wherein firms set different prices for different consumer segments (e.g., Shaffer and Zhang, 1995; Fudenberg and Tirole, 2000; Chen et al.,

---

[2] Armstrong (2006) concludes the same in a duopoly model with uniform distribution. Our analysis generalizes this insight.

2001; Iyer et al., 2005; Esteves and Resende, 2016). The rapid development of information technology has sparked interest in more granular pricing strategies (see, e.g., Acquisti et al., 2016). The seminal study of Thisse and Vives (1988) compares uniform pricing with fully personalized pricing in a spatial duopoly model, in which each consumer's location is either perfectly revealed to firms or entirely unknown. A similar approach to modeling price personalization in a duopoly is adopted by Chen and Iyer (2002); Shaffer and Zhang (2002) and Chen et al. (2020). Notably, Liu and Serfes (2004) show that finer information partitions (i.e., providing firms with more information) in a Hotelling model can generate nonmonotonic welfare and profit implications.

Most studies on personalized pricing focus on duopolistic competition. In a general discrete choice oligopoly model, Rhodes and Zhou (2024) show that personalized pricing benefits consumers and harms firms under full market coverage—which generalizes the insight of Thisse and Vives (1988)—while the comparison can be overturned otherwise. Our paper is more closely related to Rhodes and Zhou (2024) in terms of modeling approach. However, we aim to explore the ramifications of partially personalized pricing when consumers are characterized along multiple dimensions. Anderson et al. (2023) also adopt a general discrete choice model. They let firms set a listing price in the first stage and then send personalized offers in the second stage, while assuming that targeting each individual consumer is costly. Both Rhodes and Zhou (2024) and our paper assume costless targeting and a single-stage structure for pricing.

The literature on price discrimination or personalized pricing typically assumes that consumer heterogeneity can be adequately captured through variations along a single dimension of consumer characteristics. Each consumer's type is defined solely by her location in the spatial competition model of Thisse and Vives (1988). Anderson et al. (2023) and Rhodes and Zhou (2024) assume that consumers differ only in their respective gross valuations for different firms' products. As a result, these studies typically focus on a binary comparison between uniform pricing and personalized pricing. A notable exception is Armstrong (2006). In a Hotelling duopolistic setting, Armstrong (2006) considers choosiness-based pricing and compares it with uniform pricing and fully personalized pricing. We incorporate Armstrong's conceptual notions of choosiness versus loyalty and analyze all possible pricing regimes. Our study thereby highlights the qualitatively contrasting roles played by information along different dimensions of consumer characteristics.

A burgeoning literature examines how different information structures shape market outcomes and consumer and producer surplus, including Bergemann et al. (2015, 2025); Armstrong and Zhou (2022); Yang (2022) and Elliott et al. (2025). Enabled by an information-design approach, these studies accommodate rich sets of information structures about consumer values rather than relying on the traditional binary comparison between "no information" and "perfect information," and show that different information structures can have drastically different welfare implications. In these studies, as in Rhodes and Zhou (2024), a consumer's type is represented by a vector of gross valuations across firms, and information concerns this valuation vector as a whole. Our paper instead opens up this valuation vector by decomposing it into two economically distinct dimensions: brand-specific match (loyalty) and the intensity of preferences for a better match (choosiness).[3]

Our paper is also related to the literature on buyer-controlled information disclosure. Ali et al. (2023) allow consumers to voluntarily disclose verifiable information about their preferences and show that partial disclosure can improve consumer welfare relative to both uniform pricing and fully personalized pricing. They allow for multidimensional consumer types in an extension, in which the relevant heterogeneity in their disclosure problem can be summarized by a scalar valuation. Ali et al. (2023) consider a setting in which each firm provides one product, while Ichihashi (2020) and Sam (2025) consider a monopolistic and duopolistic multiproduct setting, respectively. In the latter two studies, the information disclosed by a consumer can be described by a match identity (analogous to our brand loyalty) and a precision level that can be interpreted as preference intensity (analogous to our choosiness). Preferences in these studies are essentially unidimensional, with the precision level acting as a "dial" that scales the intensity of preferences. It is worth noting that firms' information is endogenous in this strand of the literature, while it is exogenous in our paper. Further, given that we consider an oligopoly, we are able to investigate the implications of market structure for consumer and producer welfare.

The rest of the paper proceeds as follows. Section 2 sets up the model. Section 3 analyzes the duopoly case. Section 4 extends the model to an oligopoly and shows that our key insights remain valid with more firms in the market. Notably, loyalty-based pricing becomes even more likely to emerge as the consumer-optimal regime in an oligopoly. Section 5 discusses possible extensions to our baseline model, and Section 6 concludes.

2 Model and Preliminaries

We first set up the model, then present the key preliminary results of our analysis.

---

[3] Elliott et al. (2025) also allow consumer valuations to depend on multiple attributes; their information structures, however, are defined over the resulting valuation vector as a whole. Our analysis instead decomposes consumer value into brand-specific match and preference intensity, and studies how access to each dimension separately affects pricing, firm profit, and consumer welfare.

2.1 Model Setup

Each of $n \geq 2$ firms, indexed by $i \in \mathcal{N} \equiv \{1, \ldots, n\}$, offers a horizontally differentiated product at a constant marginal cost $c \geq 0$. The market involves a unit mass of consumers, and each consumer wishes to buy one unit of a product.

2.1.1 Consumer Characteristics

A consumer's gross valuation for the products supplied by these firms is given by $\boldsymbol{v} \equiv (v_1, \ldots, v_n)$. Her valuation for the product supplied by a firm i is determined by $v_i = v + tx_i$, where v is the base utility she derives from consumption of the product and tx_i measures the additional utility she gains from the product supplied by firm $i \in \mathcal{N}$.[4]

The vector $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$ captures the consumer's firm- or brand-specific preferences, which measures each product i's match to her taste. The parameter t, which is common to all firms for a given consumer, indicates the intensity of her brand-specific preferences and measures the marginal valuation for consuming a better-matched product. This also reflects the relative importance she assigns to taste vis-à-vis price in her purchasing decision. Alternatively, the parameter t can be interpreted as an indicator for income, since a consumer with a larger t tends to be less price sensitive.[5] Following the literature (Armstrong, 2006), we call the former brand-specific preferences *loyalty* and the latter *choosiness*. The distributional details of $\boldsymbol{x}$ and t will be provided in Section 2.1.3.

We assume that the base utility v is commonly known, but firms may not observe $\boldsymbol{x}$ and/or t.[6] The availability of consumer information determines firms' ability to tailor their prices, which we detail next.

2.1.2 Pricing Regimes and Equilibrium Concept

We consider four pricing regimes. Under *uniform pricing*, each firm, lacking access to consumer data, sets a single price $p_i^{\mathcal{U}}$ for all consumers. Under *loyalty-based pricing* (resp., *choosiness-based pricing*), each firm sets a (partially) personalized price $p_i^{\mathcal{L}}(\boldsymbol{x})$ (resp., $p_i^{\mathcal{C}}(t)$)

---

[4] More generally, we can allow the base utility v to depend on consumer characteristics, i.e., $v_i = v(t, \boldsymbol{x}) + tx_i$. As long as $v(t, \boldsymbol{x})$ is sufficiently large to ensure full market coverage, the specific functional form of $v(\cdot)$ is irrelevant for the competitive analysis, since a consumer's choice depends solely on $v_i - v_j$, the difference in gross valuations across products, as in a standard Hotelling model. Hence, while our setting normalizes $v(t, \boldsymbol{x}) \equiv v$ and assumes a specific form of correlation between v_i and $(t, \boldsymbol{x})$, the analysis and its implications accommodate arbitrary forms of correlations, e.g., a setting in which a consumer strongly prefers one product over another while having a low absolute valuation for both.

[5] To see this, note that a consumer's utility of buying from firm i is proportional to $v/t + x_i - p_i/t$: A larger t implies a lower price sensitivity.

[6] Allowing for heterogeneous base utility, v, has no effect in the presence of market competition under the premise of full market coverage. See Section 3.1 in Armstrong (2006) for more discussion.

based on $\boldsymbol{x}$ (resp., t). Under *fully personalized pricing*, a firm offers each consumer a price that is fully customized according to both $\boldsymbol{x}$ and t.[7]

Under each pricing regime, firms simultaneously announce their prices. Upon observing $\boldsymbol{p} = (p_1, \ldots, p_n)$, a consumer $(\boldsymbol{x}, t)$ purchases the product provided by firm i if

$$v + tx_i - p_i \geq \max_{j \neq i}\{v + tx_j - p_j\}. \tag{1}$$

In case a consumer is indifferent between multiple products, she chooses the one with the highest gross valuation. We assume throughout the paper that v is sufficiently large (i.e., $v \geq c + 2\overline{x}\overline{t}$) such that the market is fully covered.[8]

We adopt Nash equilibrium as the solution concept for all our analyses of pricing competition and focus on pure-strategy equilibrium.

2.1.3 Distributional Details of Consumer Characteristics

Each consumer's loyalty $\boldsymbol{x} \equiv (x_1, \cdots, x_n)$ is distributed on $[\underline{x}, \overline{x}]^n$, with $0 \leq \underline{x} < \overline{x} < \infty$, according to a joint cumulative distribution function (CDF) $G(\boldsymbol{x})$ with a probability density function (PDF) $g(\boldsymbol{x})$. The CDF $G(\boldsymbol{x})$ is *exchangeable*; that is, it is invariant under any permutation of $(x_1, \ldots, x_n)$. The exchangeability, as in Rhodes and Zhou (2024), assumes away systematic quality differences across firms. The marginal CDF and PDF of x_i are denoted by $G^{\mathrm{m}}(\cdot)$ and $g^{\mathrm{m}}(\cdot)$, respectively. Further, consumers' choosiness t is distributed on $[\underline{t}, \overline{t}]$, with $0 \leq \underline{t} < \overline{t} < \infty$, according to a differentiable CDF $F(\cdot)$ with a PDF $f(\cdot)$. For tractability and expositional efficiency, we assume that t and $\boldsymbol{x}$ are independent: Firms cannot draw inferences about one dimension of a consumer's preferences from data regarding the other.[9]

Under uniform pricing, a symmetric pure-strategy equilibrium requires $p_j = p^{\mathcal{U}}$ for each $j \neq i$ and (1) reduces to

$$p_i - p^{\mathcal{U}} \leq t\left(x_i - \max_{j \neq i} x_j\right).$$

Define $\hat{x}_i := x_i - \max_{j \neq i} x_j$ and denote its CDF and PDF by $\Psi(\cdot)$ and $\psi(\cdot)$, respectively. Further, define $z := t\hat{x}_i$ and denote the CDF and PDF of z by $H(\cdot)$ and $h(\cdot)$. We impose

---

[7] We take a "third-party" rather than a "first-party" approach, in the sense that a third-party data provider collects and provides information to firms. In practice, firms can learn about consumers' preferences based on their purchase history, from which they may be able to infer (partial) information about both dimensions. We leave this extension for future research.

[8] In Section 5.2.2, we relax this assumption to allow for partial market coverage. Consistent with Rhodes and Zhou (2024), the welfare rankings can reverse in the high-cost limit.

[9] In Section 5.2.1, we examine the implications of correlation between $\boldsymbol{x}$ and t. Although incorporating correlation raises significant technical complications, the central qualitative predictions of our model remain robust.

the following assumptions throughout the paper, except in Section 5.2.2:

Assumption 1 $1-\Psi(\hat{x}_i)$ *is log-concave in* $\hat{x}_i$*.*

Assumption 2 $f(t)/t$ *is log-concave in* t*.*

Assumption 3 $1-H(z)$ *is log-concave in* z*.*

Assumptions 1, 2, and 3 ensure the existence of pure-strategy equilibria in the pricing game under $\mathcal{C}$, $\mathcal{L}$, and $\mathcal{U}$, respectively.

Three remarks are in order. First, in principle, $H(\cdot)$ is determined by given $\Psi(\cdot)$ and $f(\cdot)$. For example, it can be verified that Assumption 3 follows from Assumption 2 if $\hat{x}_i$ is uniformly distributed (which satisfies Assumption 1). However, to the best of our knowledge, the literature provides no general conditions under which the survival function of the product of two independent random variables (i.e., $1-H(z)$ in our setting) is log-concave. As a result, we directly impose Assumption 3 in our analysis.

Second and relatedly, cautious readers may note that Assumption 2 is not satisfied when t follows a uniform distribution. Relaxing this restriction (e.g., assuming $f(t)$ or $1-F(t)$ to be log-concave) would render Assumption 3 less likely to be satisfied. To illustrate, suppose that both $\hat{x}_i$ and t follow uniform distributions. In that case, Assumption 3 can be shown to be violated.

Finally, despite these restrictions, a broad class of standard parametric distribution families satisfy all the required assumptions within plausible parameter ranges. We provide a detailed checklist of admissible distributions and illustrative examples in Appendix B.

2.2 Preliminary Results

We now lay out the key preliminary results of the general model.

2.2.1 Pricing Equilibrium

We first characterize the equilibrium under each pricing regime, which lays the foundation for welfare and profit comparisons. The analysis of Rhodes and Zhou (2024) can readily be adapted to characterize the pricing equilibrium under uniform pricing $\mathcal{U}$, choosiness-based pricing $\mathcal{C}$, and fully personalized pricing $\mathcal{F}$.

Lemma 1 ***(Equilibrium Characterization under*** $\boldsymbol{\mathcal{U}}$***,*** $\boldsymbol{\mathcal{C}}$***, and*** $\boldsymbol{\mathcal{F}}$***)*** *The following statements hold for the respective equilibrium under pricing regimes* $\mathcal{U}$*,* $\mathcal{C}$*, and* $\mathcal{F}$*:*

(i) *There exists a unique symmetric equilibrium under uniform pricing* $\mathcal{U}$*, in which each firm sets a price* $p^{\mathcal{U}} = c + \frac{1}{nh(0)} = c + \frac{1}{n\psi(0)\mathbb{E}[\frac{1}{t}]}$.

(ii) *Fix a realized choosiness level* t*. There exists a unique symmetric equilibrium under choosiness-based pricing* $\mathcal{C}$*, in which each firm sets a price* $p^{\mathcal{C}}(t) = c + \frac{t}{n\psi(0)}$.

(iii) *Consider fully personalized pricing* $\mathcal{F}$*. Without loss of generality, fix any given realized consumer type* $(\boldsymbol{x}, t)$ *with* $x_1 > \cdots > x_n$*. The pricing game yields a unique equilibrium outcome: The most preferred firm (i.e., firm 1) charges an equilibrium price* $p_1^{\mathcal{F}}(\boldsymbol{x}, t) = c + t(x_1 - x_2)$ *and monopolizes the market; the second most preferred firm (i.e., firm 2) charges an equilibrium price* $p_2^{\mathcal{F}}(\boldsymbol{x}, t) = c$.[10]

The equilibrium characterization for uniform pricing $\mathcal{U}$ and fully personalized pricing $\mathcal{F}$ can be obtained by adapting Lemmata 1 and 2 in Rhodes and Zhou (2024), respectively: Firms are uninformed under the former and perfectly informed in the latter. Similarly, choosiness-based pricing $\mathcal{C}$ is equivalent to uniform pricing in Rhodes and Zhou (2024) with a fixed t.

Assuming a uniform distribution, Armstrong (2006) characterizes the equilibrium under choosiness-based pricing $\mathcal{C}$ but not under loyalty-based pricing $\mathcal{L}$. Our analysis generalizes his analysis under $\mathcal{C}$ and fills the gap under $\mathcal{L}$. Fixing $\boldsymbol{x}$, denote by $k(\boldsymbol{x})$ the number of firms with positive demand in the equilibrium. The following result ensues.

Lemma 2 ***(Equilibrium Characterization under*** $\mathcal{L}$***)*** *Consider loyalty-based pricing* $\mathcal{L}$*. Without loss of generality, fix a realized profile of consumer brand preferences* $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$ *with* $x_1 > \cdots > x_n$*. The pricing game yields a unique equilibrium outcome: If* $f(\underline{t}) \geq 1/\underline{t}$*, then* $k(\boldsymbol{x}) = 1$ *and the market is monopolized by consumers' most preferred firm* 1*; otherwise,* $k(\boldsymbol{x}) \geq 2$ *and the market is segmented by a set of consumers' most preferred firms, i.e.,* $\{1, \ldots, k(\boldsymbol{x})\}$*. In particular, if* $\underline{t} = 0$*, then* $k(\boldsymbol{x}) = n$*.*

For a given realization of $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$, the interim equilibrium outcome for the case of $f(\underline{t}) < 1/\underline{t}$ can be intuitively described as follows. There exist a set of cutoffs $\big(\alpha_1(\boldsymbol{x}), \ldots, \alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\big)$, with $\alpha_0(\boldsymbol{x}) := \bar{t} > \alpha_1(\boldsymbol{x}) > \cdots > \alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x}) > \alpha_{k(\boldsymbol{x})}(\boldsymbol{x}) := \underline{t}$. A consumer purchases the product from her i-th most preferred firm (i.e., firm i in this context) if and only if her choosiness level t falls in the interval $\big[\alpha_i(\boldsymbol{x}), \alpha_{i-1}(\boldsymbol{x})\big)$, which represents the i-th choosiest consumer segment in the equilibrium. Details of the equilibrium are provided in the proof of the lemma in Appendix A.

---

[10] All other firms charge $p_i^{\mathcal{F}}(\boldsymbol{x}, t) \geq c$ for $i \in \{3, \ldots, n\}$.

Without knowing t, a firm is subject to the usual marginal–inframarginal trade-off: A competitive price allows the firm to sell to less choosy consumers—i.e., those with low t—but prevents the firm from extracting surplus from their choosier counterparts. Imagine $f(\underline{t}) \geq 1/\underline{t}$, which corresponds to a sufficiently high lower bound $\underline{t}$. That is, even the least choosy consumer is willing to pay a substantial premium for a closely matched product. In this case, losing even the least choosy consumers is excessively costly for firm 1, which forces the firm to lower its price. In Appendix A, we show that firms 2 to n each charge their marginal cost c in the most intuitive equilibrium, while the preferred firm 1 responds by charging a markup $\underline{t}(x_1 - x_2)$ to remain competitive and retain the least choosy consumers. The resulting equilibrium is efficient, since all consumers purchase their preferred product.

Suppose otherwise that $\underline{t}$ decreases such that $f(\underline{t}) < 1/\underline{t}$ holds. Retaining less choosy consumers then becomes less attractive to firm 1. The firm would raise its price to extract a premium from choosier consumers, while leaving a positive residual demand to others. This softens the price competition: As Lemma 2 and the Appendix show, at least two firms charge a price above the marginal cost c. The segmented market causes inefficiency, because some consumers end up consuming less preferred products.

For a given $\boldsymbol{x}$, the game resembles a vertical differentiation model (e.g., Shaked and Sutton, 1982, 1983): x_i can alternatively be interpreted as firm i's product quality, and t as consumers' marginal valuation of quality. However, their analysis assumes a uniform distribution. Lemma 2 generalizes the analysis by allowing for more general distribution functions, to verify that the predicted market segmentation is robust to the distribution of consumer choosiness.

2.2.2 Welfare and Profit Ranking

We now compare consumer welfare and industry profit across pricing regimes. Denote by W^j the expected equilibrium total surplus (i.e., sum of consumer welfare and industry profit) under pricing regime $j \in \{\mathcal{U}, \mathcal{C}, \mathcal{L}, \mathcal{F}\}$. The following ensues from Lemmas 1 and 2.

Proposition 1 ***(Potential Market Inefficiency with Full Market Coverage)*** *When $f(\underline{t}) \geq 1/\underline{t}$, the market achieves full efficiency, and the expected equilibrium total surplus is maximized under all pricing regimes, i.e., $W^{\mathcal{U}} = W^{\mathcal{C}} = W^{\mathcal{F}} = W^{\mathcal{L}}$ if $f(\underline{t}) \geq 1/\underline{t}$. When $f(\underline{t}) < 1/\underline{t}$, the market remains fully efficient under all pricing regimes except under loyalty-based pricing $\mathcal{L}$, i.e., $W^{\mathcal{U}} = W^{\mathcal{C}} = W^{\mathcal{F}} > W^{\mathcal{L}}$ if $f(\underline{t}) < 1/\underline{t}$.*

With full market coverage, all of these pricing regimes generate efficient outcomes except for the case of loyalty-based pricing $\mathcal{L}$ when $f(\underline{t}) < 1/\underline{t}$: As Lemma 2 shows, less choosy consumers (i.e., those with $t < \alpha_1(\boldsymbol{x})$) end up with a less preferred product, which causes efficiency loss.

Let V^j, and Π^j, respectively, denote the expected equilibrium consumer welfare and industry profit under a pricing regime j. We further compare equilibrium consumer welfare and industry profit across the three pricing regimes that generate full efficiency: fully personalized pricing $\mathcal{F}$, uniform pricing $\mathcal{U}$, and choosiness-based pricing $\mathcal{C}$.

Lemma 3 ***(Armstrong, 2006; Rhodes and Zhou, 2024)*** *Among the three pricing regimes $\mathcal{U}, \mathcal{C}$, and $\mathcal{F}$, fully personalized pricing $\mathcal{F}$ (choosiness-based pricing $\mathcal{C}$) maximizes consumer welfare (industry profit) and minimizes industry profit (consumer welfare), i.e., $V^{\mathcal{F}} > V^{\mathcal{U}} > V^{\mathcal{C}}$ and $\Pi^{\mathcal{C}} > \Pi^{\mathcal{U}} > \Pi^{\mathcal{F}}$.*

This claim directly follows from Rhodes and Zhou (2024) and Lemmata 1-2. The comparison between uniform pricing $\mathcal{U}$ and fully personalized pricing $\mathcal{F}$ is equivalent to that of Rhodes and Zhou (2024): Firms are completely uninformed in the former case and perfectly informed in the latter, which suggests $V^{\mathcal{F}} > V^{\mathcal{U}}$. This affirms the conventional wisdom in the literature: Competitive personalized pricing renders every consumer contestable, which enables firms to poach other firms' loyal consumers and thus intensifies market competition. Given $W^{\mathcal{F}} = W^{\mathcal{U}}$, we can conclude $\Pi^{\mathcal{F}} < \Pi^{\mathcal{U}}$.

However, allowing firms to acquire information about choosiness t harms consumers and renders them even worse off than under uniform pricing $\mathcal{U}$.[11] Without knowing t, a firm prices by the average, which yields $p^{\mathcal{U}} = c + 1/\left[n\psi(0)\mathbb{E}[1/t]\right]$. Upon knowing t, firms customize their prices, with $p^{\mathcal{C}}(t) = c + t/\left[n\psi(0)\right]$. Choosy consumers (i.e., those with $t > 1/\mathbb{E}[1/t]$) are charged a higher price under choosiness-based pricing $\mathcal{C}$, while the less choosy (i.e., those with $t < 1/\mathbb{E}[1/t]$) pay less. The conclusion is mathematically straightforward by Cauchy–Schwarz inequality; its economic rationale is also intuitive. Revealing t softens the price competition for choosy consumers: A larger t amplifies the advantage of the more preferred product and renders undercutting less effective when poaching others' loyal customers. In contrast, a smaller t limits the degree of perceived product differentiation and encourages competition, since a lower price is more likely to lure a consumer away from her most preferred product. Compared with uniform pricing $\mathcal{U}$, firms gain from the more valuable consumers—i.e., those with larger t and higher willingness to pay—while losing from the less valuable. The information thus benefits firms in general, but harms consumers as a whole.

Lemma 3 paves the way for our search for the consumer-optimal pricing regime. It suffices to compare the case of fully personalized pricing $\mathcal{F}$ with that of loyalty-based pricing $\mathcal{L}$. In

---

[11] This observation was first noted by Armstrong (2006) in a standard Hotelling duopoly model, but extending it to a general discrete choice model is straightforward.

what follows, we first conduct the comparison in the duopoly case ($n = 2$), then proceed to the oligopoly case ($n \geq 3$).

3 Case of Duopoly

With $n = 2$, the pricing equilibrium in Lemma 2 can be simplified as follows.

Corollary 1 *Consider loyalty-based pricing $\mathcal{L}$ and fix $\boldsymbol{x} \equiv (x_1, x_2)$, with $x_1 > x_2$. If $f(\underline{t}) \geq 1/\underline{t}$, firms charge prices*

$$p_1^{\mathcal{L}}(\boldsymbol{x}) = c + \underline{t}(x_1 - x_2) \text{ and } p_2^{\mathcal{L}}(\boldsymbol{x}) = c. \tag{2}$$

If $f(\underline{t}) < 1/\underline{t}$, firms charge prices

$$p_1^{\mathcal{L}}(\boldsymbol{x}) = p_2^{\mathcal{L}}(\boldsymbol{x}) + (x_1 - x_2)\alpha^* \text{ and } p_2^{\mathcal{L}}(\boldsymbol{x}) = c + (x_1 - x_2)\frac{F(\alpha^*)}{f(\alpha^*)}, \tag{3}$$

where $\alpha^* \in (\underline{t}, \overline{t}\,)$ uniquely solves

$$\alpha^* f(\alpha^*) = 1 - 2F(\alpha^*). \tag{4}$$

In the equilibrium, firm 2 sells to consumers with $t \in [\underline{t}, \alpha^*)$, and firm 1 sells to those with $t \in [\alpha^*, \overline{t}]$.

Firm 1 monopolizes the market when $f(\underline{t}) \geq 1/\underline{t}$; otherwise, it sells only to consumers with a choosiness level above the cutoff α^*, leaving the remaining consumers to firm 2. Notably, the cutoff α^*, determined by (4), is independent of the consumer's loyalty vector (x_1, x_2) or the degree of brand differentiation $\Delta x \equiv x_1 - x_2$. By Corollary 1, $x_1 - x_2$ is a sufficient statistic for $\boldsymbol{x} \equiv (x_1, x_2)$ in the duopoly case and plays a pivotal role in shaping the equilibrium. To see this, consider an increase in Δx. A larger Δx strengthens firm 1's relative advantage for all consumers, but it does so in a proportional way: A type-t consumer's incremental value from buying from firm 1 rather than firm 2 is $t\Delta x$. Thus, increasing Δx scales up both the premium that firm 1 can extract from choosier consumers and the competitive force exerted by marginal consumers. Equilibrium prices therefore adjust proportionally with Δx, leaving the cutoff α^*, and hence the equilibrium market shares, unchanged.

This invariance is largely an artifact of the duopoly structure, in which consumers make a binary choice and Δx determines the equilibrium. In a general oligopoly, more complex interactions arise and multiple firms may share the market; the equilibrium cutoff levels for choosiness would then depend on the consumer's entire loyalty vector $\boldsymbol{x}$.

3.1 Consumer Welfare: Loyalty-based Pricing $\mathcal{L}$ versus Fully Personalized Pricing $\mathcal{F}$

Recall that $V^{\mathcal{L}}$ and $V^{\mathcal{F}}$ denote equilibrium consumer welfare under loyalty-based pricing and fully personalized pricing, respectively. Comparing $V^{\mathcal{L}}$ and $V^{\mathcal{F}}$ in light of Lemma 3 yields the following main result.

Proposition 2 ***(Consumer Welfare Comparison under Duopoly)*** *Fix $n = 2$. Either fully personalized pricing $\mathcal{F}$ or loyalty-based pricing $\mathcal{L}$ maximizes consumer welfare. Specifically:*

(i) If either the condition $f(\underline{t}) \geq 1/\underline{t}$ or

$$\int_{\alpha^*}^{\bar{t}} [1 - F(t)]dt > \frac{F(\alpha^*)}{f(\alpha^*)} \tag{5}$$

holds, then loyalty-based pricing $\mathcal{L}$ outperforms fully personalized pricing $\mathcal{F}$ and maximizes consumer welfare, i.e., $V^{\mathcal{L}} > V^{\mathcal{F}}$.

(ii) If both the condition $f(\underline{t}) < 1/\underline{t}$ and

$$\int_{\alpha^*}^{\bar{t}} [1 - F(t)]dt < \frac{F(\alpha^*)}{f(\alpha^*)} \tag{6}$$

hold, then fully personalized pricing $\mathcal{F}$ outperforms loyalty-based pricing $\mathcal{L}$ and maximizes consumer welfare, i.e., $V^{\mathcal{F}} > V^{\mathcal{L}}$.

Proposition 2 provides necessary and sufficient conditions for loyalty-based pricing $\mathcal{L}$ to outperform fully personalized pricing $\mathcal{F}$ in terms of consumer welfare in a duopoly. With multidimensional characteristics, consumer welfare may be maximized under *partial* access to consumer data and limited inferences about consumer attributes.

In particular, loyalty-based pricing $\mathcal{L}$ maximizes consumer welfare as long as $f(\underline{t}) \geq 1/\underline{t}$ or when condition (5) holds. These conditions are intuitive. First, the result implies that $V^{\mathcal{L}}$ is more likely to exceed $V^{\mathcal{F}}$ when consumers are choosier on average. Consider, for instance, an upward shift of the entire distribution of t without varying the shape of the distribution. Such a shift renders either condition (i.e., $f(\underline{t}) \geq 1/\underline{t}$ or (5)) more likely to be satisfied, and $V^{\mathcal{L}}$ is more likely to exceed $V^{\mathcal{F}}$. The following can immediately be concluded to formalize this insight.

Corollary 2 *If consumers are sufficiently choosy, loyalty-based pricing $\mathcal{L}$ maximizes consumer welfare under duopoly. Formally, suppose that each consumer's choosiness level is* *given by $t = \tilde{t} + T$, where $\tilde{t}$ is a random variable with a fixed distribution and T is a positive constant. Then, for any distribution of $\tilde{t}$, there exists sufficiently large T such that loyalty-based pricing $\mathcal{L}$ outperforms fully personalized pricing $\mathcal{F}$.*

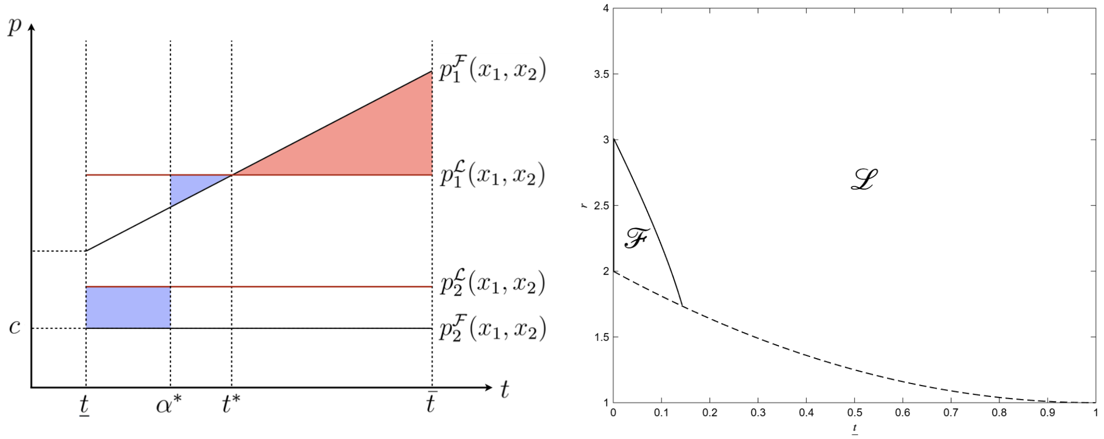
Suppose first that $f(\underline{t}) > 0$. As T increases and the distribution shifts to the right, $1/\underline{t}$ eventually falls below the fixed boundary density $f(\underline{t})$. Hence, the first condition in Proposition 2(i), namely $f(\underline{t}) \geq 1/\underline{t}$, is satisfied. Now suppose that $f(\underline{t}) = 0$. In this case, the first condition never applies. Nevertheless, the second condition in Proposition 2(i)—i.e., condition (5)—eventually holds. The reason is that Assumption 2 implies that the cutoff α^* moves to the right more slowly than the distribution itself. After removing the common rightward shift, the cutoff converges to the lower bound of the original distribution. Hence, the integral term on the left-hand side of (5) remains bounded away from zero, while the term $F(\alpha^*)/f(\alpha^*)$ on the right-hand side converges to zero. Thus (5) holds for sufficiently large shifts.

Second, loyalty-based pricing $\mathcal{L}$ is more likely to maximize consumer welfare when the market is composed of more highly choosy consumers, i.e., when the distribution of t features a heavy tail at its upper end. Under fully personalized pricing $\mathcal{F}$, any increase in valuation in the tail is fully extracted by firms through customized prices; whereas under loyalty-based pricing $\mathcal{L}$, prices are anchored by the marginal consumer α^*. This allows infra-marginal consumers (the high-t types) to capture the full surplus gain from a shift in tail. Consider, for example, a shift of probability density from a small t to a large t over the interval $t \in [\alpha^*, \bar{t}]$, holding fixed $f(\alpha^*)$ and $F(\alpha^*)$. By this construction, (4) continues to hold and thus the equilibrium market segmentation remains unchanged. Meanwhile, this shift mathematically enlarges $\int_{\alpha^*}^{\bar{t}}[1 - F(t)]dt$ and renders (5) more likely to hold. We construct the following example to illustrate this logic.

Example 1 *We set $\bar{t} = 1$ and parameterize the CDF $F(\cdot)$ as follows:*

$$F(t; \underline{t}, r) = \left(\frac{t - \underline{t}}{1 - \underline{t}}\right)^r, \text{ with } \underline{t} \in (0, 1) \text{ and } r \geq 1 + (1 - \underline{t})^2.$$

It is straightforward to verify that $f(\underline{t}) = 0 < 1/\underline{t}$ for all $\underline{t} \in (0, 1)$ and $r \geq 1 + (1 - \underline{t})^2$. As $\underline{t}$ or r increases, the probability densities would be concentrated more on large values of t, which implies that condition (5) is more likely to be satisfied. Figure 1(b) depicts the pricing regime that maximizes consumers' welfare. The horizontal axis represents $\underline{t}$ and the vertical axis r, with the former ranging from 0 to 1 and the latter from 2 to 4. The solid curve is defined by the condition $\int_{\alpha^*}^{\bar{t}}[1 - F(t)]dt = F(\alpha^*)/f(\alpha^*)$; the region to its right collects all $(\underline{t}, r)$ under which loyalty-based pricing outperforms fully personalized pricing in terms of



(a) Equilibrium pricing schedules (b) Consumer welfare

Figure 1: Loyalty-based pricing versus fully personalized pricing: $f(\underline{t}) < 1/\underline{t}$.

consumer welfare. That is, loyalty-based pricing $\mathcal{L}$ tends to prevail for large $\underline{t}$ and r, which affirms the intuition laid out above.

3.2 Discussion: Who Benefits from Loyalty-based Pricing?

We now interpret the economic logic underlying Proposition 2 using Figure 1(a). Recall from Lemma 1 that, fixing $(\boldsymbol{x}, t)$, in the equilibrium under fully personalized pricing $\mathcal{F}$, firms charge $p_1^{\mathcal{F}}(\boldsymbol{x}, t) = c + t(x_1 - x_2)$ and $p_2^{\mathcal{F}}(\boldsymbol{x}, t) = c$, with firm 1 monopolizing the market. Firm 1's price, $p_1^{\mathcal{F}}(\boldsymbol{x}, t)$, is illustrated by the upward-sloping segment in the figure.

To compare this with loyalty-based pricing $\mathcal{L}$, we begin with the case with $f(\underline{t}) < 1/\underline{t}$. By (3), under loyalty-based pricing, firms charge

$$p_1^{\mathcal{L}}(\boldsymbol{x}) = c + (x_1 - x_2)\left[\alpha^* + \frac{F(\alpha^*)}{f(\alpha^*)}\right] \text{ and } p_2^{\mathcal{L}}(\boldsymbol{x}) = c + (x_1 - x_2)\frac{F(\alpha^*)}{f(\alpha^*)}.$$

As noted above, firm 2 sells to those with $t \in [\underline{t}, \alpha^*)$, and firm 1 secures those with $t \in [\alpha^*, \bar{t}\,]$. Without knowing t, a firm has to set a flat price for all consumers: Both $p_1^{\mathcal{L}}(\boldsymbol{x})$ and $p_2^{\mathcal{L}}(\boldsymbol{x})$ are independent of t. The two prices are illustrated by the two flat segments in the respective partitions of t. This triggers the marginal–inframarginal trade-off introduced earlier.

Consumers with $t \in [\underline{t}, \alpha^*)$ are worse off under loyalty-based pricing $\mathcal{L}$. Under fully personalized pricing $\mathcal{F}$, these consumers are indifferent between purchasing from firm 1 at a price $p_1^{\mathcal{F}}(\boldsymbol{x}, t) = c + t(x_1 - x_2)$ and from firm 2 at a price $p_2^{\mathcal{F}}(\boldsymbol{x}, t) = c$. Under loyalty-based pricing $\mathcal{L}$, however, they have to purchase from their less preferred firm 2 and pay a price strictly above c.

Consumers with $t \in [\alpha^*, \bar{t}\,]$ purchase from their preferred firm 1 under loyalty-based pricing $\mathcal{L}$. The flat price they pay, $p_1^{\mathcal{L}}(\boldsymbol{x}) = c + (x_1 - x_2)[\alpha^* + F(\alpha^*)/f(\alpha^*)]$, can be either higher or lower than $p_1^{\mathcal{F}}(\boldsymbol{x}, t) = c + t(x_1 - x_2)$, as illustrated in Figure 1(a). Moderately choosy consumers (those with $t \in [\alpha^*, t^*)$) pay more, whereas very choosy consumers (those with $t \in (t^*, \bar{t}\,])$ pay less. This observation reflects the marginal–inframarginal trade-off firm 1 faces: To retain a larger market share, firm 1 must forgo the surplus it could otherwise extract from the very choosy consumers.

In summary, consumers with $t \in (t^*, \bar{t}\,]$ benefit from loyalty-based pricing $\mathcal{L}$ compared with fully personalized pricing $\mathcal{F}$, whereas all others are worse off.[12] Summing up the relative gains and losses of different consumer segments yields condition (5). The condition implies that $V^{\mathcal{L}}$ is more likely to exceed $V^{\mathcal{F}}$ when the probability mass concentrates more on these inframarginal consumers, i.e., when the upper tail of the distribution of t becomes thicker. Under fully personalized pricing $\mathcal{F}$, a firm captures increased consumer valuation by raising prices one-for-one. In contrast, under loyalty-based pricing $\mathcal{L}$, equilibrium prices are anchored to the marginal consumer and remain insensitive to the shifts toward the upper tail. Consequently, the gains from higher choosiness accrue entirely to consumers rather than firms.

This logic also explains why loyalty-based pricing always outperforms fully personalized pricing when $f(\underline{t}) \geq 1/\underline{t}$. As noted above, the condition is satisfied when even the least choosy consumers have a sufficiently high willingness to pay for a better match, i.e., when $1/\underline{t}$ is sufficiently small. This prompts the preferred firm to capture the entire market by setting a relatively low price that everyone will accept. Consequently, the marginal–inframarginal trade-off and its associated positive externality benefit every consumer. More formally, recall that under loyalty-based pricing, the cutoff type α^* in the case $f(\underline{t}) < 1/\underline{t}$ is determined by $\alpha^* f(\alpha^*) = 1 - 2F(\alpha^*)$. As the boundary case $f(\underline{t}) = 1/\underline{t}$ is approached from the region $f(\underline{t}) < 1/\underline{t}$, the solution requires $F(\alpha^*) \to 0$, which implies that both α^* and t^* depicted in Figure 1(a) converge to $\underline{t}$. This renders every consumer sufficiently choosy (i.e., $t \geq t^* = \underline{t}$) to benefit from loyalty-based pricing.

3.3 Profit Ranking

While ranking consumer welfare across pricing regimes proves complex, firms' profits can be ranked unambiguously.

---

[12] Recall that a larger t corresponds to higher income and lower price sensitivity. These observations highlight a potentially less desirable distributional implication of loyalty-based pricing: Although it may maximize aggregate consumer welfare, the gains relative to fully personalized pricing accrue to high-income consumers, whereas losses are borne by their low-income counterparts.

Proposition 3 ***(Industry Profit Comparison under Duopoly)*** *Fix $n = 2$. The equilibrium profits under the four pricing regimes can be ranked as follows: $\Pi^{\mathcal{C}} > \Pi^{\mathcal{U}} > \Pi^{\mathcal{F}} > \Pi^{\mathcal{L}}$. That is, choosiness-based pricing maximizes industry profit, while loyalty-based pricing minimizes it.*

The first two inequalities follow directly from Proposition 1 and Lemma 3. However, the comparison between $\Pi^{\mathcal{F}}$ and $\Pi^{\mathcal{L}}$ is less straightforward: Under the condition $f(\underline{t}) < 1/\underline{t}$, loyalty-based pricing $\mathcal{L}$ generates lower total surplus than fully personalized pricing $\mathcal{F}$ due to inefficient allocation (i.e., $W^{\mathcal{L}} < W^{\mathcal{F}}$); it also generates lower consumer welfare (i.e., $V^{\mathcal{L}} < V^{\mathcal{F}}$), provided condition (6) holds.

Fix $\boldsymbol{x} \equiv (x_1, x_2)$. As discussed above, under loyalty-based pricing $\mathcal{L}$, firm 1 earns more from moderately choosy consumers (those with $t \in [\alpha^*, t^*)$), but earns less from highly choosy consumers (those with $t \in [t^*, \bar{t}\,]$) relative to fully personalized pricing $\mathcal{F}$. Moreover, firm 1 relinquishes the less choosy consumers (i.e., those with $t \in [\underline{t}, \alpha^*)$), while firm 2 is now able to secure a positive profit from this consumer segment.

However, the industry's relative gain under loyalty-based pricing $\mathcal{L}$ does not offset its corresponding loss. Firm 1 surrenders surplus from its most valuable consumers, while its gain arises only from the moderately choosy segment with $t \in [\alpha^*, t^*)$—a segment whose value is constrained by its moderate willingness to pay and the competitive pressure from firm 2. Meanwhile, although firm 2 captures less choosy consumers, its profitability is also constrained: These consumers now end up with their less preferred product, which limits their willingness to pay. Proposition 3 verifies that these losses outweigh the gains under loyalty-based pricing $\mathcal{L}$, leading to $\Pi^{\mathcal{L}} < \Pi^{\mathcal{F}}$.

It is worth noting that since $\Pi^{\mathcal{C}} > \Pi^{\mathcal{U}}$ and $\Pi^{\mathcal{F}} > \Pi^{\mathcal{L}}$, the industry always benefits from observing consumers' choosiness t, regardless of the availability of information about $\boldsymbol{x}$. Intuitively, knowing t mitigates each firm's marginal–inframarginal trade-off, which enables more flexible pricing and more effective surplus extraction.

4 Case of Oligopoly

We now extend our analysis to the case with $n \geq 3$. As before, we focus on the comparison between loyalty-based pricing $\mathcal{L}$ and fully personalized pricing $\mathcal{F}$, since consumer welfare is maximized under one of these two regimes. We first outline the key preliminaries, which set the stage for our main predictions.

4.1 Preliminaries: Interim Consumer Welfare

Fix a realized profile of consumer brand preferences $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$, with $x_1 > \cdots > x_n$. With slight abuse of notation, let $V^{\mathcal{L}}(\boldsymbol{x})$ and $V^{\mathcal{F}}(\boldsymbol{x})$ denote the *interim* consumer welfare in the equilibrium under loyalty-based pricing $\mathcal{L}$ and fully personalized pricing $\mathcal{F}$, respectively. More formally, define

$$V^{\mathcal{L}}(\boldsymbol{x}) := \int_{\underline{t}}^{\bar{t}} \max\Big\{v + tx_1 - p_1^{\mathcal{L}}(\boldsymbol{x}), \ldots, v + tx_n - p_n^{\mathcal{L}}(\boldsymbol{x})\Big\}\, dF(t), \tag{7}$$

$$V^{\mathcal{F}}(\boldsymbol{x}) := \int_{\underline{t}}^{\bar{t}} (v + tx_2 - c) dF(t). \tag{8}$$

We conduct the following thought experiment. Suppose that a least preferred firm $n+1$, with $x_{n+1} < x_n$, is introduced to the market.[13] The following result ensues.

Lemma 4 ***(Interim Consumer Welfare and Number of Firms)*** *Fix $n \geq 2$. Let $\boldsymbol{x}_n \equiv (x_1, \ldots, x_n)$ with $x_1 > \cdots > x_n$ be a realized profile of consumer brand preferences, and define the augmented profile $\boldsymbol{x}_{n+1} := (\boldsymbol{x}_n, x_{n+1})$ with $x_{n+1} < x_n$. The interim equilibrium consumer welfare weakly increases under loyalty-based pricing $\mathcal{L}$ when the least preferred firm $n+1$ enters the market, while it remains unchanged under fully personalized pricing $\mathcal{F}$; that is,*

$$V^{\mathcal{L}}(\boldsymbol{x}_{n+1}) \geq V^{\mathcal{L}}(\boldsymbol{x}_n) \text{ and } V^{\mathcal{F}}(\boldsymbol{x}_{n+1}) = V^{\mathcal{F}}(\boldsymbol{x}_n).$$

For any realized $\boldsymbol{x}_n$, introducing a least preferred firm makes loyalty-based pricing more favorable relative to fully personalized pricing in terms of interim consumer welfare. Specifically, while consumer welfare remains unchanged under $\mathcal{F}$, it weakly improves under $\mathcal{L}$.

Recall from Lemma 1(iii) that firms engage in asymmetric Bertrand competition under fully personalized pricing $\mathcal{F}$; competition therefore reduces to head-to-head rivalry between the two most preferred firms. Consequently, the additional firm does not affect the market outcome. In contrast, as shown in Lemma 2, under loyalty-based pricing $\mathcal{L}$, the market can be split among *three or more firms* when $f(\underline{t}) < 1/\underline{t}$. Here, the additional firm may reshape the equilibrium, intensify market competition, and ultimately benefit consumers at the expense of firms.

---

[13] The thought experiment of "introducing or removing the least preferred firm" is used to compare the structural properties of equilibria across different market sizes. Technically, the mapping from a market of size $n+1$ to size n implied by this experiment is not measure-preserving: The marginal distribution of the top order statistics in a sample of size $n+1$ differs from that in a sample of size n. However, our subsequent welfare analysis accounts for this distinction. When evaluating ex-ante welfare, we explicitly integrate over the exact joint distribution of order statistics corresponding to the actual market size n. For results relying on pointwise dominance, the specific distribution is irrelevant; for those sensitive to distributional properties, the correct n-firm densities are applied.

This observation lays the foundation for our main results in an oligopoly.

4.2 Main Results

We now compare ex-ante expected consumer welfare under loyalty-based pricing $\mathcal{L}$ and fully personalized pricing $\mathcal{F}$ in a general oligopoly setting.

Proposition 4 ***(Consumer Welfare Comparison under Oligopoly)*** *Either fully personalized pricing $\mathcal{F}$ or loyalty-based pricing $\mathcal{L}$ maximizes consumer welfare. The following statements hold:*

(i) If $V^{\mathcal{L}} > V^{\mathcal{F}}$ when $n = 2$, then $V^{\mathcal{L}} > V^{\mathcal{F}}$ for all $n \geq 3$.[14]

(ii) If the brand-specific preferences x_i are independent and identically distributed with a weakly decreasing PDF $g^{\mathrm{m}}(\cdot)$, then $V^{\mathcal{L}} > V^{\mathcal{F}}$ whenever $n \geq 3$, regardless of the distribution $f(\cdot)$.

Comparing Proposition 4 with Proposition 2 reveals that increasing the number of firms beyond a duopoly strengthens the case for loyalty-based pricing $\mathcal{L}$.[15] By Proposition 4(i), if loyalty-based pricing $\mathcal{L}$ maximizes consumer welfare under a duopoly, this ranking remains unchanged in an oligopoly. Proposition 4(ii) further states that even if loyalty-based pricing underperforms when $n = 2$, the comparison may reverse in an oligopoly, provided that the brand-specific preferences x_i are independent and identically distributed with a weakly decreasing density.

As in the duopoly case, equilibrium industry profits can be unambiguously ranked across the four pricing regimes, and the ranking mirrors that in the duopoly setting.

Proposition 5 ***(Industry Profit Comparison under Oligopoly)*** *Fix $n \geq 3$. The equilibrium profits under the four pricing regimes can be ranked as follows: $\Pi^{\mathcal{C}} > \Pi^{\mathcal{U}} > \Pi^{\mathcal{F}} > \Pi^{\mathcal{L}}$. That is, choosiness-based pricing $\mathcal{C}$ maximizes industry profit, and loyalty-based pricing $\mathcal{L}$ minimizes it.*

---

[14] That is, $f(\underline{t}) \geq 1/\underline{t}$ or condition (5) holds.

[15] Interestingly, adopting an oligopoly framework, Zhou (2017) highlights the importance of market structure within the context of product bundling. Specifically, Zhou finds that compared with separate sales, bundling tends to raise market prices, thus benefiting firms while harming consumers as the number of firms increases.

4.3 Discussion

The observations of Proposition 4 can be intuitively interpreted in light of Lemma 4. Additional firms influence the market outcome more under loyalty-based pricing $\mathcal{L}$ than under fully personalized pricing $\mathcal{F}$, leading to greater improvement in $V^{\mathcal{L}}$ compared with $V^{\mathcal{F}}$ when the market involves more firms: Under $\mathcal{F}$, an entrant matters only if it ranks among a consumer's top two firms; in contrast, under $\mathcal{L}$, an entrant can increase market competition and benefit consumers even without being among the top two. The condition of Proposition 4(ii) reinforces this intuition. Let us continue the thought experiment depicted in Section 4.1. A weakly decreasing $g^{\mathrm{m}}(\cdot)$ implies that the additional least preferred firm's loyalty value is more likely to be close to x_n—that is, it is a closer substitute for firm n. This imposes stronger competitive pressure on existing firms from the new entrant under loyalty-based pricing $\mathcal{L}$, which intensifies price competition and renders $\mathcal{L}$ more likely to surpass fully personalized pricing $\mathcal{F}$ in terms of consumer welfare.

Furthermore, recall that the conditions in Proposition 2—which determine the comparison between $V^{\mathcal{L}}$ and $V^{\mathcal{F}}$ under duopoly—depend solely on the distribution of consumers' choosiness $f(t)$ and not on that of their brand loyalty $g(\boldsymbol{x})$. In a duopoly, the welfare comparison is pointwise, and only the difference between the two firms' loyalty values $|x_1 - x_2|$ matters (see Lemma 1(iii) and Corollary 1). This property no longer holds in an oligopoly, where the entire profile $(x_1, \ldots, x_n)$ may affect both equilibrium outcomes and the welfare comparison (see Lemma 2).

The industry profit ranking can be understood through a similar lens. Additional firms affect market outcomes more under loyalty-based pricing $\mathcal{L}$ than under fully personalized pricing $\mathcal{F}$. Consequently, the competitive forces generated by new entrants benefit consumers and reduce firms' profitability more severely under loyalty-based pricing $\mathcal{L}$ than under fully personalized pricing $\mathcal{F}$, yielding $\Pi^{\mathcal{F}} > \Pi^{\mathcal{L}}$.

5 Implications and Extensions

In this section, we first discuss the implications of our results for consumer data and competition policy. We then explore two extensions that relax our key assumptions.

5.1 Implications and Caveats

Our findings provide a normative foundation for the ongoing debate over consumer privacy and the calibration of regulatory frameworks. The central insight of our analysis is the asymmetric impact of inferences about consumer loyalty $\boldsymbol{x}$ versus choosiness t. We show that privacy protection is more effective at enhancing consumer welfare when regulatory scrutiny is tailored to the type of data being used.

A natural policy implication is that regulatory frameworks may distinguish between data types based on the inferences they enable (Tucker, 2024). While the debate over personal data trading in Europe has often polarized between two extremes (unconstrained access and a complete ban), the central policy challenge is to facilitate socially beneficial data sharing, as Heidhues et al. (2026) emphasize. Our results suggest that data that reveal brand-specific preferences (e.g., search queries or on-site interactions) may warrant less restrictive treatment than data that reveals a consumer's willingness to pay for a match (e.g., socioeconomic or demographic proxies). In our model, conditioning prices on loyalty can enhance contestability, whereas conditioning on choosiness facilitates surplus extraction by eliminating the marginal–inframarginal trade-off.

Our analysis provides a formal microfoundation consistent with distinctions articulated in recent regulatory discussions, including those by the UK's Competition and Markets Authority (CMA) that distinguish between personalization aimed at improving product matching (whereby firms learn what a consumer likes) and personalization aimed at extracting surplus (whereby firms learn how much a consumer can pay). In our framework, firms' ability to observe non-brand preference intensity t systematically raises industry profit. Similarly, recent OECD reports warn that personalized pricing may evolve from simple loyalty-based discounts toward economic profiling that infers consumers' financial sensitivity.

Our results thus align with the recent shift in regulatory discourse, identified by Tucker (2024), from behavior-based personalization ("what consumers do") toward identity-based exploitation ("who consumers are") that relies on inferences of consumer vulnerability. Legislative proposals such as New York's Algorithmic Pricing Disclosure Act reflect growing concern about pricing practices that incorporate proxies for wealth, location, or socioeconomic status. Likewise, the FTC has expressed increasing scrutiny of "surveillance pricing" systems that predict individual willingness to pay or consumer urgency.

The increasing use of algorithmic auditing by agencies such as the FTC and CMA provides a potential institutional mechanism for operationalizing such distinctions. By requiring transparency regarding the data inputs and inferential logic underlying pricing systems, regulators may reduce firms' ability to condition prices on choosiness t. In principle, limiting access to vertical willingness-to-pay signals while permitting behavioral loyalty inputs $\boldsymbol{x}$ can induce a shift toward a loyalty-based pricing regime. In our model, such a shift preserves market contestability while reinstating the marginal–inframarginal trade-off, thereby restoring downward price pressure from marginal consumers.

Nevertheless, translating these theoretical insights into practice requires caution. A key challenge arises from informational spillovers due to correlation across data sources. In reality, $\boldsymbol{x}$ and t may not be perfectly independent. For example, strong brand loyalty could be correlated with lower price sensitivity. If firms can partially infer choosiness from behavioral history, regulatory restrictions on explicit socioeconomic proxies may only imperfectly constrain surplus extraction. Platform firms with extensive interaction data may be particularly well positioned to reconstruct vertical willingness-to-pay signals indirectly. We examine such extensions in Section 5.2.1 and show that the qualitative implications of our analysis are robust to relaxing this independence assumption. Moreover, the opacity of modern machine-learning systems complicates oversight. Pricing engines may be sufficiently complex that even firms struggle to identify the precise drivers of price recommendations, which limits the effectiveness of auditing.

Despite these enforcement challenges, our framework provides a disciplined normative benchmark. Within our model, information about choosiness t is the principal driver of surplus extraction and consumer welfare losses relative to uniform pricing, whereas information about loyalty $\boldsymbol{x}$ can, under plausible conditions, enhance competitive pressure. Our analysis thus offers a structured way to think about regulating personalized pricing: not by asking whether personalization should be permitted, but by examining which dimensions of consumer characteristics firms are allowed to observe and exploit.

5.2 Robustness and Extensions

We now briefly discuss two extensions to our baseline model. First, we relax the independence assumption between consumer loyalty $\boldsymbol{x}$ and choosiness t. Second, we explore a setting with partial market coverage.

5.2.1 Correlated Consumer Characteristics

For the sake of tractability and expositional efficiency, our baseline analysis assumes independence between loyalty $\boldsymbol{x}$ and choosiness t. As stated above, we acknowledge the possibility that some data stream can support inferences about more than a single dimension of consumer characteristics, although different data streams can differ significantly in terms of how effectively they can inform each preference dimension. For instance, Miklós-Thal et al. (2024) allow firms to infer one aspect of a user's type based on data from another. Their study differs from ours in purpose: They examine users' data-sharing decisions, while we investigate the implications of personalized pricing for market and distribution outcomes under trade regulation rules.

Our qualitative rankings of consumer welfare and industry profit remain robust to correlated preferences, provided the conditional density of choosiness, $f(t \mid \boldsymbol{x})$, satisfies our baseline distributional assumptions.[16]

Proposition 6 ***(Correlated Consumer Characteristics)*** *Suppose that $\boldsymbol{x}$ and t are correlated, and the three assumptions outlined in Section 2.1.3 hold for the relevant conditional distributions. The following statements hold:*

(i) Among uniform pricing $\mathcal{U}$, choosiness-based pricing $\mathcal{C}$, and fully personalized pricing $\mathcal{F}$, consumer welfare is ranked as in the baseline model, i.e., $V^{\mathcal{F}} > V^{\mathcal{U}} > V^{\mathcal{C}}$.

(ii) Fix $n = 2$. If, for almost every realized loyalty vector $\boldsymbol{x}$, the conditional distribution $F(\cdot \mid \boldsymbol{x})$ satisfies the conditions in Proposition 2(i), then $V^{\mathcal{L}} > V^{\mathcal{F}}$. If it satisfies the conditions in Proposition 2(ii), then $V^{\mathcal{F}} > V^{\mathcal{L}}$.

(iii) Industry profit is ranked as in Proposition 3, i.e., $\Pi^{\mathcal{C}} > \Pi^{\mathcal{U}} > \Pi^{\mathcal{F}} > \Pi^{\mathcal{L}}$.

The profit ranking follows the same logic as in the baseline model and we focus below on the implications of correlation for consumer welfare. First, the comparison between uniform and fully personalized pricing regimes ($V^{\mathcal{U}}$ versus $V^{\mathcal{F}}$) follows directly from Rhodes and Zhou (2024), whose analysis does not require independence between $\boldsymbol{x}$ and t.

Second, the result whereby consumers prefer uniform pricing to choosiness-based pricing ($V^{\mathcal{U}} > V^{\mathcal{C}}$) is driven by the convexity of consumer surplus with respect to choosiness t. Variation in t induces price dispersion, which reduces aggregate consumer surplus relative to a uniform price set based on $\mathbb{E}[1/t]$. This logic continues to apply under correlation, provided that observing t does not asymmetrically update firms' beliefs about brand preferences $\boldsymbol{x}$, so that firms remain symmetric in equilibrium.

Third, the comparison between loyalty-based and fully personalized pricing ($V^{\mathcal{L}}$ versus $V^{\mathcal{F}}$) in a duopoly is pointwise in the brand-preference vector $\boldsymbol{x}$. Under loyalty-based pricing, equilibrium prices depend on the conditional distribution $f(t \mid \boldsymbol{x})$. As long as this conditional distribution satisfies condition (5), the ranking $V^{\mathcal{L}} > V^{\mathcal{F}}$ derived in Proposition 2 holds for each realization of $\boldsymbol{x}$, regardless of the strength of the correlation between $\boldsymbol{x}$ and t. However, as noted in Section 4.3, the comparison in an oligopoly is not pointwise. A full analysis allowing for correlation between t and $\boldsymbol{x}$ may involve substantial complications. Establishing the robustness of the ranking warrants further investigation, which we leave for future research.

---

[16] Imagine, for instance, that $\boldsymbol{x}$ and t are perfectly correlated, such that given $\boldsymbol{x}$, the conditional distribution of t reduces to a singleton. This scenario violates Assumption 2, and the two-dimensional heterogeneity effectively collapses to a one-dimensional model.

5.2.2 Partial Market Coverage

Our model can be readily extended to accommodate partial market coverage in equilibrium. A higher marginal cost c raises equilibrium prices and induces some consumers to opt for their outside option with zero surplus. By varying the marginal cost, we can adjust the degree of market coverage in equilibrium and examine how consumer welfare and industry profit compare across different pricing regimes.[17]

For brevity, we relegate the formal analysis of this case to the Supplemental Appendix. As the marginal cost becomes sufficiently large, uniform pricing $\mathcal{U}$ yields the highest consumer welfare, while fully personalized pricing $\mathcal{F}$ maximizes industry profit. These rankings contrast with those obtained under full market coverage, and the underlying intuition aligns with that of Rhodes and Zhou (2024) for the case of partial market coverage.

A very large c effectively filters out competition and renders each firm a local monopolist over the subset of consumers whose valuations exceed the cost: Conditional on a consumer whose value exceeds the large cost (i.e., $v + tx_i > c$ for some $i \in \mathcal{N}$), it becomes increasingly unlikely that the consumer simultaneously values another product more than the cost threshold. Consequently, for each firm, competition with other products is superseded by competition against the outside option. The conventional wisdom of monopolistic first-degree price discrimination (Pigou, 1920) is reinstated in this setting: Finer consumer information enables firms to extract more surplus, which benefits firms while harming consumers. In the absence of significant inter-firm competition, imperfect information—whether about loyalty or choosiness—prevents firms from perfectly profiling consumers and fully extracting their surplus.

6 Concluding Remarks

We analyze a general oligopoly model in which firms produce horizontally differentiated products and consumers differ along two key dimensions: brand-dependent preferences (loyalty) and brand-independent preferences (choosiness). Subject to prevailing consumer privacy regulations and competition policies, firms may draw perfect or imperfect inferences about consumer preferences and set prices accordingly. Four pricing regimes emerge: (i) uniform pricing, (ii) choosiness-based pricing, (iii) loyalty-based pricing, and (iv) fully personalized pricing.

We show that under full market coverage, consumer welfare is maximized under either fully personalized pricing (whereby firms condition prices on perfectly inferred consumer pref-

---

[17] Equivalently, market coverage can be varied by changing the base utility v. Decreasing v and increasing c are isomorphic in our setting.

erences) or loyalty-based pricing (whereby firms condition prices solely on each consumer's loyalty rather than choosiness). The latter regime is more likely to prevail in markets with a larger number of firms. In contrast, pricing based on consumers' choosiness always maximizes industry profit.

Our findings highlight the fundamentally different roles played by consumer information across distinct preference dimensions and their implications for pricing strategies. Specifically, learning about consumers' brand preferences (loyalty) renders individual consumers more contestable, whereas learning about their choosiness mitigates the marginal–inframarginal trade-off and enables more effective surplus extraction. This insight underscores the complexity of regulating consumer data protection and firms' pricing behavior and yields immediate policy implications. Evaluating the welfare consequences of commercial surveillance and personalized pricing requires moving beyond the binary question of whether consumer information should be shared with firms. Instead, a more nuanced inquiry should focus on which types of data should be shared and which should be subject to stricter regulation.

Ample room remains for future research. First, our analysis focuses on centralized information disclosure regimes and abstracts from consumers' endogenous choices to conceal or disclose personal information, e.g., through privacy settings on apps and devices. Our results suggest that consumers may have incentives to protect information that enables inferences about their non-brand preferences (t), while allowing disclosure of information that informs brand preferences ($\boldsymbol{x}$). Future research could examine consumers' optimal voluntary disclosure strategies under our two-dimensional framework.

Second, we do not consider decentralized information acquisition by firms, although our profit ranking result suggests that information regarding consumers' choosiness enables surplus extraction while information about loyalty intensifies competition. It remains an open question how competing firms optimally collect and use consumer data for their own benefit. A deeper understanding requires a comprehensive equilibrium analysis of a game in which firms collect information independently. Such an extension warrants further investigation.

Finally, we assume that each firm observes the entire loyalty vector $\boldsymbol{x}$ under loyalty-based or fully personalized pricing. A natural extension is to consider a setting in which firm i observes only consumers' loyalty toward its own product (i.e., x_i) but not toward rival products. A formal analysis of this setting is challenging, as it induces a complex Bayesian game arising from each firm's uncertainty about $\boldsymbol{x}_{-i}$. Specifically, consumer utility features a multiplicative interaction between choosiness t and brand preference x_i (i.e., tx_i). Unlike standard auction models or pricing games with additive noise, this multiplicative structure implies that a firm's probability of attracting a consumer depends on the convolution of t and the unobserved rival types $\boldsymbol{x}_{-i}$. Characterizing equilibrium pricing strategies in this environment appears analytically intractable without imposing highly restrictive assumptions. Nevertheless, such an extension remains theoretically intriguing and practically relevant, and represents a promising direction for future research.

References

Acquisti, A., C. Taylor, and L. Wagman (2016): "The economics of privacy," *Journal of Economic Literature*, 54, 442–492.

Ali, S. N., G. Lewis, and S. Vasserman (2023): "Voluntary disclosure and personalized pricing," *Review of Economic Studies*, 90, 538–571.

Anderson, S., A. Baik, and N. Larson (2023): "Price discrimination in the information age: Prices, poaching, and privacy with personalized targeted discounts," *Review of Economic Studies*, 90, 2085–2115.

Armstrong, M. (2006): *Recent Developments in the Economics of Price Discrimination*, Cambridge University Press, vol. 2 of *Econometric Society Monographs*, 97–141.

Armstrong, M. and J. Zhou (2022): "Consumer information and the limits to competition," *American Economic Review*, 112, 534–577.

Baye, M. R. and D. E. Sappington (2020): "Revealing transactions data to third parties: Implications of privacy regimes for welfare in online markets," *Journal of Economics & Management Strategy*, 29, 260–275.

Bergemann, D., B. Brooks, and S. Morris (2015): "The limits of price discrimination," *American Economic Review*, 105, 921–957.

——— (2025): "On the alignment of consumer surplus and total surplus under competitive price discrimination," *American Economic Journal: Microeconomics*, 17, 234–259.

Chen, Y. and G. Iyer (2002): "Research note consumer addressability and customized pricing," *Marketing Science*, 21, 197–208.

Chen, Y., C. Narasimhan, and Z. J. Zhang (2001): "Individual marketing with imperfect targetability," *Marketing Science*, 20, 23–41.

Chen, Z., C. Choe, and N. Matsushima (2020): "Competitive personalized pricing," *Management Science*, 66, 4003–4023.

Dubé, J.-P. and S. Misra (2023): "Personalized pricing and consumer welfare," *Journal of Political Economy*, 131, 131–189.

Elliott, M., A. Galeotti, A. Koh, and W. Li (2025): "Market segmentation through information," *Working Paper.*

Esteves, R.-B. and J. Resende (2016): "Competitive targeted advertising with price discrimination," *Marketing Science*, 35, 576–587.

Federal Trade Commission (2024): "FTC issues orders to eight companies seeking information on surveillance pricing," *Available at* `https://www.ftc.gov/news-events/news/press-releases/2024/07/ftc-issues-orders-eight-companies-seeking-information-surveillance-pricing`.

——— (2025): "FTC Surveillance Pricing 6(b) Study: Research Summaries: A Staff Perspective," *Available at* `https://www.ftc.gov/system/files/ftc_gov/pdf/p246202_surveillancepricing6bstudy_researchsummaries_redacted.pdf`.

Fudenberg, D. and J. Tirole (2000): "Customer poaching and brand switching," *RAND Journal of Economics*, 634–657.

Heidhues, P., N. Jacobson, G. Monti, and F. M. Scott Morton (2026): "Europeans should be allowed to trade personal data," Tech. Rep. 02/2026, Bruegel.

Ichihashi, S. (2020): "Online privacy and information disclosure by consumers," *American Economic Review*, 110, 569–595.

Iyer, G., D. Soberman, and J. M. Villas-Boas (2005): "The targeting of advertising," *Marketing Science*, 24, 461–476.

Liu, Q. and K. Serfes (2004): "Quality of information and oligopolistic price discrimination," *Journal of Economics & Management Strategy*, 13, 671–702.

Miklós-Thal, J., A. Goldfarb, A. M. Haviv, and C. Tucker (2024): "Frontiers: Digital hermits," *Marketing Science*, 43, 697–708.

OECD (2018): "Personalised pricing in the digital era," *Organisation for Economic Cooperation and Development Discussion Paper.*

Pigou, A. (1920): *The Economics of Welfare*, New York: MacMillan.

Rhodes, A. and J. Zhou (2024): "Personalized pricing and competition," *American Economic Review*, 114, 2141–2170.

Rott, P., J. Strycharz, and F. Alleweldt (2022): "Personalised pricing," *Available at* `https://www.europarl.europa.eu/RegData/etudes/STUD/2022/734008/IPOL_STU(2022)734008_EN.pdf`.

Sam, A. (2025): "Consumer control in competitive markets," *Working Paper.*

Shaffer, G. and Z. J. Zhang (1995): “Competitive coupon targeting,” *Marketing Science*, 14, 395–416.

——— (2002): “Competitive one-to-one promotions,” *Management Science*, 48, 1143–1160.

Shaked, A. and J. Sutton (1982): “Relaxing price competition through product differentiation,” *Review of Economic Studies*, 49, 3–13.

——— (1983): “Natural oligopolies,” *Econometrica*, 51, 1469–1483.

Shepardson, D. (2025): “Delta Air assures US lawmakers it will not personalize fares using AI,” Reuters.

Shiller, B. R. (2020): “Approximating purchase propensities and reservation prices from broad consumer tracking,” *International Economic Review*, 61, 847–870.

Siddiqui, F. (2024): “US FTC looking into targeted pricing based on personal data,” Reuters.

Spann, M., M. Bertini, O. Koenigsberg, R. Zeithammer, D. Aparicio, Y. Chen, F. Fantini, G. Z. Jin, V. Morwitz, P. P. Leszczyc, et al. (2025): “Algorithmic pricing: Implications for consumers, managers, and regulators,” *NBER Working Paper No. 32540*.

Thisse, J.-F. and X. Vives (1988): “On the strategic choice of spatial price policy,” *American Economic Review*, 78, 122–137.

Tucker, C. E. (2024): *The Economics of Privacy: An Agenda*, Chicago: University of Chicago Press, 5–20.

Yang, K. H. (2022): “Selling consumer data for profit: Optimal market-segmentation design and its consequences,” *American Economic Review*, 112, 1364–1393.

Zhou, J. (2017): “Competitive bundling,” *Econometrica*, 85, 145–172.

Appendix A: Proofs

We first state several intermediate results (the proofs can be found in the Supplemental Appendix).

Lemma A1 *Suppose that Assumption 2 holds. Then* $f(t)$*,* $F(t)$*, and* $1-F(t)$ *are all log-concave in* t*. Moreover, both* $\frac{1-F(t)}{f(t)}-t$ *and* $\frac{1-F(t)}{f(t)}-tF(t)$ *are strictly decreasing.*

Lemma A2 *Suppose that Assumption 2 holds and* $f(\underline{t})<1/\underline{t}$*. There exists a unique* $\alpha^*\in(\underline{t},\bar{t})$ *that solves* $\alpha^* f(\alpha^*)=1-2F(\alpha^*)$*.*

Lemma A3 *Fix* $\delta_1,\delta_2>0$ *and* $\bar{t}\geq\alpha>\alpha'\geq\underline{t}$*. The function*

$$B_{\delta_1,\delta_2}\left(\alpha,\alpha'\right):=\frac{F(\alpha)-F(\alpha')}{f(\alpha)/\delta_1+f(\alpha')/\delta_2}$$

is strictly increasing in α *and is strictly decreasing in* α'*.*

Proof of Lemma 1

Proof. We characterize firms' equilibrium pricing strategies under $\mathcal{U}$, $\mathcal{C}$, and $\mathcal{F}$, respectively.

Uniform pricing We focus on the symmetric equilibrium. Fixing $i\in\{1,\dots,n\}$ and $p_j=p^{\mathcal{U}}$ for each $j\neq i$, firm i's profit for charging price p_i is

$$\pi_i(p_i;\boldsymbol{p}_{-i})\big|_{p_j=p^{\mathcal{U}},\forall j\neq i}=(p_i-c)\Pr\left(p_i-p^{\mathcal{U}}\leq t\big(x_i-\max_{j\neq i}x_j\big)\right)=(p_i-c)\left[1-H(p_i-p^{\mathcal{U}})\right].$$

By Assumption 3, $1-H(p_i-p^{\mathcal{U}})$ is log-concave in p_i. This implies that $\pi_i(p_i;\boldsymbol{p}_{-i})$ is also log-concave in p_i, holding fixed $p_j=p^{\mathcal{U}}$ for all $j\neq i$. Therefore, the symmetric equilibrium is uniquely determined by the following first-order condition at $p_i=p^{\mathcal{U}}$:

$$0=\frac{\partial\pi_i(p_i,\boldsymbol{p}_{-i})}{\partial p_i}\bigg|_{p_j=p^{\mathcal{U}},\forall j\in\mathcal{N}}=1-H(0)-\left(p^{\mathcal{U}}-c\right)h(0),$$

from which we can obtain that $p^{\mathcal{U}}=c+\frac{1-H(0)}{h(0)}=c+\frac{1}{nh(0)}$.

The last equality follows from the assumption that the distribution of $\boldsymbol{x}$ is exchangeable, which implies that $1-H(0)=\Pr\left(x_i=\max_{j\in\mathcal{N}}x_j\right)=1/n$.

Recall that $h(\cdot)$ is the PDF of $z=t\hat{x}$. It holds that $h(0)=\int_0^1\frac{f(t)}{t}\psi(0)dt=\psi(0)\mathbb{E}\left[\frac{1}{t}\right]$, which implies that $p^{\mathcal{U}}=c+\frac{1}{nh(0)}=c+\frac{1}{n\psi(0)\mathbb{E}[\frac{1}{t}]}$.

Choosiness-based pricing Fix t and suppose $p_j = p^{\mathcal{C}}(t)$ for each $j \neq i$. Similar to the case of uniform pricing, firm i's profit under Assumption 1 is log-concave. By the first-order condition for firm i, the equilibrium price can be solved as $p^{\mathcal{C}}(t) = c + \frac{t[1-\Psi(0)]}{\psi(0)} = c + \frac{t}{n\psi(0)}$.

Fully personalized pricing Fixing $(\boldsymbol{x}, t)$, with $x_1 > \cdots > x_n$, the firms engage in standard asymmetric Bertrand competition and the equilibrium pricing schedules are $p_1^{\mathcal{F}}(\boldsymbol{x}, t) = c + t(x_1 - x_2)$, $p_2^{\mathcal{F}}(\boldsymbol{x}, t) = c$, and $p_i^{\mathcal{F}}(\boldsymbol{x}, t) \geq c$ for $i \in \{3, \ldots, n\}$. ■

Proof of Lemma 2

Proof. We show that firms' equilibrium strategies are as follows:

(i) When $f(\underline{t}) \geq 1/\underline{t}$, it holds that $k(\boldsymbol{x}) = 1$ and the equilibrium prices satisfy

$$p_1^{\mathcal{L}}(\boldsymbol{x}) = c + \underline{t}(x_1 - x_2), p_2^{\mathcal{L}}(\boldsymbol{x}) = c, \text{ and } p_i^{\mathcal{L}}(\boldsymbol{x}) \geq c \text{ for all } i \in \{3, \ldots, n\}. \tag{9}$$

(ii) When $f(\underline{t}) < 1/\underline{t}$, it holds that $k(\boldsymbol{x}) \geq 2$. In equilibrium, firms 1 to $k(\boldsymbol{x})$ have positive demand, while firms $k(\boldsymbol{x})+1$ to n have zero demand and thus earn zero profits. Without loss of generality, we set $p_{k(\boldsymbol{x})+1}^{\mathcal{L}}(\boldsymbol{x}) = \cdots = p_n^{\mathcal{L}}(\boldsymbol{x}) = c$. By arguments similar to the case of $f(\underline{t}) \geq 1/\underline{t}$, there may exist other equilibria. For example, when $k(\boldsymbol{x}) \leq n-2$, firms $k(\boldsymbol{x}) + 2$ to n can charge an arbitrary price above their marginal cost c. Again, all these equilibria are outcome equivalent.

For firms with positive demand, there exist a set of cutoffs $\big(\alpha_1(\boldsymbol{x}), \ldots, \alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\big)$, with $\alpha_0(\boldsymbol{x}) := \bar{t} > \alpha_1(\boldsymbol{x}) > \cdots > \alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x}) > \alpha_{k(\boldsymbol{x})}(\boldsymbol{x}) := \underline{t}$, such that a consumer purchases from firm i if and only if her choosiness level $t \in \big[\alpha_i(\boldsymbol{x}), \alpha_{i-1}(\boldsymbol{x})\big)$. Moreover, the equilibrium prices $\big(p_1^{\mathcal{L}}(\boldsymbol{x}), \ldots, p_{k(\boldsymbol{x})}^{\mathcal{L}}(\boldsymbol{x})\big)$, the set of cutoffs $\big(\alpha_1(\boldsymbol{x}), \ldots, \alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\big)$, and the number of firms with positive demand $k(\boldsymbol{x})$ are uniquely pinned down by the following conditions:

(a) First-order conditions for profit maximization for the first $k(\boldsymbol{x}) - 1$ firms:

$$p_1^{\mathcal{L}}(\boldsymbol{x}) = c + \frac{1 - F\left(\alpha_1(\boldsymbol{x})\right)}{\frac{f\left(\alpha_1(\boldsymbol{x})\right)}{x_2 - x_1}}, \text{ and} \tag{10}$$

$$p_i^{\mathcal{L}}(\boldsymbol{x}) = c + \frac{F\left(\alpha_{i-1}(\boldsymbol{x})\right) - F\left(\alpha_i(\boldsymbol{x})\right)}{\frac{f\left(\alpha_{i-1}(\boldsymbol{x})\right)}{x_{i-1} - x_i} + \frac{f\left(\alpha_i(\boldsymbol{x})\right)}{x_{i+1} - x_i}}, i \in \big\{2, \ldots, k(\boldsymbol{x}) - 1\big\}. \tag{11}$$

(b) Karush-Kuhn-Tucker (KKT) conditions for the $k(\boldsymbol{x})$-th firm:

$$p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x}) = c + \frac{F\left(\alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\right)}{\frac{f\left(\alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\right)}{x_{k(\boldsymbol{x})-1}-x_{k(\boldsymbol{x})}} + \xi f(\underline{t})}, \xi \in \left[0, \frac{1}{x_{k(\boldsymbol{x})} - x_{k(\boldsymbol{x})+1}}\right], \tag{12}$$

$$p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x}) \leq c + \underline{t}\left(x_{k(\boldsymbol{x})} - x_{k(\boldsymbol{x})+1}\right), \tag{13}$$

$$\xi \times \left[\underline{t}\left(x_{k(\boldsymbol{x})} - x_{k(\boldsymbol{x})+1}\right) + c - p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x})\right] = 0, \tag{14}$$

where $x_{n+1} := -\infty$ if $k(\boldsymbol{x}) = n$.

(c) Indifference conditions of consumers with pivotal choosiness levels:

$$p^{\mathcal{L}}_i(\boldsymbol{x}) = p^{\mathcal{L}}_{i+1}(\boldsymbol{x}) + \alpha_i(\boldsymbol{x})(x_i - x_{i+1}), i \in \left\{1, \ldots, k(\boldsymbol{x}) - 1\right\}. \tag{15}$$

Fix $\boldsymbol{x} = (x_1, \ldots, x_n)$ and, without loss of generality, order firms so that $x_1 > \cdots > x_n$. The proof proceeds in six steps. Steps I and II establish the demand geometry under loyalty-based pricing: In any equilibrium, the active firms must be the highest-loyalty firms, and their market shares are separated by cutoffs in the choosiness dimension. Step III establishes the log-concavity of each active firm's profit. Step IV derives the first-order conditions for all non-marginal active firms and the KKT condition for the marginal active firm. Steps V and VI then prove existence and uniqueness, separately for the cases $f(\underline{t}) \geq 1/\underline{t}$ and $f(\underline{t}) < 1/\underline{t}$.

Step I We first show that the active firms, if any, must be consecutive from the top of the loyalty ranking. Suppose, to the contrary, that firm i is active while firm $j < i$ is inactive. Since $x_j > x_i$, firm j offers a higher gross value than firm i to every consumer with $t > 0$. By deviating to firm i's price, firm j attracts all consumers served by firm i and earns strictly positive profit, a contradiction. Hence the active firms must be firms $1, \ldots, k(\boldsymbol{x})$ for some $k(\boldsymbol{x})$.

Step II Fix the number of active firms $k(\boldsymbol{x})$ and their price profile $(p^{\mathcal{L}}_1(\boldsymbol{x}), \ldots, p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x}))$. For each $i = 1, \ldots, k(\boldsymbol{x}) - 1$, define $\alpha_i(\boldsymbol{x}) := \frac{p^{\mathcal{L}}_i(\boldsymbol{x}) - p^{\mathcal{L}}_{i+1}(\boldsymbol{x})}{x_i - x_{i+1}}$. A consumer prefers firm i to firm $i+1$ if and only if $t \geq \alpha_i(\boldsymbol{x})$. Hence, by single crossing, if all firms $1, \ldots, k(\boldsymbol{x})$ are active, then their demand intervals are ordered by cutoffs

$$\alpha_0(\boldsymbol{x}) := \overline{t} > \alpha_1(\boldsymbol{x}) > \cdots > \alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x}) > \alpha_{k(\boldsymbol{x})}(\boldsymbol{x}) := \underline{t},$$

and firm i serves consumers with $t \in [\alpha_i(\boldsymbol{x}), \alpha_{i-1}(\boldsymbol{x}))$. The definition of $\alpha_i(\boldsymbol{x})$ is equivalent to the indifference condition (15).

Step III. We next show that each active firm's profit is log-concave in its own price. For $i \leq k(\boldsymbol{x})$, firm i's profit is

$$\pi_i(p_i, \boldsymbol{p}_{-i}; \boldsymbol{x}) = (p_i - c) \Pr\left(\min_{j<i} \left\{ \frac{p_j - p_i}{x_j - x_i}, \bar{t} \right\} > t \geq \max_{j>i} \left\{ \frac{p_i - p_j}{x_i - x_j}, \underline{t} \right\} \right).$$

Consider a region in which the two bounds are determined by unique firms $j_1 < i$ and $j_2 > i$. Then $\pi_i(p_i, \boldsymbol{p}_{-i}; \boldsymbol{x}) = (p_i - c)\mathcal{D}_i(p_i, \boldsymbol{p}_{-i}; \boldsymbol{x})$, where

$$\mathcal{D}_i(p_i, \boldsymbol{p}_{-i}; \boldsymbol{x}) := F\left(\frac{p_{j_1} - p_i}{x_{j_1} - x_i} \right) - F\left(\frac{p_i - p_{j_2}}{x_i - x_{j_2}} \right).$$

The boundary cases are handled by the same one-sided derivative argument. We use the following auxiliary lemma, proved in the Supplemental Appendix.

Lemma A4 *Under Assumption 2, $\mathcal{D}_i(p_i, \boldsymbol{p}_{-i}; \boldsymbol{x})$ is log-concave in p_i.*

Since $p_i - c$ is also log-concave, Lemma A4 implies that $\pi_i(p_i, \boldsymbol{p}_{-i}; \boldsymbol{x})$ is log-concave in p_i.

Step IV. We derive the first-order and KKT conditions. By Step II, the relevant competitors of any interior active firm $i = 2, \ldots, k(\boldsymbol{x}) - 1$ are firms $i - 1$ and $i + 1$. Hence firm i's profit is differentiable at $p_i^{\mathcal{L}}(\boldsymbol{x})$, and its first-order condition is

$$\begin{aligned} 0 = &\Big[F\left(\alpha_{i-1}(\boldsymbol{x})\right) - F\left(\alpha_i(\boldsymbol{x})\right) \Big] \\ &- \left(p_i^{\mathcal{L}}(\boldsymbol{x}) - c \right) \left[\frac{f\left(\alpha_{i-1}(\boldsymbol{x})\right)}{x_{i-1} - x_i} + \frac{f\left(\alpha_i(\boldsymbol{x})\right)}{x_i - x_{i+1}} \right], \end{aligned}$$

which is equivalent to (11). The same argument for firm 1 gives (10).

It remains to derive the KKT condition for the marginal active firm $k(\boldsymbol{x})$. First, (13) must hold. Otherwise, if $p_{k(\boldsymbol{x})}^{\mathcal{L}}(\boldsymbol{x}) > c + \underline{t}\left(x_{k(\boldsymbol{x})} - x_{k(\boldsymbol{x})+1}\right)$, firm $k(\boldsymbol{x}) + 1$ could charge a price in $\left(c,\, p_{k(\boldsymbol{x})}^{\mathcal{L}}(\boldsymbol{x}) - \underline{t}\left(x_{k(\boldsymbol{x})} - x_{k(\boldsymbol{x})+1}\right)\right)$ and obtain positive demand and positive profit, a contradiction.

It remains to derive (12) and (14). Let $\tilde{p}_{k(\boldsymbol{x})}$ denote a deviation price of firm $k(\boldsymbol{x})$. If (13) is slack, then for all $\tilde{p}_{k(\boldsymbol{x})}$ in a neighborhood of $p_{k(\boldsymbol{x})}^{\mathcal{L}}(\boldsymbol{x})$, firm $k(\boldsymbol{x})$'s profit is

$$\left(\tilde{p}_{k(\boldsymbol{x})} - c\right) F\left(\frac{p_{k(\boldsymbol{x})-1}^{\mathcal{L}}(\boldsymbol{x}) - \tilde{p}_{k(\boldsymbol{x})}}{x_{k(\boldsymbol{x})-1} - x_{k(\boldsymbol{x})}} \right).$$

The first-order condition is therefore

$$F\left(\alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\right) - \left(p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x}) - c\right) \frac{f\left(\alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\right)}{x_{k(\boldsymbol{x})-1} - x_{k(\boldsymbol{x})}} = 0,$$

which is (12) with $\xi = 0$; (14) also holds because (13) is slack.

If (13) binds, firm $k(\boldsymbol{x})$ is at a kink. The left and right derivatives of its profit at $p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x})$ must satisfy

$$F\left(\alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\right) - \left(p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x}) - c\right) \frac{f\left(\alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\right)}{x_{k(\boldsymbol{x})-1} - x_{k(\boldsymbol{x})}} \geq 0$$

and

$$F\left(\alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\right) - \left(p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x}) - c\right) \left[\frac{f\left(\alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\right)}{x_{k(\boldsymbol{x})-1} - x_{k(\boldsymbol{x})}} + \frac{f(\underline{t})}{x_{k(\boldsymbol{x})} - x_{k(\boldsymbol{x})+1}}\right] \leq 0.$$

These two inequalities are equivalent to (12) for some

$$\xi \in \left[0, \frac{1}{x_{k(\boldsymbol{x})} - x_{k(\boldsymbol{x})+1}}\right],$$

and (14) follows from the binding constraint.

Step V. Suppose $f(\underline{t}) \geq 1/\underline{t}$. We show that no equilibrium can have $k(\boldsymbol{x}) \geq 2$. Suppose, to the contrary, that such an equilibrium exists. Combining (10) and (15) gives

$$c + \frac{1 - F(\alpha_1(\boldsymbol{x}))}{f(\alpha_1(\boldsymbol{x}))}(x_1 - x_2) = p^{\mathcal{L}}_1(\boldsymbol{x}) = p^{\mathcal{L}}_2(\boldsymbol{x}) + \alpha_1(\boldsymbol{x})(x_1 - x_2) \geq c + \alpha_1(\boldsymbol{x})(x_1 - x_2),$$

and hence $\frac{1-F(\alpha_1(\boldsymbol{x}))}{f(\alpha_1(\boldsymbol{x}))} - \alpha_1(\boldsymbol{x}) \geq 0$. However, by Lemma A1, $(1 - F(\alpha))/f(\alpha) - \alpha$ is strictly decreasing in α. Since $\alpha_1(\boldsymbol{x}) > \underline{t}$ and $f(\underline{t}) \geq 1/\underline{t}$, $\frac{1-F(\alpha_1(\boldsymbol{x}))}{f(\alpha_1(\boldsymbol{x}))} - \alpha_1(\boldsymbol{x}) < \frac{1-F(\underline{t})}{f(\underline{t})} - \underline{t} \leq 0$, a contradiction. Thus $k(\boldsymbol{x}) = 1$.

Step VI. Suppose $f(\underline{t}) < 1/\underline{t}$. We prove existence and uniqueness by reducing the equilibrium conditions to a one-dimensional shooting problem in α_1. Fix $\boldsymbol{x}$ and fix an arbitrary $\alpha_1 \in [\underline{t}, \overline{t}]$. We recursively construct candidate prices and cutoffs as follows. First, let

$$\acute{p}_1(\alpha_1) := c + \frac{1 - F(\alpha_1)}{f(\alpha_1)}(x_1 - x_2), \qquad \acute{\alpha}_1(\alpha_1) := \alpha_1. \tag{16}$$

Given $\acute{p}_j(\alpha_1)$ and $\acute{\alpha}_j(\alpha_1)$, define

$$\acute{p}_{j+1}(\alpha_1) := \acute{p}_j(\alpha_1) - \acute{\alpha}_j(\alpha_1)(x_j - x_{j+1}). \tag{17}$$

If firm $j+1$ is treated as an interior active firm, define $\acute{\alpha}_{j+1}(\alpha_1)$ as the unique solution a to

$$\acute{p}_{j+1}(\alpha_1) = c + \frac{F(\acute{\alpha}_j(\alpha_1)) - F(a)}{\frac{f(\acute{\alpha}_j(\alpha_1))}{x_j - x_{j+1}} + \frac{f(a)}{x_{j+1} - x_{j+2}}}. \tag{18}$$

If such an interior continuation is not possible, we set $\acute{\alpha}_{j+1}(\alpha_1) = \underline{t}$ and treat firm $j+1$ as the candidate marginal active firm. Equivalently, the recursion stops at the first j for which either

$$\acute{p}_j(\alpha_1) - \acute{\alpha}_j(\alpha_1)(x_j - x_{j+1}) \leq c \tag{19}$$

or

$$\acute{p}_j(\alpha_1) \geq c + \frac{F(\acute{\alpha}_{j-1}(\alpha_1))}{\frac{f(\acute{\alpha}_{j-1}(\alpha_1))}{x_{j-1} - x_j} + \frac{f(\underline{t})}{x_j - x_{j+1}}}. \tag{20}$$

Let $k(\alpha_1)$ denote the candidate marginal active firm generated by this recursion.

By construction, for any fixed α_1, the candidate prices and cutoffs generated above satisfy (10), (11), and (15) for all non-marginal active firms. Therefore, the candidate profile is an equilibrium if and only if the terminal KKT conditions (12)–(14) are satisfied by firm $k(\alpha_1)$.

The following monotonicity property is the key to uniqueness. Its proof is in the Supplemental Appendix.

Lemma A5 *For each candidate active firm i, $\acute{p}_i(\alpha_1)$ is decreasing in α_1, and $\acute{\alpha}_i(\alpha_1)$ is increasing in α_1.*

It remains to show that there is a unique value of α_1 for which the terminal KKT conditions hold. We use the following shooting lemma, whose full proof is in the Supplemental Appendix.

Lemma A6 *Suppose $f(\underline{t}) < 1/\underline{t}$. There exists a unique $\alpha_1^* \in [\underline{t}, \overline{t}]$ such that the candidate prices and cutoffs generated by* (16)–(20) *satisfy the terminal KKT conditions* (12)–(14)*.*

The idea behind Lemma A6 is as follows. If α_1 is too small, then by Lemma A5 the constructed prices are too high and the constructed cutoffs too low. The candidate marginal active firm then cannot deter entry by the next firm, so (13) fails. If α_1 is too large, the constructed prices are too low and the cutoffs are too high, so the marginal firm's own-price optimality condition (12) fails. Hence only an intermediate interval of values of α_1 can support an equilibrium. On this interval, the candidate active set is fixed; denote the marginal active firm by k^*. Define the terminal FOC residual by

$$R^{\text{FOC}}(\alpha_1) := \acute{p}_{k^*}(\alpha_1) - c - \frac{F(\acute{\alpha}_{k^*-1}(\alpha_1))}{f(\acute{\alpha}_{k^*-1}(\alpha_1))/(x_{k^*-1} - x_{k^*})}.$$

The monotonicity above implies that R^{FOC} is decreasing. If (13) is slack, the terminal KKT condition is $R^{\mathrm{FOC}}(\alpha_1) = 0$, which has at most one solution. If (13) binds, the solution is the unique value of α_1 at which

$$\acute{p}_{k^*}(\alpha_1) = c + \underline{t}(x_{k^*} - x_{k^*+1}).$$

The endpoint comparisons establish existence, and the monotonicity argument establishes uniqueness.

By Lemma A6, let α_1^* be the unique value that satisfies the terminal KKT conditions. The candidate prices and cutoffs generated by α_1^* satisfy (10), (11), (12)–(14), and (15). By the log-concavity established in Step III, these conditions are sufficient for equilibrium. Conversely, any equilibrium must satisfy (10), (11), and (15), and hence must be generated by the above recursion for its value of α_1. Lemma A6 then implies that this value must be α_1^*. Therefore, the equilibrium outcome is unique. ■

Proof of Proposition 1

Proof. See main text. ■

Proof of Lemma 3

Proof. Applying Proposition 1 in Rhodes and Zhou (2024) yields $V^{\mathcal{U}} < V^{\mathcal{F}}$. It remains to show $V^{\mathcal{C}} < V^{\mathcal{U}}$. By Proposition 1, $W^{\mathcal{U}} = W^{\mathcal{C}}$. Moreover, note that

$$p^{\mathcal{U}} = c + \frac{1}{n\psi(0)\mathbb{E}[\frac{1}{t}]} < c + \frac{\mathbb{E}(t)}{n\psi(0)} = \mathbb{E}\big[p^{\mathcal{C}}(t)\big],$$

from which we can conclude that $V^{\mathcal{C}} < V^{\mathcal{U}}$. ■

Proof of Proposition 2

Proof. Without loss of generality, consider the case of $x_1 > x_2$. Recall from (7) and (8) that $V^{\mathcal{L}}(\boldsymbol{x})$ and $V^{\mathcal{F}}(\boldsymbol{x})$ are the interim consumer welfare. It suffices to show that the comparison between $V^{\mathcal{L}}(\boldsymbol{x})$ and $V^{\mathcal{F}}(\boldsymbol{x})$ holds pointwisely. Consider the following two cases:

Case (a): $\boldsymbol{f(\underline{t}) \geq 1/\underline{t}}$.

By Lemma 1 and Corollary 1, fixing $x_1 > x_2$, consumers with (x_1, x_2) always buy from firm 1; moreover, we have that

$$p_1^{\mathcal{F}}(\boldsymbol{x}, t) = c + t(x_1 - x_2) \geq c + \underline{t}(x_1 - x_2) = p_1^{\mathcal{L}}(\boldsymbol{x}) \text{ and } p_2^{\mathcal{F}}(\boldsymbol{x}, t) = p_2^{\mathcal{L}}(\boldsymbol{x}) = c, \forall\, t \in [\underline{t}, \overline{t}].$$

Therefore, $V^{\mathcal{L}} > V^{\mathcal{F}}$.

Case (b): $f(\underline{t}) < 1/\underline{t}$.

By Corollary 1, under loyalty-based pricing, the consumer buys from firm 1 if and only if $t \geq \alpha^*$. The consumer welfare is

$$V^{\mathcal{L}}(\boldsymbol{x}) = \int_{\underline{t}}^{\alpha^*} \left[v + tx_2 - p_2^{\mathcal{L}}(\boldsymbol{x})\right] f(t)dt + \int_{\alpha^*}^{\bar{t}} \left[v + tx_1 - p_1^{\mathcal{L}}(\boldsymbol{x})\right] f(t)dt. \tag{21}$$

Similarly, consumers' welfare at $\boldsymbol{x}$ under fully personalized pricing amounts to

$$V^{\mathcal{F}}(\boldsymbol{x}) = \int_{\underline{t}}^{\bar{t}} \left[v + tx_1 - p_1^{\mathcal{F}}(\boldsymbol{x}, t)\right] f(t)dt = \int_{\underline{t}}^{\bar{t}} \left[v + tx_2 - c\right] f(t)dt. \tag{22}$$

Subtracting (22) from (21) and carrying out the algebra, we can obtain that

$$V^{\mathcal{L}}(\boldsymbol{x}) - V^{\mathcal{F}}(\boldsymbol{x}) = (x_1 - x_2)\left[\int_{\alpha^*}^{\bar{t}} \left[1 - F(t)\right] dt - \frac{F(\alpha^*)}{f(\alpha^*)}\right].$$

Recall the postulated $x_1 > x_2$. Therefore, $V^{\mathcal{L}}(\boldsymbol{x}) > V^{\mathcal{F}}(\boldsymbol{x})$ is equivalent to (5). This concludes the proof. ■

Proof of Proposition 3

Proof. By Proposition 1, $W^{\mathcal{F}} = W^{\mathcal{U}} = W^{\mathcal{C}}$. By Lemma 3, $V^{\mathcal{F}} > V^{\mathcal{U}} > V^{\mathcal{C}}$. Therefore, $\Pi^{\mathcal{F}} < \Pi^{\mathcal{U}} < \Pi^{\mathcal{C}}$ and it suffices to show $\Pi^{\mathcal{F}} > \Pi^{\mathcal{L}}$.

Denote firms' interim equilibrium profits at $\boldsymbol{x}$ under $\mathcal{L}$ and $\mathcal{F}$ by $\Pi^{\mathcal{L}}(\boldsymbol{x})$ and $\Pi^{\mathcal{F}}(\boldsymbol{x})$, respectively. It suffices to show that the comparison between $\Pi^{\mathcal{L}}(\boldsymbol{x})$ and $\Pi^{\mathcal{F}}(\boldsymbol{x})$ holds pointwisely. The proof for the case of $f(\underline{t}) \geq 1/\underline{t}$ is obvious and we focus on the case of $f(\underline{t}) < 1/\underline{t}$. $\Pi^{\mathcal{L}}(\boldsymbol{x})$ and $\Pi^{\mathcal{F}}(\boldsymbol{x})$ can be expressed as

$$\Pi^{\mathcal{L}}(\boldsymbol{x}) = |x_1 - x_2| \times \left(\frac{F(\alpha^*)}{f(\alpha^*)} + \alpha^* \left[1 - F(\alpha^*)\right]\right) - c,$$

and

$$\Pi^{\mathcal{F}}(\boldsymbol{x}) = \mathbb{E}_t\left[t|x_1 - x_2|\right] - c = |x_1 - x_2| \times \int_{\underline{t}}^{\bar{t}} tf(t)dt - c.$$

Therefore, it suffices to show that

$$\int_{\underline{t}}^{\bar{t}} tf(t)dt \geq \frac{F(\alpha^*)}{f(\alpha^*)} + \alpha^* \left[1 - F(\alpha^*)\right]. \tag{23}$$

It is useful to state the following two lemmata (whose proofs can be found in the Supplemental Appendix).

Lemma A7 *Fixing $f(\cdot)$, there exists a distribution supported on $[0,\bar{t}^\dagger]$ with CDF $F^\dagger(\cdot)$ and PDF $f^\dagger(\cdot)$, such that $\ln\frac{f^\dagger(t)}{t}$ is linear on $[0,\bar{t}^\dagger]$ and satisfies*

$$f^\dagger(\alpha^*) = f(\alpha^*), F^\dagger(\alpha^*) = F(\alpha^*), \tag{24}$$

and

$$\int_{\underline{t}}^{\bar{t}} tf(t)dt \geq \int_0^{\bar{t}^\dagger} tf^\dagger(t)dt. \tag{25}$$

Lemma A8 *Let $F^\dagger$ be the distribution constructed in Lemma A7. Then*

$$\int_0^{\bar{t}^\dagger} tf^\dagger(t)dt \geq \frac{F^\dagger(\alpha^*)}{f^\dagger(\alpha^*)} + \alpha^*\left[1 - F^\dagger(\alpha^*)\right]. \tag{26}$$

By Lemma A7, the left-hand side of (23) decreases and the right-hand side remains unchanged if we replace $F(\cdot)$ and $f(\cdot)$ with $F^\dagger(\cdot)$ and $f^\dagger(\cdot)$, respectively. By Lemma A8, (23) holds for the constructed distribution $F^\dagger(\cdot)$. Therefore, (23) holds for the distribution $F(\cdot)$. This completes the proof. ■

Proof of Lemma 4

Proof. Fix $\boldsymbol{x}_n \equiv (x_1, \ldots, x_n)$ with $x_1 > \cdots > x_n$, and consider adding one more firm with $x_{n+1} < x_n$. Under fully personalized pricing, consumers always buy from firm 1, and firm 1's price depends only on $x_1 - x_2$. Hence$V^{\mathcal{F}}(\boldsymbol{x}_{n+1}) = V^{\mathcal{F}}(\boldsymbol{x}_n)$. It therefore remains to show that $V^{\mathcal{L}}(\boldsymbol{x}_{n+1}) \geq V^{\mathcal{L}}(\boldsymbol{x}_n)$.

Define the auxiliary welfare

$$\widetilde{V}^{\mathcal{L}}(\boldsymbol{x}_{n+1}) := \mathbb{E}_t\left[\max_{i\in\{1,\ldots,n\}}\left\{v + tx_i - p_i^{\mathcal{L}}(\boldsymbol{x}_{n+1})\right\}\right].$$

That is, $\widetilde{V}^{\mathcal{L}}(\boldsymbol{x}_{n+1})$ is the consumer welfare when consumers can choose only among the original n firms, but these firms charge their equilibrium loyalty-based prices in the $(n+1)$-firm market. Then

$$V^{\mathcal{L}}(\boldsymbol{x}_{n+1}) - V^{\mathcal{L}}(\boldsymbol{x}_n) = \underbrace{V^{\mathcal{L}}(\boldsymbol{x}_{n+1}) - \widetilde{V}^{\mathcal{L}}(\boldsymbol{x}_{n+1})}_{\text{choice effect}} + \underbrace{\widetilde{V}^{\mathcal{L}}(\boldsymbol{x}_{n+1}) - V^{\mathcal{L}}(\boldsymbol{x}_n)}_{\text{price effect}}.$$

We show that both effects are nonnegative.

First, the choice effect is nonnegative because the first maximum below is taken over a larger set:

$$\begin{aligned}&V^{\mathcal{L}}(\boldsymbol{x}_{n+1})-\widetilde{V}^{\mathcal{L}}(\boldsymbol{x}_{n+1})\\&=\mathbb{E}_t\left[\max_{i\in\{1,\ldots,n+1\}}\left\{v+tx_i-p_i^{\mathcal{L}}(\boldsymbol{x}_{n+1})\right\}-\max_{i\in\{1,\ldots,n\}}\left\{v+tx_i-p_i^{\mathcal{L}}(\boldsymbol{x}_{n+1})\right\}\right]\geq 0.\end{aligned}$$

Next, consider the price effect. We use the following auxiliary lemma, whose proof is in the Supplemental Appendix.

Lemma A9 *For each* $i=1,\ldots,n$, $p_i^{\mathcal{L}}(\boldsymbol{x}_n,x_{n+1})$ *is weakly decreasing in* x_{n+1}. *Consequently,* $\widetilde{V}^{\mathcal{L}}(\boldsymbol{x}_n,x_{n+1})$ *is weakly increasing in* x_{n+1}.

Note that $\widetilde{V}^{\mathcal{L}}(\boldsymbol{x}_n,-\infty)=V^{\mathcal{L}}(\boldsymbol{x}_n)$. By Lemma A9, we have that $\widetilde{V}^{\mathcal{L}}(\boldsymbol{x}_n,x_{n+1})-V^{\mathcal{L}}(\boldsymbol{x}_n)\geq 0$. This completes the proof. ■

Proof of Proposition 4

Proof. We first prove part (i) of the proposition. Fix $\boldsymbol{x}\equiv(x_1,\ldots,x_n)$ and assume $x_1>\cdots>x_n$ without loss of generality. By Lemma 4, we have that

$$V^{\mathcal{L}}(\boldsymbol{x})\geq V^{\mathcal{L}}(\boldsymbol{x}_{n-1})\geq\cdots\geq V^{\mathcal{L}}(x_1,x_2)\text{ and }V^{\mathcal{F}}(\boldsymbol{x})=V^{\mathcal{F}}(\boldsymbol{x}_{n-1})=\cdots=V^{\mathcal{F}}(x_1,x_2).$$

Recall from Proposition 2 that the consumer welfare comparison between $\mathcal{L}$ and $\mathcal{F}$ holds pointwisely for the case of $n=2$. Therefore, we have that $V^{\mathcal{L}}(x_1,x_2)>V^{\mathcal{F}}(x_1,x_2)$, which in turn implies that $V^{\mathcal{L}}(\boldsymbol{x})>V^{\mathcal{F}}(\boldsymbol{x})$.

Next, we prove part (ii) of the proposition. If $V^{\mathcal{L}}>V^{\mathcal{F}}$ in the duopoly case, the result follows from part (i). Hence, in the rest of the proof, suppose that $V^{\mathcal{F}}>V^{\mathcal{L}}$ in the duopoly case. We first state the following intermediate result (whose proof is obvious and is omitted for brevity):

Lemma A10 *The consumer welfare* $V^{\mathcal{L}}(\boldsymbol{x})$, $\widetilde{V}^{\mathcal{L}}(\boldsymbol{x})$, *and* $V^{\mathcal{F}}(\boldsymbol{x})$ *are homogeneous of degree one and translation invariant—i.e., for each* $k>0$, *we have that*

$$\begin{gathered}V^{\mathcal{L}}(k\boldsymbol{x})=kV^{\mathcal{L}}(\boldsymbol{x}),\widetilde{V}^{\mathcal{L}}(k\boldsymbol{x})=kV^{\mathcal{F}}(\boldsymbol{x}),V^{\mathcal{F}}(k\boldsymbol{x})=kV^{\mathcal{F}}(\boldsymbol{x}),\\V^{\mathcal{L}}(\boldsymbol{x}+k)=V^{\mathcal{L}}(\boldsymbol{x}),\widetilde{V}^{\mathcal{L}}(\boldsymbol{x}+k)=V^{\mathcal{F}}(\boldsymbol{x}),\text{and }V^{\mathcal{F}}(\boldsymbol{x}+k)=V^{\mathcal{F}}(\boldsymbol{x}).\end{gathered}$$

For notational convenience, define $\boldsymbol{x}^{(3)}:=(x^{(1)},x^{(2)},x^{(3)})$, $\boldsymbol{x}^{(2)}:=(x^{(1)},x^{(2)})$, and $r(\boldsymbol{x}):=\frac{x^{(2)}-x^{(3)}}{x^{(1)}-x^{(2)}}$. Set $\kappa:=1/2$. Carrying out the algebra, we have that

$$V^{\mathcal{L}}-V^{\mathcal{F}}=\mathbb{E}_{\boldsymbol{x}\sim g}\left[V^{\mathcal{L}}(\boldsymbol{x})-V^{\mathcal{F}}(\boldsymbol{x})\right]$$

$$
\begin{aligned}
&\geq \mathbb{E}_{\boldsymbol{x}\sim g}\left[V^{\mathcal{L}}\left(\boldsymbol{x}^{(3)}\right)-V^{\mathcal{F}}\left(\boldsymbol{x}^{(3)}\right)\right]\\
&=\mathbb{E}_{\boldsymbol{x}\sim g}\left[\mathbb{1}\left(r(\boldsymbol{x})\leq\kappa\right)\times\left[V^{\mathcal{L}}\left(\boldsymbol{x}^{(3)}\right)-V^{\mathcal{F}}\left(\boldsymbol{x}^{(3)}\right)\right]\right]\\
&\quad+\mathbb{E}_{\boldsymbol{x}\sim g}\left[\mathbb{1}\left(r(\boldsymbol{x})\geq\kappa\right)\times\left[V^{\mathcal{L}}\left(\boldsymbol{x}^{(3)}\right)-V^{\mathcal{F}}\left(\boldsymbol{x}^{(3)}\right)\right]\right]\\
&\geq \mathbb{E}_{\boldsymbol{x}\sim g}\left[\mathbb{1}\left(r(\boldsymbol{x})\leq\kappa\right)\times\left[\widetilde{V}^{\mathcal{L}}\left(\boldsymbol{x}^{(3)}\right)-V^{\mathcal{F}}\left(\boldsymbol{x}^{(3)}\right)\right]\right]\\
&\quad+\mathbb{E}_{\boldsymbol{x}\sim g}\left[\mathbb{1}\left(r(\boldsymbol{x})\geq\kappa\right)\times\left[V^{\mathcal{L}}\left(\boldsymbol{x}^{(3)}\right)-V^{\mathcal{F}}\left(\boldsymbol{x}^{(3)}\right)\right]\right], \qquad (27)
\end{aligned}
$$

where the first inequality follows from Lemma 4 and the second inequality from Lemma A9.

By Lemma A10, we have that

$$
\widetilde{V}^{\mathcal{L}}\left(\boldsymbol{x}^{(3)}\right)-V^{\mathcal{F}}\left(\boldsymbol{x}^{(3)}\right)=\left(x^{(1)}-x^{(2)}\right)\times\left[\widetilde{V}^{\mathcal{L}}\left(1,0,-r(\boldsymbol{x})\right)-V^{\mathcal{F}}\left(1,0,-r(\boldsymbol{x})\right)\right]. \qquad (28)
$$

By Lemmas 4 and A9, $\widetilde{V}^{\mathcal{L}}(\boldsymbol{x}^{(3)})$ is increasing in $x^{(3)}$ and $W^{\mathcal{F}}(\boldsymbol{x}^{(3)})$ is independent of $x^{(3)}$; together with the postulated $r(\boldsymbol{x})\leq\kappa$, we have that

$$
\begin{aligned}
\widetilde{V}^{\mathcal{L}}\left(1,0,-r(\boldsymbol{x})\right)-V^{\mathcal{F}}\left(1,0,-r(\boldsymbol{x})\right)&\geq\widetilde{V}^{\mathcal{L}}\left(1,0,-\kappa\right)-V^{\mathcal{F}}\left(1,0,-\kappa\right)\\
&=\widetilde{V}^{\mathcal{L}}\left(\kappa+1,\kappa,0\right)-V^{\mathcal{F}}\left(\kappa+1,\kappa,0\right), \qquad (29)
\end{aligned}
$$

where the equality follows from Lemma A10. Further, note that

$$
\begin{aligned}
V^{\mathcal{L}}\left(\boldsymbol{x}^{(3)}\right)-V^{\mathcal{F}}\left(\boldsymbol{x}^{(3)}\right)\geq&V^{\mathcal{L}}\left(\boldsymbol{x}^{(2)}\right)-V^{\mathcal{F}}\left(\boldsymbol{x}^{(2)}\right)\\
=&\left(x^{(1)}-x^{(2)}\right)\times\left[V^{\mathcal{L}}\left(\kappa+1,\kappa\right)-V^{\mathcal{F}}\left(\kappa+1,\kappa\right)\right], \qquad (30)
\end{aligned}
$$

where the equality follows from Lemma A10 and the inequality from Lemma 4.

Recall that we are now in the remaining case in which $V^{\mathcal{F}}(\kappa+1,\kappa)-V^{\mathcal{L}}(\kappa+1,\kappa)>0$. Together with (27), (28), (29), and (30), we have that

$$
\begin{aligned}
V^{\mathcal{L}}-V^{\mathcal{F}}\geq&\,\mathbb{E}_{\boldsymbol{x}\sim g}\left[\mathbb{1}\left(r(\boldsymbol{x})\leq\kappa\right)\times(x^{(1)}-x^{(2)})\right]\times\left[\widetilde{V}^{\mathcal{L}}\left(\kappa+1,\kappa,0\right)-V^{\mathcal{F}}\left(\kappa+1,\kappa,0\right)\right]\\
&+\mathbb{E}_{\boldsymbol{x}\sim g}\left[\mathbb{1}\left(r(\boldsymbol{x})\geq\kappa\right)\times(x^{(1)}-x^{(2)})\right]\times\left[V^{\mathcal{L}}\left(\kappa+1,\kappa\right)-V^{\mathcal{F}}\left(\kappa+1,\kappa\right)\right]\\
=&\,\mathbb{E}_{\boldsymbol{x}\sim g}\left[\mathbb{1}\left(r(\boldsymbol{x})\geq\kappa\right)\times(x^{(1)}-x^{(2)})\right]\times\left[V^{\mathcal{F}}\left(\kappa+1,\kappa\right)-V^{\mathcal{L}}\left(\kappa+1,\kappa\right)\right]\\
&\times\left[\mathcal{C}_1(g,\kappa)\mathcal{C}_2(f,\kappa)-1\right], \qquad (31)
\end{aligned}
$$

where

$$\mathcal{C}_1(g,\kappa) := \frac{\mathbb{E}_{\boldsymbol{x}\sim g}\left[\mathbb{I}\{r(\boldsymbol{x}) \le \kappa\}(x^{(1)} - x^{(2)})\right]}{\mathbb{E}_{\boldsymbol{x}\sim g}\left[\mathbb{I}\{r(\boldsymbol{x}) \ge \kappa\}(x^{(1)} - x^{(2)})\right]}$$

and

$$\mathcal{C}_2(f,\kappa) := \frac{\widetilde{V}^{\mathcal{L}}(\kappa+1,\kappa,0) - V^{\mathcal{F}}(\kappa+1,\kappa,0)}{V^{\mathcal{F}}(\kappa+1,\kappa) - V^{\mathcal{L}}(\kappa+1,\kappa)}.$$

The following auxiliary lemma, proved in the Supplemental Appendix, gives the needed estimate.

Lemma A11 *Suppose that the brand-specific preferences x_i are independent and identically distributed with a weakly decreasing density g. Set $\kappa = 1/2$. If $V^{\mathcal{F}} > V^{\mathcal{L}}$ for $n = 2$, then*

$$\mathcal{C}_1(g,\kappa)\mathcal{C}_2(f,\kappa) > 1.$$

By Lemma A11, the last bracket in (31) is strictly positive. This concludes the proof. ■

Proof of Proposition 5

Proof. Similar to the proof of Proposition 3, we focus on the case of $f(\underline{t}) < 1/\underline{t}$ and show that $\Pi^{\mathcal{F}}(\boldsymbol{x}) > \Pi^{\mathcal{L}}(\boldsymbol{x})$ holds pointwisely. Note that $\Pi^{\mathcal{L}}(\boldsymbol{x})$ and $\Pi^{\mathcal{F}}(\boldsymbol{x})$ can be expressed as

$$\Pi^{\mathcal{L}}(\boldsymbol{x}) = \sum_{i=1}^{k(\boldsymbol{x})} \left(p_i^{\mathcal{L}}(\boldsymbol{x}) - c\right)\left[F\left(\alpha_{i-1}(\boldsymbol{x})\right) - F\left(\alpha_i(\boldsymbol{x})\right)\right], \tag{32}$$

$$\Pi^{\mathcal{F}}(\boldsymbol{x}) = \mathbb{E}\left[t\right](x_1 - x_2) = (x_1 - x_2)\int_{\underline{t}}^{\bar{t}} t f(t)dt. \tag{33}$$

Carrying out the algebra, we can obtain that

$$\begin{aligned}\Pi^{\mathcal{L}}(\boldsymbol{x}) &= p_1^{\mathcal{L}}(\boldsymbol{x}) - c - \sum_{i=1}^{k(\boldsymbol{x})-1}(x_i - x_{i+1})\alpha_i(\boldsymbol{x})F\left(\alpha_i(\boldsymbol{x})\right)\\ &\le p_1^{\mathcal{L}}(\boldsymbol{x}) - c - (x_1 - x_2)\alpha_1(\boldsymbol{x})F\left(\alpha_1(\boldsymbol{x})\right)\\ &= (x_1 - x_2)\times\left[\frac{1 - F\left(\alpha_1(\boldsymbol{x})\right)}{f\left(\alpha_1(\boldsymbol{x})\right)} - \alpha_1(\boldsymbol{x})F\left(\alpha_1(\boldsymbol{x})\right)\right],\end{aligned} \tag{34}$$

where the first equality follows from (32) and (15); the inequality follows from the fact that $(x_i - x_{i+1})\alpha_i(\boldsymbol{x})F(\alpha_i(\boldsymbol{x}))$ is nonnegative for each $i \in \{2, \ldots, k(\boldsymbol{x}) - 1\}$; and the last equality follows from (10).

Recall that α^* is the solution to (4). It is useful to state the following lemma (whose proof can be found in the Supplemental Appendix):

Lemma A12 *Suppose that* $f(\underline{t}) < 1/\underline{t}$. *Fixing* $\boldsymbol{x} = (x_1, \ldots, x_n)$ *with* $x_1 > \cdots > x_n$, *it holds that* $\alpha_1(\boldsymbol{x}) \geq \alpha^* \geq \alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})$. *Moreover, both inequalities are strict if* $k(\boldsymbol{x}) = n$.

By Lemma A12, we can obtain that

$$\frac{1 - F\left(\alpha_1(\boldsymbol{x})\right)}{f\left(\alpha_1(\boldsymbol{x})\right)} - \alpha_1(\boldsymbol{x}) F\left(\alpha_1(\boldsymbol{x})\right) \leq \frac{1 - F(\alpha^*)}{f(\alpha^*)} - \alpha^* F(\alpha^*) = \frac{F(\alpha^*)}{f(\alpha^*)} + \alpha^* \left(1 - F(\alpha^*)\right), \tag{35}$$

where the inequality follows from Lemmas A1 and A12, and the equality follows from (4). Combining (34) and (35) yields that

$$\Pi^{\mathcal{L}}(\boldsymbol{x}) \leq (x_1 - x_2) \times \left[\frac{F(\alpha^*)}{f(\alpha^*)} + \alpha^* \left(1 - F(\alpha^*)\right)\right] \leq (x_1 - x_2) \times \int_{\underline{t}}^{\bar{t}} t f(t) dt = \Pi^{\mathcal{F}}(\boldsymbol{x}),$$

where the second inequality follows from (23) and the equality from (33). This concludes the proof. ■

Proof of Proposition 6

Proof. First, we consider part (i). The ranking $V^{\mathcal{F}} > V^{\mathcal{U}}$ follows from Proposition 1 in Rhodes and Zhou (2024). It remains to show that $V^{\mathcal{U}} > V^{\mathcal{C}}$. Since both uniform pricing and choosiness-based pricing allocate each consumer to the firm with the highest loyalty draw, total surplus is the same under the two regimes. Hence it suffices to show that industry profit is higher under $\mathcal{C}$ than under $\mathcal{U}$.

Recall that $\hat{x}_i := x_i - \max_{j \neq i} x_j$ and let $\psi(0 \mid t)$ denote the conditional density of $\hat{x}_i$ at zero given t. Recall $z_i = t\hat{x}_i$, and $h(0)$ denotes the unconditional density of z_i at zero. By the same argument as in Lemma 1, the symmetric uniform price satisfies

$$p^{\mathcal{U}} = c + \frac{1}{nh(0)}.$$

Because $z_i = t\hat{x}_i$, we have

$$h(0) = \mathbb{E}_t \left[\frac{\psi(0 \mid t)}{t}\right].$$

Hence

$$p^{\mathcal{U}} - c = \frac{1}{n\mathbb{E}_t[\psi(0 \mid t)/t]}.$$

Under choosiness-based pricing, firms observe t. Conditional on t, the same first-order condition as in Lemma 1 gives

$$p^{\mathcal{C}}(t) = c + \frac{t}{n\psi(0 \mid t)}.$$

Note that

$$\mathbb{E}_t\left[p^{\mathcal{C}}(t)-c\right]=\frac{1}{n}\mathbb{E}_t\left[\frac{1}{\psi(0\mid t)/t}\right]<\frac{1}{n\mathbb{E}_t[\psi(0\mid t)/t]}=p^{\mathcal{U}}-c,$$

where the inequality follows from Jensen's inequality. This implies that $V^{\mathcal{U}}>V^{\mathcal{C}}$.

Next, we consider part (ii). Fix a realized loyalty vector $\boldsymbol{x}$. Conditional on this realization, the loyalty-based pricing game is exactly the loyalty-based game studied in Proposition 2, with the distribution of t replaced by the conditional distribution $F(\cdot\mid\boldsymbol{x})$. The proof of Proposition 2 is pointwise in $\boldsymbol{x}$ and does not use independence between t and $\boldsymbol{x}$. Therefore, for almost every $\boldsymbol{x}$, if $F(\cdot\mid\boldsymbol{x})$ satisfies the conditions in Proposition 2(i), then $V^{\mathcal{L}}(\boldsymbol{x})>V^{\mathcal{F}}(\boldsymbol{x})$. Taking expectations over $\boldsymbol{x}$ gives $V^{\mathcal{L}}>V^{\mathcal{F}}$. Similarly, if $F(\cdot\mid\boldsymbol{x})$ satisfies the conditions in Proposition 2(ii) for almost every $\boldsymbol{x}$, then $V^{\mathcal{F}}(\boldsymbol{x})>V^{\mathcal{L}}(\boldsymbol{x})$ holds for almost every $\boldsymbol{x}$, and hence $V^{\mathcal{F}}>V^{\mathcal{L}}$.

Finally, we consider part (iii). Under $\mathcal{U}$, $\mathcal{C}$, and $\mathcal{F}$, the allocation is efficient under full market coverage: Each consumer buys from the firm with the highest realized loyalty. Hence total surplus is the same under these three regimes. The consumer-welfare ranking in part (i) therefore implies the reverse ranking of industry profit, i.e., $\Pi^{\mathcal{C}}>\Pi^{\mathcal{U}}>\Pi^{\mathcal{F}}$.

It remains to compare $\mathcal{F}$ and $\mathcal{L}$. The proof of Proposition 3 compares profits under fully personalized and loyalty-based pricing pointwise in the realized loyalty vector $\boldsymbol{x}$. Replacing the distribution of t with the conditional distribution $F(\cdot\mid\boldsymbol{x})$ leaves the argument unchanged. Thus $\Pi^{\mathcal{F}}(\boldsymbol{x})>\Pi^{\mathcal{L}}(\boldsymbol{x})$ holds for almost every $\boldsymbol{x}$. Taking expectations over $\boldsymbol{x}$ yields $\Pi^{\mathcal{F}}>\Pi^{\mathcal{L}}$. This completes the proof. ■

Appendix B: Admissible Distributions for Equilibrium Existence

In this appendix, we provide concrete examples of distributions that simultaneously satisfy Assumptions 1, 2, and 3.

Table 1 focuses on the standard benchmark whereby relative brand preferences follow a uniform distribution.

Table 1: Admissible Distributions of Choosiness (t) when $\hat{x}_i \sim U[0,1]$

Distribution of t	**Support**	**Sufficient Parameter Conditions**
Power Function	$[0,1]$	$f(t) \propto t^k$ with $k \geq 1$
Beta Distribution	$[0,1]$	Beta(α, β) with $\alpha \geq 2$, $\beta \geq 1$
Shifted Beta	$[\underline{t}, \bar{t}]$	Beta(α, β) shifted to $[\underline{t}, \bar{t}]$ with $(\alpha-1)^{\frac{1}{3}} + \left(\frac{\underline{t}(\beta-1)}{\bar{t}(\alpha-1)}\right)^{\frac{1}{3}} \geq \left(\frac{\bar{t}-\underline{t}}{\bar{t}}\right)^{\frac{2}{3}}$
Truncated Normal	$[\underline{t}, \bar{t}]$	$N(\mu, \sigma^2)$ truncated to $[\underline{t}, \bar{t}]$ with $\underline{t} \geq \sigma$

Table 2 generalizes this benchmark to allow for varying degrees of brand loyalty concentration by assuming that the preferences follow a power distribution.

Table 2: Admissible Distributions of Choosiness (t) when $x_i \sim$ Power(m)

Distribution of t	**Support**	**Sufficient Parameter Conditions**
Power Function	$[0,1]$	$f(t) \propto t^k$ with $k \geq nm$
Beta Distribution	$[0,1]$	Beta(α, β) with $\alpha \geq nm+1$, $\beta \geq 1$
Shifted Beta	$[\underline{t}, \bar{t}]$	Beta(α, β) shifted to $[\underline{t}, \bar{t}]$ with $(\alpha-1)^{\frac{1}{3}} + \left(\frac{\underline{t}(\beta-1)}{\bar{t}(\alpha-1)}\right)^{\frac{1}{3}} \geq \left(\frac{\bar{t}-\underline{t}}{\bar{t}}\right)^{\frac{2}{3}} (nm)^{\frac{1}{3}}$
Truncated Normal	$[\underline{t}, \bar{t}]$	$N(\mu, \sigma^2)$ truncated to $[\underline{t}, \bar{t}]$ with $\underline{t} \geq \sigma\sqrt{nm}$

Supplemental Appendix

A Omitted Proofs

Proof of Lemma A1

Proof. Clearly, $\ln f(t) = \ln \frac{f(t)}{t} + \ln t$ is concave. The log-concavity of $F(t)$ and $1 - F(t)$ follows from the Prékopa-Borell theorem. Further, the log-concavity of $1 - F(t)$ implies that $\frac{1-F(t)}{f(t)}$ is weakly decreasing. Therefore, both $\frac{1-F(t)}{f(t)} - t$ and $\frac{1-F(t)}{f(t)} - tF(t)$ are strictly decreasing. ■

Proof of Lemma A2

Proof. Let $\mathcal{Q}(\alpha) := \frac{1-F(\alpha)}{f(\alpha)} - \frac{F(\alpha)}{f(\alpha)} - \alpha$. By Lemma A1, both $F(t)$ and $1 - F(t)$ are log-concave, which implies that $\frac{F(\alpha)}{f(\alpha)}$ is increasing in α and $\frac{1-F(\alpha)}{f(\alpha)}$ is decreasing in α. Therefore, $\mathcal{Q}$ is strictly decreasing in α. It follows from $f(\underline{t}) < 1/\underline{t}$ that $\mathcal{Q}(\underline{t}) = \frac{1}{f(\underline{t})} - \underline{t} > 0$; moreover, $\mathcal{Q}(\bar{t}) = -\frac{1}{f(\bar{t})} - \bar{t} < 0$. Therefore, there exists a unique $\alpha^* \in (\underline{t}, \bar{t})$ such that $\mathcal{Q}(\alpha^*) = 0$. This concludes the proof. ■

Proof of Lemma A3

Proof. Carrying out the algebra, we have that

$$\begin{aligned}
\frac{\partial B_{\delta_1,\delta_2}}{\partial \alpha} &= B_{\delta_1,\delta_2}(\alpha,\alpha') \times \left[\frac{f(\alpha)}{F(\alpha) - F(\alpha')} - \frac{f'(\alpha)}{f(\alpha) + \frac{f(\alpha')\delta_1}{\delta_2}} \right] \\
&> B_{\delta_1,\delta_2}(\alpha,\alpha') \times \left[\frac{f(\alpha)}{F(\alpha)} - \frac{f'(\alpha)}{f(\alpha) + \frac{f(\alpha')\delta_1}{\delta_2}} \right] \\
&= \frac{B_{\delta_1,\delta_2}(\alpha,\alpha')}{F(\alpha) \times \left[f(\alpha) + \frac{f(\alpha')\delta_1}{\delta_2} \right]} \times \left\{ f(\alpha) \left[f(\alpha) + \frac{f(\alpha')\delta_1}{\delta_2} \right] - f'(\alpha)F(\alpha) \right\} \\
&\geq \frac{B_{\delta_1,\delta_2}(\alpha,\alpha')}{F(\alpha) \times \left[f(\alpha) + \frac{f(\alpha')\delta_1}{\delta_2} \right]} \times \left\{ \left[f(\alpha) \right]^2 - f'(\alpha)F(\alpha) \right\}.
\end{aligned}$$

By Lemma A1, $F(\cdot)$ is log-concave, which implies that $\left[f(\alpha)\right]^2 - f'(\alpha)F(\alpha) \geq 0$ and thus $\frac{\partial B_{\delta_1,\delta_2}}{\partial \alpha} > 0$. Similarly, we have that

$$\frac{\partial B_{\delta_1,\delta_2}}{\partial \alpha'} = - B_{\delta_1,\delta_2}(\alpha,\alpha') \times \left[\frac{f(\alpha')}{F(\alpha) - F(\alpha')} + \frac{f'(\alpha')}{\frac{f(\alpha)\delta_2}{\delta_1} + f(\alpha')} \right]$$

$$
\begin{aligned}
&< -B_{\delta_1,\delta_2}(\alpha,\alpha') \times \left[\frac{f(\alpha')}{1-F(\alpha')} + \frac{f'(\alpha')}{\frac{f(\alpha)\delta_2}{\delta_1} + f(\alpha')}\right] \\
&= -\frac{B_{\delta_1,\delta_2}(\alpha,\alpha')}{\left[1-F(\alpha')\right] \times \left[\frac{f(\alpha)\delta_2}{\delta_1} + f(\alpha')\right]} \times \left\{f(\alpha') \times \left[\frac{f(\alpha)\delta_2}{\delta_1} + f(\alpha')\right] + f'(\alpha') \times \left[1-F(\alpha')\right]\right\} \\
&\le -\frac{B_{\delta_1,\delta_2}(\alpha,\alpha')}{\left[1-F(\alpha')\right] \times \left[\frac{f(\alpha)\delta_2}{\delta_1} + f(\alpha')\right]} \times \left\{f(\alpha')^2 + f'(\alpha') \times \left[1-F(\alpha')\right]\right\}.
\end{aligned}
$$

By Lemma A1, $1-F(\cdot)$ is log-concave, which implies that $f(\alpha')^2 + f'(\alpha') \times \left[1-F(\alpha')\right] \ge 0$ and thus $\frac{\partial B_{\delta_1,\delta_2}}{\partial \alpha'} < 0$. This concludes the proof. ■

Proof of Lemma A4

Proof. For notational convenience, denote $\alpha_+ := \frac{p_{j_1}-p_i}{x_{j_1}-x_i}$ and $\alpha_- := \frac{p_i-p_{j_2}}{x_i-x_{j_2}}$. It suffices to show that $\frac{\mathcal{D}_i(p_i,\boldsymbol{p}_{-i};\boldsymbol{x})}{\partial\mathcal{D}_i(p_i,\boldsymbol{p}_{-i};\boldsymbol{x})/\partial p_i}$ is increasing in p_i. Set $\delta_1 = x_{j_1} - x_i$ and $\delta_2 = x_i - x_{j_2}$ in the function $B_{\delta_1,\delta_2}(\alpha,\alpha')$ defined in Lemma A3. Simple algebra would verify that

$$
\frac{\mathcal{D}_i(p_i,\boldsymbol{p}_{-i};\boldsymbol{x})}{\partial\mathcal{D}_i(p_i,\boldsymbol{p}_{-i};\boldsymbol{x})/\partial p_i} = -B_{\delta_1,\delta_2}(\alpha_+,\alpha_-).
$$

By Lemma A3, $B_{\delta_1,\delta_2}(\alpha_+,\alpha_-)$ is increasing in α_+ and decreasing in α_-. This, together with the fact that α_+ is decreasing in p_i and α_- is increasing in p_i, implies that $\frac{\mathcal{D}_i(p_i,\boldsymbol{p}_{-i};\boldsymbol{x})}{\partial\mathcal{D}_i(p_i,\boldsymbol{p}_{-i};\boldsymbol{x})/\partial p_i}$ increases with p_i. This completes the proof. ■

Proof of Lemma A5

Proof. We prove the lemma by induction.

Base case: $\acute{\alpha}_1(\alpha_1)$ is obviously increasing in α_1. By Lemma A3, $\acute{p}_1(\alpha_1)$—which we define in (16)—is decreasing in α_1.

Inductive step: Suppose that $\acute{p}_i(\alpha_1)$ is decreasing in α_1 and $\acute{\alpha}_i(\alpha_1)$ is increasing in α_1. By (17), $\acute{p}_{i+1}(\alpha_1)$ is decreasing in α_1. Set $\delta_1 = x_i - x_{i+1}$ and $\delta_2 = x_{i+1} - x_{i+2}$ in the function $B_{\delta_1,\delta_2}(\alpha,\alpha')$ defined in Lemma A3. It follows from (18) that

$$
\acute{p}_{i+1}(\alpha_1) = c + B_{\delta_1,\delta_2}\left(\acute{\alpha}_i(\alpha_1), \acute{\alpha}_{i+1}(\alpha_1)\right).
$$

Recall that $\acute{p}_{i+1}(\alpha_1)$ is decreasing in α_1 and $\acute{\alpha}_i(\alpha_1)$ is increasing in α_1 by assumption. Therefore,

$$
0 > \frac{d\acute{p}_{i+1}(\alpha_1)}{d\alpha_1} = \underbrace{\frac{\partial B_{\delta_1,\delta_2}}{\partial \alpha}}_{\ge 0} \times \underbrace{\frac{d\acute{\alpha}_i}{d\alpha_1}}_{>0} + \underbrace{\frac{\partial B_{\delta_1,\delta_2}}{\partial \alpha'}}_{\le 0} \times \frac{d\acute{\alpha}_{i+1}}{d\alpha_1},
$$

from which we can conclude that $\frac{d\acute{\alpha}_{i+1}}{d\alpha_1} > 0$.

Conclusion: By the principle of induction, $\acute{p}_i(\alpha_1)$ is decreasing in α_1 and $\acute{\alpha}_i(\alpha_1)$ is increasing in α_1 for all $i \in \{1, \ldots, k(\alpha_1)\}$. This concludes the proof. ■

Proof of Lemma A6

Proof. Fix $\boldsymbol{x}$ and consider the recursive construction in (16)–(20). By Lemma A3, each recursive step is well defined. Let $k(\alpha_1)$ be the candidate marginal active firm generated by the recursion.

Define

$$k^* := \max_{\alpha_1 \in [\underline{t}, \overline{t}]} k(\alpha_1), \qquad \mathcal{X} := \{\alpha_1 \in [\underline{t}, \overline{t}] : k(\alpha_1) = k^*\}.$$

We first show that $\mathcal{X}$ is an interval. Let $\underline{\alpha}_1 := \inf \mathcal{X}$ and $\overline{\alpha}_1 := \sup \mathcal{X}$. If $k^* = 2$, then $k(\alpha_1) = 2$ for all $\alpha_1 \in [\underline{t}, \overline{t}]$, and hence $\mathcal{X} = [\underline{t}, \overline{t}]$. Suppose now that $k^* \geq 3$.

For $\alpha_1 \in \mathcal{X}$, the recursion does not stop at $j = k^* - 1$. Hence the two stopping conditions (19) and (20) both fail at $j = k^* - 1$, which gives

$$\acute{p}_{k^*-1}(\alpha_1) - \acute{\alpha}_{k^*-1}(\alpha_1)(x_{k^*-1} - x_{k^*}) \geq c, \tag{A1}$$

$$\acute{p}_{k^*-1}(\alpha_1) \leq c + \frac{F(\acute{\alpha}_{k^*-2}(\alpha_1))}{\frac{f(\acute{\alpha}_{k^*-2}(\alpha_1))}{x_{k^*-2} - x_{k^*-1}} + \frac{f(\underline{t})}{x_{k^*-1} - x_{k^*}}}. \tag{A2}$$

At the endpoints $\underline{\alpha}_1$ and $\overline{\alpha}_1$, these inequalities hold by continuity. By Lemma A5, they then hold for every $\alpha_1 \in [\underline{\alpha}_1, \overline{\alpha}_1]$. Thus $k(\alpha_1) = k^*$ throughout this interval, so

$$\mathcal{X} = [\underline{\alpha}_1, \overline{\alpha}_1].$$

Moreover, (A2) binds at $\alpha_1 = \underline{\alpha}_1$, while (A1) binds at $\alpha_1 = \overline{\alpha}_1$.

Next, within $\mathcal{X}$, the equilibrium must have the last cutoff equal to $\underline{t}$. Since $\acute{\alpha}_{k^*}(\alpha_1)$ is increasing in α_1 by Lemma A5, there exists a threshold $\hat{\alpha}_1 \in \mathcal{X}$ such that

$$\acute{\alpha}_{k^*}(\alpha_1) = \underline{t} \quad \text{if and only if} \quad \alpha_1 \in [\underline{\alpha}_1, \hat{\alpha}_1].$$

Equivalently, $\hat{\alpha}_1$ is characterized by

$$\acute{p}_{k^*}(\hat{\alpha}_1) = c + \frac{F(\acute{\alpha}_{k^*-1}(\hat{\alpha}_1))}{\frac{f(\acute{\alpha}_{k^*-1}(\hat{\alpha}_1))}{x_{k^*-1} - x_{k^*}} + \frac{f(\underline{t})}{x_{k^*} - x_{k^*+1}}}, \tag{A3}$$

with the convention stated in the main proof when $k^* = n$. Therefore, no $\alpha_1 \in (\hat{\alpha}_1, \overline{\alpha}_1]$ can satisfy the terminal KKT conditions, because the candidate last cutoff would exceed $\underline{t}$.

It remains to find the unique value of α_1 in $[\underline{\alpha}_1, \hat{\alpha}_1]$ that satisfies the terminal KKT conditions. Define the terminal FOC residual

$$R^{\text{FOC}}(\alpha_1) := \acute{p}_{k^*}(\alpha_1) - c - \frac{F(\acute{\alpha}_{k^*-1}(\alpha_1))}{f(\acute{\alpha}_{k^*-1}(\alpha_1))/(x_{k^*-1} - x_{k^*})}.$$

By Lemma A5, $R^{\text{FOC}}(\alpha_1)$ is decreasing in α_1. We now distinguish two cases.

First, suppose

$$\acute{p}_{k^*}(\underline{\alpha}_1) < c + \underline{t}(x_{k^*} - x_{k^*+1}).$$

Then, since $\acute{p}_{k^*}(\alpha_1)$ is decreasing in α_1, (13) is slack for every $\alpha_1 \in [\underline{\alpha}_1, \hat{\alpha}_1]$. Thus the terminal KKT conditions reduce to

$$R^{\text{FOC}}(\alpha_1) = 0.$$

At $\alpha_1 = \underline{\alpha}_1$, (A2) binds, which implies

$$R^{\text{FOC}}(\underline{\alpha}_1) \geq 0.$$

At $\alpha_1 = \hat{\alpha}_1$, (A3) implies

$$R^{\text{FOC}}(\hat{\alpha}_1) \leq 0.$$

Since R^{FOC} is decreasing, there exists a unique $\alpha_1^* \in [\underline{\alpha}_1, \hat{\alpha}_1]$ such that $R^{\text{FOC}}(\alpha_1^*) = 0$. This value satisfies (12)–(14) with $\xi = 0$.

Second, suppose

$$\acute{p}_{k^*}(\underline{\alpha}_1) \geq c + \underline{t}(x_{k^*} - x_{k^*+1}).$$

Since $\acute{p}_{k^*}(\alpha_1)$ is decreasing in α_1, there exists a unique $\tilde{\alpha}_1 \in [\underline{\alpha}_1, \hat{\alpha}_1]$ such that

$$\acute{p}_{k^*}(\tilde{\alpha}_1) = c + \underline{t}(x_{k^*} - x_{k^*+1}).$$

If

$$R^{\text{FOC}}(\tilde{\alpha}_1) < 0,$$

then the terminal firm's unconstrained optimal price lies above the entry-deterrence cap, so the cap binds. Hence $\alpha_1^* = \tilde{\alpha}_1$ satisfies the KKT conditions with a positive multiplier. If instead

$$R^{\text{FOC}}(\tilde{\alpha}_1) \geq 0,$$

then, since $R^{\text{FOC}}(\hat{\alpha}_1) \leq 0$ and R^{FOC} is decreasing, there exists a unique $\alpha_1^* \in [\tilde{\alpha}_1, \hat{\alpha}_1]$ such that

$$R^{\text{FOC}}(\alpha_1^*) = 0.$$

At this value the entry-deterrence cap is slack or just binding, and the KKT conditions hold with $\xi = 0$. Thus in either subcase there is a unique $\alpha_1^* \in [\underline{\alpha}_1, \hat{\alpha}_1]$ that satisfies the terminal KKT conditions.

Finally, we rule out values of α_1 outside $\mathcal{X}$. If $\alpha_1' < \underline{\alpha}_1$, then $k(\alpha_1') =: k' < k^*$. Since the recursion does not stop at $j = k'$ when $\alpha_1 = \underline{\alpha}_1$, we have

$$\acute{p}_{k'}(\underline{\alpha}_1) - \acute{\alpha}_{k'}(\underline{\alpha}_1)(x_{k'} - x_{k'+1}) > c.$$

By Lemma A5, the left-hand side is decreasing in α_1, and hence

$$\acute{p}_{k'}(\alpha_1') - \acute{\alpha}_{k'}(\alpha_1')(x_{k'} - x_{k'+1}) > c.$$

Thus the candidate marginal firm cannot deter entry by firm $k' + 1$, so (13) fails.

Similarly, if $\alpha_1' > \overline{\alpha}_1$, then the recursion stops before reaching k^* because the candidate price is too low to satisfy the terminal firm's own-price optimality condition. Equivalently, (12) fails at the candidate marginal firm. Therefore no $\alpha_1' \notin \mathcal{X}$ can satisfy the terminal KKT conditions.

We conclude that there exists a unique $\alpha_1^* \in [\underline{t}, \overline{t}]$ such that the candidate prices and cutoffs generated by the shooting construction satisfy (12)–(14). ■

Proof of Lemma A7

Proof. The proof consists of two steps. In the first step, we construct the PDF $f^\dagger$ and the CDF $F^\dagger$ such that (24) holds. In the second step, we prove that the constructed $f^\dagger$ satisfies (25).

Step I Define $f^\dagger(t) := \mathcal{M} t e^{\beta t}$ and

$$F^\dagger(t) = \int_0^t f^\dagger(s) ds = \mathcal{M} \frac{e^{\beta t}(\beta t - 1) + 1}{\beta^2}, \forall t \in \big[0, \overline{t}^\dagger\big],$$

where the parameters $(\mathcal{M}, \beta, \overline{t}^\dagger)$ are to be constructed to satisfy (24) and $F^\dagger(\overline{t}^\dagger) = 1$.

We first construct $(\mathcal{M}, \beta)$. Note that

$$\frac{F^\dagger(\alpha^*)}{f^\dagger(\alpha^*)} = \frac{e^{\beta\alpha^*}(\beta\alpha^* - 1) + 1}{\beta^2 \alpha^* e^{\beta\alpha^*}} = \alpha^* \times \underbrace{\frac{e^u(u-1) + 1}{u^2 e^u}}_{=:\phi(u)},$$

where $u := \beta\alpha^*$. Further, we have that $\phi(+\infty) = 0$, $\phi(-\infty) = +\infty$, and $\phi'(\cdot) < 0$ on $\mathbb{R}$.

Therefore, there exists a unique u to satisfy

$$\phi(u) = \frac{F^\dagger(\alpha^*)}{\alpha^* f^\dagger(\alpha^*)}.$$

It can be verified that $(\mathcal{M}, \beta) = \left(\frac{f(\alpha^*)}{\alpha^* e^{\beta\alpha^*}}, \frac{u}{\alpha^*}\right)$ satisfies (24). By construction, β can be positive or negative. Fixing $\mathcal{M}$ and β, it can be verified that $\mathcal{M}\frac{e^{u'}(u'-1)+1}{\beta^2}$ is strictly decreasing in u' for $u' < 0$ and increasing in u' for $u' > 0$, and thus there exist two solutions to $\mathcal{M}\frac{e^{u'}(u'-1)+1}{\beta^2} = 1$. Pick the solution that has the same sign as β and denote it u' with slight abuse of notation. To complete the construction, we set $\bar{t}^\dagger = u'/\beta$.

Step II We prove (25). Let

$$\beta' := \frac{d}{dt}\left[\ln\frac{f(t)}{t}\right]\bigg|_{t=\alpha^*}.$$

We first show that $\beta' \leq \beta$. Suppose, to the contrary, that $\beta' > \beta$. From the log-concavity of $f(t)/t$, we have that

$$\ln\frac{f(t)}{t} \leq \ln\frac{f(\alpha^*)}{\alpha^*} + \beta'(t-\alpha^*) < \ln\frac{f(\alpha^*)}{\alpha^*} + \beta(t-\alpha^*), \forall\, t \in [\underline{t}, \alpha^*),$$

which in turn implies that

$$f(t) < t\frac{f(\alpha^*)}{\alpha^*}e^{\beta(t-\alpha^*)} = f^\dagger(t), \forall\, t \in [\underline{t}, \alpha^*).$$

It follows from the above inequality that

$$F(\alpha^*) = \int_{\underline{t}}^{\alpha^*} f(t)dt < \int_0^{\alpha^*} f^\dagger(t)dt = F^\dagger(\alpha^*) = F(\alpha^*),$$

which is a contradiction. Therefore, we must have $\beta' \leq \beta$.

Next, recall that $\ln\frac{f(t)}{t}$ is concave in t and $\ln\frac{f^\dagger(t)}{t}$ is linear in t by construction. Further, $f(\alpha^*) = f^\dagger(\alpha^*)$ and $\beta' \leq \beta$. Therefore, there exists $\alpha^\dagger < \alpha^*$ such that

$$f(t) \leq f^\dagger(t), \forall\, t \in [0, \alpha^\dagger], \text{ and } f(t) \geq f^\dagger(t), \forall\, t \in [\alpha^\dagger, \alpha^*].$$

The above condition, together with $F(\alpha^*) = F^\dagger(\alpha^*)$, implies that $F(\cdot|t \leq \alpha^*)$ first-order stochastically dominates $F^\dagger(\cdot|t \leq \alpha^*)$—i.e., $F(\cdot|t \leq \alpha^*) \geq_{FOSD} F^\dagger(\cdot|t \leq \alpha^*)$—from which we can conclude that

$$\int_{\underline{t}}^{\alpha^*} tf(t)dt \geq \int_{0}^{\alpha^*} tf^\dagger(t)dt. \tag{A4}$$

Similarly, it follows from $\beta' \leq \beta$ that

$$\ln\frac{f(t)}{t} \leq \ln\frac{f(\alpha^*)}{\alpha^*} + \beta'(t-\alpha^*) \leq \ln\frac{f(\alpha^*)}{\alpha^*} + \beta(t-\alpha^*), \forall\, t \in \left[\alpha^*, \bar{t}^\dagger\right],$$

which implies that

$$f(t) < t\frac{f(\alpha^*)}{\alpha^*}e^{\beta(t-\alpha^*)} \leq f^\dagger(t), \forall\, t \in \left[\alpha^*, \bar{t}^\dagger\right].$$

Because $F(\alpha^*) = F^\dagger(\alpha^*)$, we have that $\bar{t}^\dagger \leq \bar{t}$. Moreover, $f(t) \leq f^\dagger(t), \forall\, t \in \left[\alpha^*, \bar{t}^\dagger\right]$ and $f(t) \geq f^\dagger(t), \forall\, t \in \left[\bar{t}^\dagger, \bar{t}\right]$. Therefore, $F(\cdot|t \geq \alpha^*) \geq_{FOSD} F^\dagger(\cdot|t \geq \alpha^*)$, which implies that

$$\int_{\alpha^*}^{\bar{t}} tf(t)dt \geq \int_{\alpha^*}^{\bar{t}^\dagger} tf^\dagger(t)dt. \tag{A5}$$

Summing (A4) and (A5) completes the proof. ■

Proof of Lemma A8

Proof. Recall from the construction in Lemma A7 that $u' = \beta\bar{t}^\dagger$ and $u = \beta\alpha^*$. Further, we have that

$$\int_{0}^{\bar{t}^\dagger} tf^\dagger(t)dt = \int_{0}^{\bar{t}^\dagger} \mathcal{M}t^2e^{\beta t}dt = \mathcal{M}\frac{e^{u'}\left[(u')^2 - 2u' + 2\right] - 2}{\beta^3}, \tag{A6}$$

$$F(\alpha^*) = F^\dagger(\alpha^*) = \mathcal{M}\frac{e^u(u-1)+1}{\beta^2}, \text{ and } f(\alpha^*) = f^\dagger(\alpha^*) = \mathcal{M}\frac{ue^u}{\beta}. \tag{A7}$$

Substituting $\alpha^* = u/\beta$, (A6), and (A7) into (26), it remains to prove

$$\mathcal{M}\frac{e^{u'}\left[(u')^2 - 2u' + 2\right] - 2}{\beta^2} \geq \frac{e^u(u-1)+1}{ue^u} + u\left[1 - \mathcal{M}\frac{e^u(u-1)+1}{\beta^2}\right]. \tag{A8}$$

Further, from (4), we have that

$$1 = \alpha^* f^\dagger(\alpha^*) + 2F^\dagger(\alpha^*) = \frac{\mathcal{M}}{\beta^2}\left[u^2e^u + 2e^u(u-1) + 2\right]. \tag{A9}$$

Therefore, (A8) can be rewritten as

$$\frac{e^{u'}\left[(u')^2 - 2u' + 2\right] - 2}{u^2e^u + 2e^u(u-1) + 2} \geq \frac{e^u(u-1)+1}{ue^u} + u\frac{u^2e^u + e^u(u-1) + 1}{u^2e^u + 2e^u(u-1) + 2}. \tag{A10}$$

Recall that u' is the solution to $\mathcal{M}\frac{e^{u'}(u'-1)+1}{\beta^2} = 1$; together with (A9), u and u' must satisfy

$$u^2 e^u + 2e^u(u-1) + 2 = \frac{\beta^2}{\mathcal{M}} = e^{u'}(u'-1) + 1. \tag{A11}$$

It can be verified that (A10) holds for all (u, u') satisfying (A11). This completes the proof. ■

Proof of Lemma A9

Proof. By Lemma 2, the equilibrium is independent of x_{n+1} when $k(\boldsymbol{x}_{n+1}) \leq n-1$. Therefore, $k(\boldsymbol{x}_{n+1}) \in \{n, n+1\}$. We consider the following two cases.

Case I: $\boldsymbol{k(x_{n+1}) = n}$. If $\xi = 0$, then the equilibrium price $p_i^{\mathcal{L}}(\boldsymbol{x}_{n+1})$ is independent of x_{n+1} and is thus weakly decreasing in x_{n+1} for $i \in \{1, \ldots, n\}$. If otherwise $\xi > 0$, by (14), $p_n^{\mathcal{L}}(\boldsymbol{x}_{n+1}) = c + \underline{t}(x_n - x_{n+1})$. For each i, we slightly abuse notation and denote the functions defined in (17) and (18) by $\acute{p}_{i+1}(\alpha_1; \boldsymbol{x}_{i+1})$ and $\acute{\alpha}_{i+1}(\alpha_1; \boldsymbol{x}_{i+2})$, respectively, where $\boldsymbol{x}_i := (x_1, \ldots, x_i)$. Similarly, denote $\acute{p}_1(\alpha_1)$ by $\acute{p}_1(\alpha_1; \boldsymbol{x}_2)$.

Note that $\acute{p}_n(\alpha_1(\boldsymbol{x}_{n+1}); \boldsymbol{x}_n) = p_n^{\mathcal{L}}(\boldsymbol{x}_{n+1}) = c + \underline{t}(x_n - x_{n+1})$. By the implicit function theorem, we have that $0 > -\underline{t} = \frac{dp_n^{\mathcal{L}}(\boldsymbol{x}_{n+1})}{dx_{n+1}} = \frac{\partial \acute{p}_n}{\partial \alpha_1} \times \frac{d\alpha_1(\boldsymbol{x}_{n+1})}{dx_{n+1}}$. Recall from Lemma A5 that $\frac{\partial \acute{p}_n}{\partial \alpha_1} < 0$. We can conclude that

$$\frac{d\alpha_1(\boldsymbol{x}_{n+1})}{dx_{n+1}} > 0. \tag{A12}$$

Therefore, for each $i \in \{1, \ldots, n-1\}$, we have that

$$\frac{dp_i^{\mathcal{L}}(\boldsymbol{x}_{n+1})}{dx_{n+1}} = \left.\frac{\partial \acute{p}_i}{\partial \alpha_1}\right|_{\alpha_1 = \alpha_1(\boldsymbol{x}_{n+1})} \times \frac{d\alpha_1(\boldsymbol{x}_{n+1})}{dx_{n+1}}. \tag{A13}$$

By Lemma A5 and (A12), the right-hand side of (A13) is negative, which implies that $p_i^{\mathcal{L}}(\boldsymbol{x}_{n+1})$ is decreasing in x_{n+1}.

Case II: $\boldsymbol{k(x_{n+1}) = n+1}$. By Lemma A5 and (A13), it suffices to prove (A12). The KKT condition (12) becomes

$$p_{n+1}^{\mathcal{L}}(\boldsymbol{x}_{n+1}) = c + \frac{F\big(\alpha_n(\boldsymbol{x}_{n+1})\big)}{\frac{f\left(\alpha_n(\boldsymbol{x}_{n+1})\right)}{x_n - x_{n+1}}} = c + (x_n - x_{n+1}) \times \frac{F\big(\alpha_n(\boldsymbol{x}_{n+1})\big)}{f\,\big(\alpha_n(\boldsymbol{x}_{n+1})\big)}. \tag{A14}$$

Setting $i = n$ in (15), we can obtain that

$$p_n^{\mathcal{L}}(\boldsymbol{x}_{n+1}) = p_{n+1}^{\mathcal{L}}(\boldsymbol{x}_{n+1}) + \alpha_n(\boldsymbol{x}_{n+1})(x_n - x_{n+1}). \tag{A15}$$

Combining (A14) and (A15) yields that

$$p_n^{\mathcal{L}}(\boldsymbol{x}_{n+1}) = c + \left[\frac{F\left(\alpha_n(\boldsymbol{x}_{n+1})\right)}{f\left(\alpha_n(\boldsymbol{x}_{n+1})\right)} + \alpha_n(\boldsymbol{x}_{n+1})\right] \times (x_n - x_{n+1}). \tag{A16}$$

Setting $j = n-1$ in (18) yields that

$$\acute{p}_n(\alpha_1, \boldsymbol{x}_n) = c + \frac{F\left(\acute{\alpha}_{n-1}(\alpha_1;\boldsymbol{x}_n)\right) - F\left(\acute{\alpha}_n(\alpha_1;\boldsymbol{x}_{n+1})\right)}{\frac{f\left(\acute{\alpha}_{n-1}(\alpha_1;\boldsymbol{x}_n)\right)}{x_{n-1}-x_n} + \frac{f\left(\acute{\alpha}_n(\alpha_1;\boldsymbol{x}_{n+1})\right)}{x_n - x_{n+1}}}.$$

Taking the derivative of $\acute{p}_n(\alpha_1;\boldsymbol{x}_n)$ with respect to x_{n+1}, we have that

$$\begin{aligned} &\underbrace{\frac{\partial}{\partial \alpha_n}\left[\frac{F\left(\acute{\alpha}_{n-1}(\alpha_1;\boldsymbol{x}_n)\right) - F(\alpha_n)}{\frac{f\left(\acute{\alpha}_{n-1}(\alpha_1;\boldsymbol{x}_n)\right)}{x_{n-1}-x_n} + \frac{f(\alpha_n)}{x_n - x_{n+1}}}\right]\Bigg|_{\alpha_n = \acute{\alpha}_n(\alpha_1;\boldsymbol{x}_{n+1})}}_{<0} \times \frac{\partial \acute{\alpha}_n}{\partial x_{n+1}} \\ &\qquad + \underbrace{\frac{\partial}{\partial x_{n+1}}\left[\frac{F\left(\acute{\alpha}_{n-1}(\alpha_1;\boldsymbol{x}_n)\right) - F(\alpha_n)}{\frac{f\left(\acute{\alpha}_{n-1}(\alpha_1;\boldsymbol{x}_n)\right)}{x_{n-1}-x_n} + \frac{f(\alpha_n)}{x_n - x_{n+1}}}\right]\Bigg|_{\alpha_n = \acute{\alpha}_n(\alpha_1;\boldsymbol{x}_{n+1})}}_{<0} = 0, \end{aligned}$$

where the equality follows from the implicit function theorem. The above equation implies that $\frac{\partial \acute{\alpha}_n}{\partial x_{n+1}} < 0$.

Recall that $p_n^{\mathcal{L}}(\boldsymbol{x}_{n+1}) = \acute{p}_n(\alpha_1(\boldsymbol{x}_{n+1});\boldsymbol{x}_n)$ and $\alpha_n(\boldsymbol{x}_{n+1}) = \acute{\alpha}_n(\alpha_1(\boldsymbol{x}_{n+1});\boldsymbol{x}_{n+1})$; together with (A16), we have that

$$\acute{p}_n(\alpha_1(\boldsymbol{x}_{n+1});\boldsymbol{x}_n) = c + \left[\frac{F\left(\acute{\alpha}_n(\alpha_1(\boldsymbol{x}_{n+1});\boldsymbol{x}_{n+1})\right)}{f\left(\acute{\alpha}_n(\alpha_1(\boldsymbol{x}_{n+1});\boldsymbol{x}_{n+1})\right)} + \acute{\alpha}_n\left(\alpha_1(\boldsymbol{x}_{n+1});\boldsymbol{x}_{n+1}\right)\right] \times (x_n - x_{n+1}).$$

Taking the derivative of $\acute{p}_n(\alpha_1(\boldsymbol{x}_{n+1});\boldsymbol{x}_n)$ with respect to x_{n+1}, we can obtain that

$$\begin{aligned} &\underbrace{\frac{\partial \acute{p}_n}{\partial \alpha_1}}_{<0} \times \frac{d\alpha_1(\boldsymbol{x}_{n+1})}{dx_{n+1}} \\ = &- \underbrace{\left[\frac{F\left(\acute{\alpha}_n(\alpha_1(\boldsymbol{x}_{n+1});\boldsymbol{x}_{n+1})\right)}{f\left(\acute{\alpha}_n(\alpha_1(\boldsymbol{x}_{n+1});\boldsymbol{x}_{n+1})\right)} + \acute{\alpha}_n\left(\alpha_1(\boldsymbol{x}_{n+1});\boldsymbol{x}_{n+1}\right)\right]}_{>0} \end{aligned}$$

$$+\underbrace{\frac{\partial}{\partial\alpha_n}\left[\frac{F(\alpha_n)}{f(\alpha_n)}+\alpha_n\right]\Bigg|_{\alpha_n=\acute{\alpha}_n\left(\alpha_1(\boldsymbol{x}_{n+1});\boldsymbol{x}_{n+1}\right)}}_{>0}\times\left[\underbrace{\frac{\partial\acute{\alpha}_n}{\partial\alpha_1}}_{>0}\times\frac{d\alpha_1(\boldsymbol{x}_{n+1})}{dx_{n+1}}+\underbrace{\frac{\partial\acute{\alpha}_n}{\partial x_{n+1}}}_{<0}\right],$$

where the equality again follows from the implicit function theorem. This implies that $\frac{d\alpha_1(\boldsymbol{x}_{n+1})}{dx_{n+1}} > 0$ and completes the proof. ■

Proof of Lemma A11

Proof. We first show that $\mathcal{C}_1(g,\kappa) \geq \kappa^2 + 2\kappa$. Note that the PDF of the conditional distribution $x^{(2)}|(x^{(1)}, x^{(3)})$ is $\frac{g^{\mathrm{m}}(x^{(2)})}{G^{\mathrm{m}}(x^{(1)})-G^{\mathrm{m}}(x^{(3)})}$ and is weakly decreasing in $x^{(2)}$. Therefore,

$$\begin{aligned}
&\mathbb{E}_{\boldsymbol{x}\sim g}\left[\mathbb{1}\left(\frac{x^{(2)}-x^{(3)}}{x^{(1)}-x^{(2)}}\geq\kappa\right)\times(x^{(1)}-x^{(2)})\middle|\left(x^{(1)},x^{(3)}\right)\right]\\
&=\left(x^{(1)}-x^{(3)}\right)\times\mathbb{E}_{\boldsymbol{x}\sim g}\left[\mathbb{1}\left(\frac{x^{(2)}-x^{(3)}}{x^{(1)}-x^{(2)}}\geq\kappa\right)\times\frac{x^{(1)}-x^{(2)}}{x^{(1)}-x^{(3)}}\middle|\left(x^{(1)},x^{(3)}\right)\right]\\
&=\left(x^{(1)}-x^{(3)}\right)^2\times\int_{\frac{\kappa}{\kappa+1}}^{1}(1-s)\frac{g^{\mathrm{m}}\big(sx^{(1)}+(1-s)x^{(3)}\big)}{G^{\mathrm{m}}(x^{(1)})-G^{\mathrm{m}}(x^{(3)})}ds\\
&\leq\left(x^{(1)}-x^{(3)}\right)^2\times\frac{g^{\mathrm{m}}\big(\frac{\kappa x^{(1)}+x^{(3)}}{\kappa+1}\big)}{G^{\mathrm{m}}(x^{(1)})-G^{\mathrm{m}}(x^{(3)})}\times\int_{\frac{\kappa}{\kappa+1}}^{1}(1-s)ds\\
&=\left(x^{(1)}-x^{(3)}\right)^2\times\frac{g^{\mathrm{m}}\big(\frac{\kappa x^{(1)}+x^{(3)}}{\kappa+1}\big)}{G^{\mathrm{m}}(x^{(1)})-G^{\mathrm{m}}(x^{(3)})}\times\frac{1}{2(\kappa+1)^2}.
\end{aligned}\tag{A17}$$

Similarly, we have that

$$\begin{aligned}
&\mathbb{E}_{\boldsymbol{x}\sim g}\left[\mathbb{1}\left(\frac{x^{(2)}-x^{(3)}}{x^{(1)}-x^{(2)}}\leq\kappa\right)\times\left(x^{(1)}-x^{(2)}\right)\middle|\left(x^{(1)},x^{(3)}\right)\right]\\
&\geq\left(x^{(1)}-x^{(3)}\right)^2\times\frac{g^{\mathrm{m}}\big(\frac{\kappa x^{(1)}+x^{(3)}}{\kappa+1}\big)}{G^{\mathrm{m}}(x^{(1)})-G^{\mathrm{m}}(x^{(3)})}\times\int_{0}^{\frac{\kappa}{\kappa+1}}(1-s)ds\\
&=\left(x^{(1)}-x^{(3)}\right)^2\times\frac{g^{\mathrm{m}}\big(\frac{\kappa x^{(1)}+x^{(3)}}{\kappa+1}\big)}{G^{\mathrm{m}}(x^{(1)})-G^{\mathrm{m}}(x^{(3)})}\times\frac{\kappa^2+2\kappa}{2(\kappa+1)^2}.
\end{aligned}\tag{A18}$$

Combining (A17) and (A18) yields that

$$\mathbb{E}_{\boldsymbol{x}\sim g}\left[\mathbb{1}\left(\frac{x^{(2)}-x^{(3)}}{x^{(1)}-x^{(2)}}\leq\kappa\right)\times(x^{(1)}-x^{(2)})\middle|\left(x^{(1)},x^{(3)}\right)\right]$$

$$\geq (\kappa^2 + 2\kappa) \times \mathbb{E}_{\boldsymbol{x}\sim g}\left[\mathbb{1}\left(\frac{x^{(2)} - x^{(3)}}{x^{(1)} - x^{(2)}} \geq \kappa \right) \times (x^{(1)} - x^{(2)}) \middle| (x^{(1)}, x^{(3)}) \right].$$

By the law of iterated expectations, we can obtain that

$$\begin{aligned} &\mathbb{E}_{\boldsymbol{x}\sim g}\left[\mathbb{1}\left(\frac{x^{(2)} - x^{(3)}}{x^{(1)} - x^{(2)}} \leq \kappa \right) \times (x^{(1)} - x^{(2)}) \right] \\ &\geq (\kappa^2 + 2\kappa) \times \mathbb{E}_{\boldsymbol{x}\sim g}\left[\mathbb{1}\left(\frac{x^{(2)} - x^{(3)}}{x^{(1)} - x^{(2)}} \geq \kappa \right) \times (x^{(1)} - x^{(2)}) \right], \end{aligned}$$

which implies that $\mathcal{C}_1(g, \kappa) \geq \kappa^2 + 2\kappa$. Setting $\kappa = \frac{1}{2}$ yields that $\mathcal{C}_1(g, \frac{1}{2}) \geq \frac{5}{4}$.

Next, we show that $\min_f \mathcal{C}_2(f, \frac{1}{2}) > \frac{4}{5}$. It is useful to state the following lemma:

Lemma B1 *Fix κ and suppose that $f^\ddagger$ minimizes $\mathcal{C}_2(f, \kappa)$ among all density functions f that satisfy Assumption 2, $f(\underline{t}) < 1/\underline{t}$, and (6). Then $\ln \frac{f^\ddagger(t)}{t}$ is piecewise linear with at most three segments.*

Proof. Fix $\boldsymbol{x} = (\kappa+1, \kappa, 0)$. For each PDF $f(\cdot)$ that satisfies Assumption 2, $f(\underline{t}) < 1/\underline{t}$, and (6), we construct an auxiliary distribution with PDF $f^\ddagger(\cdot)$ that takes the following form:

$$f^\ddagger(t) := \begin{cases} 0, & t < t_0^\ddagger \\ \mathcal{M}_1 t e^{\beta_1(t - \alpha_1(\boldsymbol{x}))}, & t_0^\ddagger \leq t \leq t_1^\ddagger, \\ \mathcal{M}_2 t e^{\beta_2(t - \alpha^*)}, & t_1^\ddagger \leq t \leq t_2^\ddagger, \\ \mathcal{M}_3 t e^{\beta_3(t - \alpha_2(\boldsymbol{x}))}, & t_2^\ddagger \leq t \leq t_3^\ddagger, \\ 0, & t > t_3^\ddagger. \end{cases} \tag{A19}$$

Further, let $F^\ddagger(t) := \int_0^t f^\ddagger(s)ds$.

The set of parameters $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \beta_1, \beta_2, \beta_3, t_0^\ddagger, t_1^\ddagger, t_2^\ddagger, t_3^\ddagger)$ in (A19) are to be constructed to satisfy

$$f^\ddagger(t) = f(t), t \in \left\{\alpha^*, \alpha_1(\boldsymbol{x}), \alpha_2(\boldsymbol{x})\right\}, \tag{A20}$$

$$F^\ddagger(t) = F(t), t \in \left\{\alpha^*, \alpha_1(\boldsymbol{x}), \alpha_2(\boldsymbol{x})\right\}, \tag{A21}$$

$$f^{\ddagger\prime}(\alpha^*) = f'(\alpha^*), \tag{A22}$$

$$F^\ddagger(t_3^\ddagger) = 1, \tag{A23}$$

and ensure that $f^\ddagger(t)$ is continuous at $t \in \{t_1^\ddagger, t_2^\ddagger\}$. Moreover, we require that $t_0^\ddagger \in [\underline{t}, \alpha_2(\boldsymbol{x})]$, $t_1^\ddagger \in [\alpha_2(\boldsymbol{x}), \alpha^*]$, $t_2^\ddagger \in [\alpha^*, \alpha_1(\boldsymbol{x})]$, and $t_3^\ddagger \in [\alpha_1(\boldsymbol{x}), \overline{t}]$. As will be clear later, under loyalty-based pricing, the equilibrium pricing schedules—i.e., $\big(p_i^{\mathcal{L}}(\boldsymbol{x})\big)_{i=1,2,3}$ and $\big(p_i^{\mathcal{L}}(\kappa+1,\kappa)\big)_{i=1,2}$—and the equilibrium cutoffs—i.e., $\big(\alpha_1(\boldsymbol{x}),\alpha_2(\boldsymbol{x})\big)$ under triopoly and α^* under duopoly with density f—are the same as those with the constructed density $f^\ddagger(t)$. Moreover, $\mathcal{C}_2(f^\ddagger,\kappa)\leq\mathcal{C}_2(f,\kappa)$.

Step I We first prove the existence of the set of parameters. First, fixing $\big(\alpha_1(\boldsymbol{x}),\alpha^*,\alpha_2(\boldsymbol{x})\big)$, we set $\mathcal{M}_1=f(\alpha_2(\boldsymbol{x}))/\alpha_2(\boldsymbol{x})$, $\mathcal{M}_2=f(\alpha^*)/\alpha^*$, and $\mathcal{M}_3=f(\alpha_1(\boldsymbol{x}))/\alpha_1(\boldsymbol{x})$. It is straightforward to verify that the constructed $(\mathcal{M}_1,\mathcal{M}_2,\mathcal{M}_3)$ satisfies (A20).

Second, we construct β_2 such that (A22) is satisfied. By (A19), we have that $\frac{d}{dt}\ln\frac{f^\ddagger(t)}{t}=\beta_2$. Evidently, (A22) is satisfied when we set $\beta_2=\frac{d}{dt}\ln\frac{f(t)}{t}|_{t=\alpha^*}$.

Third, we construct $(t_1^\ddagger,t_2^\ddagger)$—which depends on β_1 and β_3—such that $f^\ddagger$ is continuous at $t_1^\ddagger$ and $t_2^\ddagger$. Again, by (A19), the continuity of $f^\ddagger$ at $t\in\{t_1^\ddagger,t_2^\ddagger\}$ is equivalent to

$$\begin{aligned}&\beta_1\left(t_1^\ddagger-\alpha_2(\boldsymbol{x})\right)+\ln\mathcal{M}_1=\beta_2\left(t_1^\ddagger-\alpha^*\right)+\ln\mathcal{M}_2,\text{ and}\\&\beta_2\left(t_2^\ddagger-\alpha^*\right)+\ln\mathcal{M}_2=\beta_3\left(t_2^\ddagger-\alpha_1(\boldsymbol{x})\right)+\ln\mathcal{M}_3,\end{aligned}$$

from which we can solve $(t_1^\ddagger,t_2^\ddagger)$ as follows:

$$\begin{aligned}t_1^\ddagger=&\frac{1}{\beta_1-\beta_2}\times\left[-\beta_2\alpha^*+\ln\mathcal{M}_2+\beta_1\alpha_2(\boldsymbol{x})-\ln\mathcal{M}_1\right],\text{ and}\\t_2^\ddagger=&\frac{1}{\beta_2-\beta_3}\times\left[\beta_2\alpha^*-\ln\mathcal{M}_2-\beta_3\alpha_1(\boldsymbol{x})+\ln\mathcal{M}_3\right].\end{aligned}$$

Note that our construction requires that $t_1^\ddagger\in\big[\alpha_2(\boldsymbol{x}),\alpha^*\big]$ and $t_2^\ddagger\in\big[\alpha^*,\alpha_1(\boldsymbol{x})\big]$, which will be verified after β_1 and β_3 are pinned down later.

From the above analysis, it remains to construct $(\beta_1,\beta_3,t_0^\ddagger,t_3^\ddagger)$ to satisfy (A21) and (A23), which are equivalent to the following equations:

$$F^\ddagger(\alpha^*)-F^\ddagger\left(\alpha_2(\boldsymbol{x})\right)=F(\alpha^*)-F\left(\alpha_2(\boldsymbol{x})\right),\tag{A24}$$

$$F^\ddagger\left(\alpha_1(\boldsymbol{x})\right)-F^\ddagger(\alpha^*)=F\left(\alpha_1(\boldsymbol{x})\right)-F(\alpha^*),\tag{A25}$$

$$F\left(\alpha_2(\boldsymbol{x})\right)=F^\ddagger\left(\alpha_2(\boldsymbol{x})\right),\text{ and}\tag{A26}$$

$$1-F\left(\alpha_1(\boldsymbol{x})\right)=F^\ddagger(t_3^\ddagger)-F^\ddagger\left(\alpha_1(\boldsymbol{x})\right).\tag{A27}$$

We first construct β_1 and β_3 to satisfy (A24) and (A25), respectively. By Lemma A12, we have that $\alpha^* > \alpha_2(\boldsymbol{x})$. Therefore, (A24) can be expressed as

$$\int_{\alpha_2(\boldsymbol{x})}^{\alpha^*} f(t)dt = \int_{\alpha_2(\boldsymbol{x})}^{\alpha^*} f^\ddagger(t)dt = \int_{\alpha_2(\boldsymbol{x})}^{t_1^\ddagger} \mathcal{M}_1 t e^{\beta_1(t-\alpha_1(\boldsymbol{x}))}dt + \int_{t_1^\ddagger}^{\alpha^*} \mathcal{M}_2 t e^{\beta_2(t-\alpha^*)}dt =: \mathcal{A}_1(\beta_1),$$

where the second equality follows from (A19). It suffices to show that there exists β_1 such that $\mathcal{A}_1(\beta_1) = \int_{\alpha_2(\boldsymbol{x})}^{\alpha^*} f(t)dt$.

Define $\underline{\beta}_1 := \frac{\ln\frac{f(\alpha^*)}{\alpha^*} - \ln\frac{f(\alpha_2(\boldsymbol{x}))}{\alpha_2(\boldsymbol{x})}}{\alpha^*-\alpha_2(\boldsymbol{x})}$ and $\overline{\beta}_1 := \frac{d}{dt}\ln\frac{f(t)}{t}\Big|_{t=\alpha_2(\boldsymbol{x})}$. Fix $\beta_1 = \underline{\beta}_1$. By the concavity of the function $\ln\frac{f(t)}{t}$ and the constructed $(\mathcal{M}_1, \mathcal{M}_2, t_1^\ddagger, \beta_2)$, we can verify that $f^\ddagger(t) \leq f(t)$ for all $t \in [\alpha_2(\boldsymbol{x}), \alpha^*]$, which implies that $\mathcal{A}_1(\underline{\beta}_1) = \int_{\alpha_2(\boldsymbol{x})}^{\alpha^*} f^\ddagger(t)dt \leq \int_{\alpha_2(\boldsymbol{x})}^{\alpha^*} f(t)dt$. Similarly, fixing $\beta_1 = \overline{\beta}_1$, we can verify that $f^\ddagger(t) \geq f(t)$ for $t \in [\alpha_2(\boldsymbol{x}), \alpha^*]$, which implies that $\mathcal{A}_1(\overline{\beta}_1) \geq \int_{\alpha_2(\boldsymbol{x})}^{\alpha^*} f(t)dt$. Therefore, there exists β_1 with

$$\frac{\ln\frac{f(\alpha^*)}{\alpha^*} - \ln\frac{f\left(\alpha_2(\boldsymbol{x})\right)}{\alpha_2(\boldsymbol{x})}}{\alpha^* - \alpha_2(\boldsymbol{x})} \equiv \underline{\beta_1} \leq \beta_1 \leq \overline{\beta}_1 \equiv \frac{d}{dt}\ln\frac{f(t)}{t}\bigg|_{t=\alpha_2(\boldsymbol{x})}, \tag{A28}$$

such that (A24) is satisfied. From (A28) and the construction of $t_1^\ddagger$, it can be verified that $t_1^\ddagger \in [\alpha_2(\boldsymbol{x}), \alpha^*]$. Similarly, we can show that there exists β_3 such that (A25) is satisfied; moreover, $t_2^\ddagger \in [\alpha^*, \alpha_1(\boldsymbol{x})]$.

Next, we construct $t_0^\ddagger$ and $t_3^\ddagger$ to satisfy (A26) and (A27), respectively. By (A19), we have that

$$F^\ddagger\left(\alpha_2(\boldsymbol{x})\right) = \int_{t_0^\ddagger}^{\alpha_2(\boldsymbol{x})} f^\ddagger(t)dt = \int_{t_0^\ddagger}^{\alpha_2(\boldsymbol{x})} \mathcal{M}_1 t e^{\beta_1\left(t-\alpha_2(\boldsymbol{x})\right)}dt.$$

Clearly, $F^\ddagger(\alpha_2(\boldsymbol{x}))$ is decreasing in $t_0^\ddagger$ and $F^\ddagger(\alpha_2(\boldsymbol{x})) = 0 < F(\alpha_2(\boldsymbol{x}))$ at $t_0^\ddagger = \alpha_2(\boldsymbol{x})$. Therefore, it suffices to show that $F^\ddagger\left(\alpha_2(\boldsymbol{x})\right) \geq F\left(\alpha_2(\boldsymbol{x})\right)$ at $t_0^\ddagger = \underline{t}$, which is equivalent to

$$\int_{\underline{t}}^{\alpha_2(\boldsymbol{x})} \mathcal{M}_1 t e^{\beta_1\left(t-\alpha_2(\boldsymbol{x})\right)}dt \geq \int_{\underline{t}}^{\alpha_2(\boldsymbol{x})} f(t)dt. \tag{A29}$$

In fact, from the concavity of $\ln\frac{f(t)}{t}$, $f\left(\alpha_2(\boldsymbol{x})\right) = f^\ddagger\left(\alpha_2(\boldsymbol{x})\right)$, and (A28), we can conclude that

$$\mathcal{M}_1 t e^{\beta_1\left(t-\alpha_2(\boldsymbol{x})\right)} \geq f(t), \forall\, t \in \left[\underline{t}, \alpha_2(\boldsymbol{x})\right], \tag{A30}$$

which implies (A29). Therefore, there exists $t_0^\ddagger \in [\underline{t}, \alpha_2^*(\boldsymbol{x})]$ to satisfy (A26). Similarly, we can show that there exists $t_3^\ddagger \in [\alpha_1^*(\boldsymbol{x}), \overline{t}\,]$ to satisfy (A27). This completes the construction.

Step II We show that $\mathcal{C}_2(f^\ddagger, \kappa) \leq \mathcal{C}_2(f, \kappa)$. It is useful to prove the following lemma:

Lemma B2 *Suppose that $f^{\ddagger}$ is defined in* (A19) *such that* (A20)-(A23) *hold. The following statements hold:*

(i) $\frac{f^{\ddagger}(t)}{t}$ *is piecewise linear with at most three segments.*

(ii) $F^{\ddagger}(\cdot|t \in [\underline{t}, \alpha_2(\boldsymbol{x})]) \geq_{FOSD} F(\cdot|t \in [\underline{t}, \alpha_2(\boldsymbol{x})])$ *and* $F(\cdot|t \in [\alpha_1(\boldsymbol{x}), \bar{t}]) \geq_{FOSD} F^{\ddagger}(\cdot|t \in [\alpha_1(\boldsymbol{x}), \bar{t}])$.

(iii) $F^{\ddagger}(\cdot|t \in [\underline{t}, \alpha^*]) \geq_{FOSD} F(\cdot|t \in [\underline{t}, \alpha^*])$ *and* $F(\cdot|t \in [\alpha^*, \bar{t}]) \geq_{FOSD} F^{\ddagger}(\cdot|t \in [\alpha^*, \bar{t}])$.

Proof. Part (i) is obvious. For part (ii), $F^{\ddagger}(\cdot|t \in [\underline{t}, \alpha_2(\boldsymbol{x})]) \geq_{FOSD} F(\cdot|t \in [\underline{t}, \alpha_2(\boldsymbol{x})])$ follows immediately from (A30). Similarly, we can show that $F(\cdot|t \in [\alpha_1(\boldsymbol{x}), \bar{t}]) \geq_{FOSD} F^{\ddagger}(\cdot|t \in [\alpha_1(\boldsymbol{x}), \bar{t}])$. It remains to prove part (iii). Next, we prove $F^{\ddagger}(\cdot|t \in [\underline{t}, \alpha^*]) \geq_{FOSD} F(\cdot|t \in [\underline{t}, \alpha^*])$. The proof of $F(\cdot|t \in [\alpha^*, \bar{t}]) \geq_{FOSD} F^{\ddagger}(\cdot|t \in [\alpha^*, \bar{t}])$ is similar.

Note that $F^{\ddagger}(\alpha_2(\boldsymbol{x})) = F(\alpha_2(\boldsymbol{x}))$, $F^{\ddagger}(\alpha^*) = F(\alpha^*)$, and $F^{\ddagger}(\cdot|t \in [\underline{t}, \alpha_2(\boldsymbol{x})]) \geq_{FOSD} F(\cdot|t \in [\underline{t}, \alpha_2(\boldsymbol{x})])$. It suffices to show that $F^{\ddagger}(\cdot|t \in [\alpha_2(\boldsymbol{x}), \alpha^*]) \geq_{FOSD} F(\cdot|t \in [\alpha_2(\boldsymbol{x}), \alpha^*])$, which holds if we can show that there exists $\xi \in [\alpha_2(\boldsymbol{x}), \alpha^*]$ such that $f^{\ddagger}(t) \leq f(t)$ for $t \in [\alpha_2(\boldsymbol{x}), \xi]$ and $f^{\ddagger}(t) \geq f(t)$ for $t \in [\xi, \alpha^*]$.

Recall $t_1^{\ddagger} \in [\alpha_2(\boldsymbol{x}), \alpha^*]$. By (A22) and the concavity of $\ln \frac{f(t)}{t}$, we have that

$$f^{\ddagger}(t) = \mathcal{M}_2 t e^{\beta_2(t-\alpha^*)} \geq f(t), \forall\, t \in [t_1^{\ddagger}, \alpha^*].$$

Therefore, it suffices to show that there exists $\xi \in [\alpha_2(\boldsymbol{x}), t_1^{\ddagger}]$ such that $f^{\ddagger}(t) \leq f(t)$ for $t \in [\alpha_2(\boldsymbol{x}), \xi]$ and $f^{\ddagger}(t) \geq f(t)$ for $t \in [\xi, t_1^{\ddagger}]$.

Note that the difference $\ln \frac{f(t)}{t} - \ln \frac{f^{\ddagger}(t)}{t}$ is concave in t for $t \in [\alpha_2(\boldsymbol{x}), t_1^{\ddagger}]$. Moreover, by (A20) and (A28), this difference equals zero at $t = \alpha_2(\boldsymbol{x})$ and has a strictly positive right derivative there. Therefore, there exists $\xi \in [\alpha_2(\boldsymbol{x}), t_1^{\ddagger}]$ such that $\ln \frac{f(t)}{t} - \ln \frac{f^{\ddagger}(t)}{t} \geq 0$—or equivalently, $f^{\ddagger}(t) \leq f(t)$—for $t \in [\alpha_2(\boldsymbol{x}), \xi]$ and $\ln \frac{f(t)}{t} - \ln \frac{f^{\ddagger}(t)}{t} \leq 0$—or equivalently, $f^{\ddagger}(t) \geq f(t)$—for $t \in [\xi, t_1^{\ddagger}]$. This concludes the proof. ■

Recall $\mathcal{C}_2(f, \kappa)$. It suffices to show that

$$\left[\widetilde{V}^{\mathcal{L}}(\boldsymbol{x}) - V^{\mathcal{F}}(\boldsymbol{x})\right]\Big|_{t\sim F^{\ddagger}} \leq \left[\widetilde{V}^{\mathcal{L}}(\boldsymbol{x}) - V^{\mathcal{F}}(\boldsymbol{x})\right]\Big|_{t\sim F} \quad \text{and} \tag{A31}$$

$$\left[V^{\mathcal{F}}(\kappa+1, \kappa) - \widetilde{V}^{\mathcal{L}}(\kappa+1, \kappa)\right]\Big|_{t\sim F^{\ddagger}} \geq \left[V^{\mathcal{F}}(\kappa+1, \kappa) - \widetilde{V}^{\mathcal{L}}(\kappa+1, \kappa)\right]\Big|_{t\sim F}. \tag{A32}$$

We first prove (A31). Carrying out the algebra, we have that

$$\left[\widetilde{V}^{\mathcal{L}}(\boldsymbol{x}) - V^{\mathcal{F}}(\boldsymbol{x})\right]\Big|_{t\sim F}$$

$$
\begin{aligned}
=&\int_{\alpha_1(\boldsymbol{x})}^{\bar{t}} \left[v + tx_1 - p_1^{\mathcal{L}}(\boldsymbol{x})\right] f(t)dt + \int_{\underline{t}}^{\alpha_1(\boldsymbol{x})} \left[v + tx_2 - p_2^{\mathcal{L}}(\boldsymbol{x})\right] f(t)dt \\
&- \int_{\underline{t}}^{\bar{t}} \left[v + tx_1 - t(x_1 - x_2)\right] f(t)dt \\
=&(x_1 - x_2)\int_{\alpha_1(\boldsymbol{x})}^{\bar{t}} tf(t)dt - p_1^{\mathcal{L}}(\boldsymbol{x})\left[1 - F(\alpha_1(\boldsymbol{x}))\right] - p_2^{\mathcal{L}}(\boldsymbol{x})F\left(\alpha_1(\boldsymbol{x})\right). \qquad \text{(A33)}
\end{aligned}
$$

By (A20) and (A21), under loyalty-based pricing, the equilibrium pricing schedules $\left(p_i^{\mathcal{L}}(\boldsymbol{x})\right)_{i=1,2,3}$ and the cutoffs $\left(\alpha_1(\boldsymbol{x}), \alpha_2(\boldsymbol{x})\right)$ with density f are the same as those with density $f^{\ddagger}$. Therefore, we have that

$$
\left[\widetilde{V}^{\mathcal{L}}(\boldsymbol{x}) - V^{\mathcal{F}}(\boldsymbol{x})\right]\Big|_{t\sim F^{\ddagger}} = (x_1 - x_2)\int_{\alpha_1(\boldsymbol{x})}^{\bar{t}} tf^{\ddagger}(t)dt - p_1^{\mathcal{L}}(\boldsymbol{x})\left[1 - F(\alpha_1(\boldsymbol{x}))\right] - p_2^{\mathcal{L}}(\boldsymbol{x})F\left(\alpha_1(\boldsymbol{x})\right). \qquad \text{(A34)}
$$

By (A33) and (A34), it suffices to show that $\int_{\alpha_1(\boldsymbol{x})}^{\bar{t}} tf(t)dt \geq \int_{\alpha_1(\boldsymbol{x})}^{\bar{t}} tf^{\ddagger}(t)dt$, which follows immediately from $F(\cdot|t \in [\alpha_1(\boldsymbol{x}), \bar{t}]) \geq_{FOSD} F^{\ddagger}(\cdot|t \in [\alpha_1(\boldsymbol{x}), \bar{t}])$, as shown in Lemma B2.

Next, we prove (A32). Fix $(x_1, x_2) = (\kappa + 1, \kappa)$. Note that

$$
\begin{aligned}
&\left[V^{\mathcal{F}}(\kappa+1,\kappa) - \widetilde{V}^{\mathcal{L}}(\kappa+1,\kappa)\right]\Big|_{t\sim F} \\
=&-\int_{\alpha^*}^{\bar{t}} \left[v + t(\kappa+1) - p_1^{\mathcal{L}}(\kappa+1,\kappa)\right] f(t)dt \\
&- \int_{\underline{t}}^{\alpha^*} \left[v + t\kappa - p_2^{\mathcal{L}}(\kappa+1,\kappa)\right] f(t)dt + \int_{\underline{t}}^{\bar{t}} \left[v + t\kappa\right] f(t)dt \\
=&-\int_{\alpha^*}^{\bar{t}} tf(t)dt + p_1^{\mathcal{L}}(\kappa+1,\kappa)\left[1 - F(\alpha^*)\right] + p_2^{\mathcal{L}}(\kappa+1,\kappa)F(\alpha^*). \qquad \text{(A35)}
\end{aligned}
$$

By (A20) and (A21), under loyalty-based pricing, the equilibrium pricing schedules $\left(p_i^{\mathcal{L}}(\kappa+1,\kappa)\right)_{i=1,2}$ and the cutoff α^* with density f are the same as those with density $f^{\ddagger}$. Therefore,

$$
\begin{aligned}
&\left[V^{\mathcal{F}}(\kappa+1,\kappa) - \widetilde{V}^{\mathcal{L}}(\kappa+1,\kappa)\right]\Big|_{t\sim F^{\ddagger}} \\
=&-\int_{\alpha^*}^{\bar{t}} tf^{\ddagger}(t)dt + p_1^{\mathcal{L}}(\kappa+1,\kappa)\left[1 - F(\alpha^*)\right] + p_2^{\mathcal{L}}(\kappa+1,\kappa)F(\alpha^*). \qquad \text{(A36)}
\end{aligned}
$$

By (A35) and (A36), it suffices to show that $\int_{\alpha^*}^{\bar{t}} tf(t)dt \geq \int_{\alpha^*}^{\bar{t}} tf^{\ddagger}(t)dt$, which follows immediately from $F(\cdot|t \in [\alpha^*, \bar{t}]) \geq_{FOSD} F^{\ddagger}(\cdot|t \in [\alpha^*, \bar{t}])$, as shown in Lemma B2. This completes the proof of Lemma B1. ■

By Lemma B1, to search for f that minimizes $\mathcal{C}_2(f,\kappa)$, it suffices to look over density functions that satisfy Assumption 2 and are piecewise linear with at most three segments. Note that these functions can be parameterized by seven parameters and it can be verified that $\min_f \mathcal{C}_2(f,\frac{1}{2}) > \frac{4}{5}$. This completes the whole proof of Lemma A11. ■

Proof of Lemma A12

Proof. By $f(\underline{t}) < 1/\underline{t}$, we have that $k(\boldsymbol{x}) \geq 2$. If $k(\boldsymbol{x}) = 2$, then we have that $\alpha_1(\boldsymbol{x}) = \alpha^* > \alpha_2(\boldsymbol{x}) = \underline{t}$. It remains to consider the case of $k(\boldsymbol{x}) \geq 3$.

From the proof of Lemma A2, we can conclude that $tf(t) + 2F(t) > 1$ if and only if $t > \alpha^*$. Therefore, it suffices to show that

$$\alpha_1(\boldsymbol{x}) f\left(\alpha_1(\boldsymbol{x})\right) + 2F\left(\alpha_1(\boldsymbol{x})\right) > 1, \tag{A37}$$

and

$$\alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x}) f\left(\alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\right) + 2F\left(\alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\right) < 1. \tag{A38}$$

In what follows, we prove (A37). The proof of (A38) is similar and omitted for brevity. Carrying out the algebra, we can obtain that

$$\begin{aligned}
\frac{1 - F\left(\alpha_1(\boldsymbol{x})\right)}{\frac{f\left(\alpha_1(\boldsymbol{x})\right)}{x_1 - x_2}} = p_1^{\mathcal{L}}(\boldsymbol{x}) - c = p_2^{\mathcal{L}}(\boldsymbol{x}) + \alpha_1(\boldsymbol{x})(x_1 - x_2) - c \\
&= \frac{F\left(\alpha_1(\boldsymbol{x})\right) - F\left(\alpha_2(\boldsymbol{x})\right)}{\frac{f\left(\alpha_1(\boldsymbol{x})\right)}{x_1 - x_2} + \frac{f\left(\alpha_2(\boldsymbol{x})\right)}{x_2 - x_3}} + \alpha_1(\boldsymbol{x})(x_1 - x_2) \\
&< \frac{F\left(\alpha_1(\boldsymbol{x})\right)}{\frac{f\left(\alpha_1(\boldsymbol{x})\right)}{x_1 - x_2}} + \alpha_1(\boldsymbol{x})(x_1 - x_2),
\end{aligned}$$

where the first equality follows from (10); the second equality follows from (15); the third equality follows from (11); and the inequality follows from the fact that $F(\alpha_2(\boldsymbol{x})) \geq 0$ and $f(\alpha_2(\boldsymbol{x})) > 0$. Simplifying the above condition gives (A37). This concludes the proof. ■

B Partial Coverage

In this part, we relax the assumption of full market coverage in equilibrium. A higher marginal cost c raises equilibrium prices and potentially leads some consumers to opt for their outside option with zero surplus. By varying the marginal cost, we can adjust the degree of market coverage in equilibrium and examine its implications for consumer welfare and industry profit across different pricing regimes.

A consumer purchases from firm i if and only if

$$v + tx_i - p_i \geq \max_{j \neq i} \left\{ v + tx_j - p_j, 0 \right\}.$$

A symmetric pure-strategy equilibrium under uniform pricing requires $p_j = p^{\mathcal{U}}$ for each $j \neq i$ in equilibrium. This condition can be rewritten as

$$p_i - p^{\mathcal{U}} \leq t \left[x_i - \max_{j \neq i} \left\{ x_j, \frac{p^{\mathcal{U}} - v}{t} \right\} \right].$$

With slight abuse of notation, we define

$$\hat{x}_i(\hat{y}) := x_i - \max_{j \neq i} \{x_j, \hat{y}\} \quad \text{and} \quad z(y) := tx_i - \max_{j \neq i} \{tx_j, y\}.$$

The CDF and PDF of $\hat{x}_i(\hat{y})$ are denoted by $\Psi(\cdot; \hat{y})$ and $\psi(\cdot; \hat{y})$, respectively; those of $z(y)$ are similarly denoted by $H(\cdot; y)$ and $h(\cdot; y)$. We impose the following regularity conditions in parallel with Assumption 1 and Assumption 3.

Assumption 1′ *$1 - \Psi(\cdot; \hat{y})$ is log-concave for each $\hat{y} \in \mathbb{R}$. Moreover,*

$$\frac{1 - \Psi(0; \hat{y})}{\psi(0; \hat{y})}$$

is weakly decreasing in $\hat{y}$.

Assumption 3′ *$1 - H(\cdot; y)$ is log-concave for each $y \in \mathbb{R}$. Moreover,*

$$\frac{1 - H(0; y)}{h(0; y)}$$

is weakly decreasing in y.

The following result can be obtained.

Proposition B1 ***(Welfare Comparison under Partial Coverage)*** *Suppose that Assumptions 1′, 2, and 3′ hold. There exists a unique symmetric equilibrium under $\mathcal{U}$ and $\mathcal{C}$ and a unique equilibrium outcome under $\mathcal{F}$ and $\mathcal{L}$. Moreover, if $f(\bar{t}) > 0$ and $g^{\mathrm{m}}(\bar{x}) > 0$, there exists a threshold $\underline{c} < v + \bar{t}\bar{x}$ such that the following results hold:*

(i) Uniform pricing maximizes consumer welfare when $c > \underline{c}$. Moreover,

$$\lim_{c \nearrow v + \bar{t}\bar{x}} V^{\mathcal{U}} : V^{\mathcal{C}} : V^{\mathcal{L}} : V^{\mathcal{F}} = 32 : 27 : 27 : 0.$$

(ii) Fully personalized pricing maximizes industry profit when $c > \underline{c}$. *Moreover,*

$$\lim_{c \nearrow v + \bar{t}\bar{x}} \Pi^{\mathcal{U}} : \Pi^{\mathcal{C}} : \Pi^{\mathcal{L}} : \Pi^{\mathcal{F}} = 48 : 54 : 54 : 108.$$

Proof of Proposition B1. We first characterize equilibrium under the four pricing regimes.

Under uniform pricing $\mathcal{U}$, firms choose a common price $p^{\mathcal{U}}$. Recall that

$$z(y) := tx_i - \max_{j \neq i}\{tx_j, y\},$$

with CDF and PDF $H(\cdot; y)$ and $h(\cdot; y)$. Firm i's demand at price p_i when all other firms charge $p^{\mathcal{U}}$, is

$$1 - H(p_i - p^{\mathcal{U}}; p^{\mathcal{U}} - v).$$

Thus the symmetric equilibrium price is characterized by

$$p^{\mathcal{U}} = c + \frac{1 - H(0; p^{\mathcal{U}} - v)}{h(0; p^{\mathcal{U}} - v)}. \tag{B1}$$

Assumption $3'$ guarantees existence and uniqueness.

Under choosiness-based pricing $\mathcal{C}$, firms observe t. Conditional on t, the game is a symmetric uniform-pricing game with effective outside threshold $(p^{\mathcal{C}}(t) - v)/t$. Hence

$$p^{\mathcal{C}}(t) = c + t\frac{1 - \Psi(0; (p^{\mathcal{C}}(t) - v)/t)}{\psi(0; (p^{\mathcal{C}}(t) - v)/t)}. \tag{B2}$$

Assumption $1'$ guarantees existence and uniqueness.

Under fully personalized pricing $\mathcal{F}$, firms observe both t and $\boldsymbol{x}$. The winning firm extracts the consumer's entire surplus whenever the consumer's highest valuation exceeds cost. Hence consumer welfare is zero up to ties, and industry profit is total realized surplus above cost:

$$\Pi^{\mathcal{F}} = \mathbb{E}_{t,\boldsymbol{x}}\left[\left(v + t\max_i x_i - c\right)_+\right], \qquad V^{\mathcal{F}} = 0. \tag{B3}$$

Under loyalty-based pricing $\mathcal{L}$, firms observe $\boldsymbol{x}$ but not t. Fix $\boldsymbol{x}$ and order firms so that $x_1 > \cdots > x_n$. The equilibrium continues to have a cutoff structure. Firms $1, \ldots, k(\boldsymbol{x})$ are active, and consumers with $t \in [\alpha_i(\boldsymbol{x}), \alpha_{i-1}(\boldsymbol{x}))$ purchase from firm i, with $\alpha_0(\boldsymbol{x}) = \bar{t}$. The first $k(\boldsymbol{x}) - 1$ firms' first-order conditions and the indifference conditions between adjacent active firms are the same as those in Lemma 2.

It remains to describe the terminal active firm. If $v+\underline{t}x_{k(\boldsymbol{x})+1}>0$, then even the lowest-$t$ consumer obtains positive gross surplus from firm $k(\boldsymbol{x})+1$. Hence the outside option is never relevant at the lower end of the market, and the equilibrium is characterized by the same KKT conditions as in Lemma 2, with $\alpha_{k(\boldsymbol{x})}(\boldsymbol{x})=\underline{t}$.

If instead $v+\underline{t}x_{k(\boldsymbol{x})+1}\leq 0$, the outside option may bind at the bottom of the market. In this case the terminal active firm $k(\boldsymbol{x})$ satisfies the following KKT conditions:

$$p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x})=c+\frac{F\left(\alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\right)-F\left(\alpha_{k(\boldsymbol{x})}(\boldsymbol{x})\right)}{\frac{f\left(\alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\right)}{x_{k(\boldsymbol{x})-1}-x_{k(\boldsymbol{x})}}+\xi f\left(\alpha_{k(\boldsymbol{x})}(\boldsymbol{x})\right)},\qquad \xi\in\left[0,\frac{1}{x_{k(\boldsymbol{x})}}\right], \tag{B4}$$

$$p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x})\leq v+\alpha_{k(\boldsymbol{x})}(\boldsymbol{x})x_{k(\boldsymbol{x})}, \tag{B5}$$

$$\xi\left[v+\alpha_{k(\boldsymbol{x})}(\boldsymbol{x})x_{k(\boldsymbol{x})}-p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x})\right]=0, \tag{B6}$$

$$\left(\frac{1}{x_{k(\boldsymbol{x})}}-\xi\right)\left(\alpha_{k(\boldsymbol{x})}(\boldsymbol{x})-\underline{t}\right)=0. \tag{B7}$$

To see this, suppose first that $\alpha_{k(\boldsymbol{x})}(\boldsymbol{x})>\underline{t}$. Then the marginal consumer with $t=\alpha_{k(\boldsymbol{x})}(\boldsymbol{x})$ is indifferent between buying from firm $k(\boldsymbol{x})$ and taking the outside option, so

$$p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x})=v+\alpha_{k(\boldsymbol{x})}(\boldsymbol{x})x_{k(\boldsymbol{x})}.$$

Firm $k(\boldsymbol{x})$'s first-order condition is

$$p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x})=c+\frac{F\left(\alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\right)-F\left(\alpha_{k(\boldsymbol{x})}(\boldsymbol{x})\right)}{\frac{f\left(\alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\right)}{x_{k(\boldsymbol{x})-1}-x_{k(\boldsymbol{x})}}+\frac{f\left(\alpha_{k(\boldsymbol{x})}(\boldsymbol{x})\right)}{x_{k(\boldsymbol{x})}}},$$

which is (B4) with $\xi=1/x_{k(\boldsymbol{x})}$. This also gives (B6) and (B7).

Now suppose that $\alpha_{k(\boldsymbol{x})}(\boldsymbol{x})=\underline{t}$. Then (B7) holds. If (B5) is slack, the outside-option constraint does not bind locally, and the terminal firm's first-order condition is

$$p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x})=c+\frac{F\left(\alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\right)-F\left(\alpha_{k(\boldsymbol{x})}(\boldsymbol{x})\right)}{\frac{f\left(\alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\right)}{x_{k(\boldsymbol{x})-1}-x_{k(\boldsymbol{x})}}},$$

which is (B4) with $\xi=0$; then (B6) also holds. If (B5) binds, the terminal firm is at a kink. The left and right derivatives imply (B4) for some $\xi\in[0,1/x_{k(\boldsymbol{x})}]$, and (B6) follows from the binding constraint. Existence and outcome uniqueness follow by the same shooting argument as in Lemma 2.

We now derive the limiting welfare and profit comparisons. Let

$$\Delta_c := v + \bar{t}\overline{x} - c.$$

We first consider uniform pricing. As $c \nearrow v + \bar{t}\overline{x}$, only consumers with (t, x_i) close to $(\bar{t}, \overline{x})$ purchase. Since $f(\bar{t}) > 0$ and $g^{\mathrm{m}}(\overline{x}) > 0$, the upper tail of $v + tx_i$ satisfies

$$\Pr(v + tx_i \geq c + s) = \frac{f(\bar{t})g^{\mathrm{m}}(\overline{x})}{2\bar{t}\overline{x}}(\Delta_c - s)^2(1 + o(1))$$

uniformly for $s \in [0, \Delta_c]$. In this limit, the probability that the same consumer values two distinct products above the cost threshold is of smaller order and can be ignored in the leading term. Thus firm i's profit under uniform pricing is

$$(p^{\mathcal{U}} - c)\frac{f(\bar{t})g^{\mathrm{m}}(\overline{x})}{2\bar{t}\overline{x}}(\Delta_c - (p^{\mathcal{U}} - c))^2(1 + o(1)).$$

Maximizing this expression gives

$$p^{\mathcal{U}} - c = \frac{1}{3}\Delta_c(1 + o(1)). \tag{B8}$$

Therefore,

$$\Pi^{\mathcal{U}} = \frac{2n}{27}\frac{f(\bar{t})g^{\mathrm{m}}(\overline{x})}{\bar{t}\overline{x}}\Delta_c^3(1 + o(1)). \tag{B9}$$

Consumer welfare under uniform pricing is

$$V^{\mathcal{U}} = n\mathbb{E}_{t,x_i}\left[(v + tx_i - p^{\mathcal{U}})_+\right](1 + o(1)).$$

Using (B8), we obtain

$$V^{\mathcal{U}} = \frac{4n}{81}\frac{f(\bar{t})g^{\mathrm{m}}(\overline{x})}{\bar{t}\overline{x}}\Delta_c^3(1 + o(1)). \tag{B10}$$

The same tail calculation applies to choosiness-based and loyalty-based pricing. Under $\mathcal{C}$, for each observed t close to $\bar{t}$, firms choose the monopoly price against the outside option in the brand-loyalty dimension. Under $\mathcal{L}$, for each observed x_i close to $\overline{x}$, firms choose the monopoly price against the outside option in the choosiness dimension. In both cases, the limiting markup is one half of the remaining surplus margin. Consequently,

$$\Pi^{\mathcal{C}} = \Pi^{\mathcal{L}} = \frac{n}{12}\frac{f(\bar{t})g^{\mathrm{m}}(\overline{x})}{\bar{t}\overline{x}}\Delta_c^3(1 + o(1)), \tag{B11}$$

$$V^{\mathcal{C}} = V^{\mathcal{L}} = \frac{n}{24}\frac{f(\bar{t})g^{\mathrm{m}}(\overline{x})}{\bar{t}\overline{x}}\Delta_c^3(1 + o(1)). \tag{B12}$$

Finally, under fully personalized pricing, firms extract all surplus from served consumers. Therefore,

$$\Pi^{\mathcal{F}} = \frac{n}{6}\frac{f(\bar{t})g^{\mathrm{m}}(\bar{x})}{\bar{t}\bar{x}}\Delta_c^3(1+o(1)), \tag{B13}$$

whereas

$$V^{\mathcal{F}} = o(\Delta_c^3). \tag{B14}$$

Combining (B10), (B12), and (B14), we obtain

$$\lim_{c \nearrow v+\bar{t}\bar{x}} V^{\mathcal{U}} : V^{\mathcal{C}} : V^{\mathcal{L}} : V^{\mathcal{F}} = \frac{4}{81} : \frac{1}{24} : \frac{1}{24} : 0 = 32 : 27 : 27 : 0.$$

Similarly, combining (B9), (B11), and (B13), we obtain

$$\lim_{c \nearrow v+\bar{t}\bar{x}} \Pi^{\mathcal{U}} : \Pi^{\mathcal{C}} : \Pi^{\mathcal{L}} : \Pi^{\mathcal{F}} = \frac{2}{27} : \frac{1}{12} : \frac{1}{12} : \frac{1}{6} = 48 : 54 : 54 : 108.$$

Thus, for all sufficiently large c, uniform pricing maximizes consumer welfare and fully personalized pricing maximizes industry profit. This completes the proof. ■

With a large marginal cost c, Proposition B1 states that, in the limiting case, uniform pricing $\mathcal{U}$ generates the highest consumer welfare, while fully personalized pricing $\mathcal{F}$ yields the highest industry profit. These rankings contrast with those under full market coverage. The intuition aligns with that of Rhodes and Zhou (2024). A large c effectively filters out competition and renders each firm a local monopolist: Conditional on a consumer whose value exceeds the large cost—i.e., $v + tx_i > c$ for some $i \in \mathcal{N}$—it is highly unlikely that this consumer values another product more than the cost threshold. Consequently, for each firm, competition with other products is overshadowed by competition with the outside option. The conventional wisdom of monopolistic first-degree price discrimination (Pigou, 1920) is reinstated in this setting, which suggests that finer consumer information benefits firms while harming consumers. In the absence of significant inter-firm competition, lacking either type of information would prevent a firm from perfectly profiling its consumers and fully extracting their surplus.

It is worth noting that when c is sufficiently large, information about consumers' choosiness plays a role analogous to that of brand loyalty. By Proposition B1, consumer welfare and industry profit under loyalty-based pricing $\mathcal{L}$ converge to those under choosiness-based pricing $\mathcal{C}$—i.e.,

$$\lim_{c \nearrow v+\bar{t}\bar{x}} \frac{V^{\mathcal{C}}}{V^{\mathcal{L}}} = \lim_{c \nearrow v+\bar{t}\bar{x}} \frac{\Pi^{\mathcal{C}}}{\Pi^{\mathcal{L}}} = 1.$$

To see this, consider choosiness-based pricing $\mathcal{C}$ for a fixed t. Firms will not price below c, and consumers with $v + tx_i \geq p_i$ purchase from firm i. When $c \nearrow v + \bar{t}\bar{x}$, only consumers with x_i close to $\bar{x}$ consider a purchase. This causes firm i to face a linear demand in this limiting case, regardless of the marginal distribution $g^{\mathrm{m}}(x_i)$. A similar rationale applies to loyalty-based pricing $\mathcal{L}$ for a fixed x_i. Each firm again faces linear demand in the limit, independent of the distribution $f(t)$. Consequently, both pricing regimes yield the same asymptotic consumer welfare and industry profit.