On Equilibrium Uniqueness in Generalized Multi-Prize Nested Lottery Contests*

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Abstract

The generalized multiple-prize nested lottery contest framework has been broadly applied to model noisy competitions that award prizes to multiple recipients. Equilibrium existence was not formally established until the recent contribution of Fu, Wu, and Zhu (2021b). This paper comprehensively examines the equilibrium uniqueness of the contest model. We first consider a multi-prize contest with identical players, which are commonly assumed in the literature. We verify that a symmetric equilibrium exists and that the equilibrium is unique, which lays a foundation for the numerous studies that adopt multi-prize nested lottery contest models. We then proceed to an asymmetric setting in which players may differ in their prize valuations, impact functions, and/or cost functions. We show that the equilibrium uniqueness persists when players are weakly heterogeneous. However, equilibrium uniqueness may fail—and multiple equilibria may arise—when players are sufficiently heterogeneous.

Keywords: Multi-prize Contest; Equilibrium Uniqueness; Discontinuous Game

JEL Classification Codes: C72, D72.

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1 Introduction

Contests are widespread in practice as an effective mechanism to mobilize focused efforts. Contenders in a contest strive to leapfrog, and their efforts are nonrefundable regardless of the outcome. A wide array of competitive activities exemplify such a phenomenon, ranging from college admissions and innovation challenges to the labor market inside firms. Two observations are prevalent in these scenarios. First, one's win or loss depends on not only his effort, but also various noisy and random factors. Second, although the majority of the relevant economics literature assumes a winner-take-all prize structure, such contests in reality often reward several top performers instead of concentrating the entire purse on a single grand prize. Consider, for instance, various sporting tournaments in which silver and bronze medals are awarded to the runner-up and second runner-up, respectively. Multiple prizes are also typical in R&D challenges and crowdsourcing competitions. The First Responder Unmanned Aircraft System (UAS) Endurance prize competition from the National Institute of Standard Technology (NIST)—which invited submissions of drone prototypes—not only rewarded the three top-ranked teams, but also set aside prizes for eight "best in class winners." The economics literature has also increasingly recognized the critical role played by the prevailing prize structure in shaping players' incentives and espoused the merit of multiple prizes (see, e.g., Kireyev, 2020; Fu, Wang, and Wu, 2021a) in diverse contexts.

The multiple-prize nested lottery contest framework (Clark and Riis, 1996, 1998b) generalizes the popularly adopted lottery contest model and has been broadly used to model noisy competitions that award prizes to multiple recipients (e.g., Szymanski and Valletti, 2005; Azmat and Möller, 2009; Brown, 2011; Fu and Lu, 2009, 2012a; Fu and Wu, 2021; Fu, Wang, and Wu, 2021a).[1] With a single prize and a given effort profile $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$, one wins the contest with a probability

$$p_i(\boldsymbol{x}) = \frac{f_i(x_i)}{\sum_{j=1}^{n} f_j(x_j)}, \tag{1}$$

which boils down to a generalized lottery contest. The function $f_i(\cdot)$, conventionally called the impact function, converts a player's effort x_i into his effective output in the contest.[2] When multiple prizes are to be awarded, the contest model literally resembles a sequential lottery process. Suppose that $l \geq 2$ prizes are available to $n \geq l$ players, with each eligible for at most one prize. The winner of the first prize is determined by the ratio-form contest

[1] All-pay auction models are also popularly adopted to model contest-like competitions, in which a higher effort ensures a win. Multiple prizes can readily be accommodated in an all-pay auction, as players are precisely ranked by their bids and each is awarded a prize according to their respective rank (see e.g., Clark and Riis, 1998a; Barut and Kovenock, 1998; Bulow and Levin, 2006; Siegel, 2009, 2010, 2014; Xiao, 2016; and Fang, Noe, and Strack, 2020).

[2] See Skaperdas (1996) for an axiomatic foundation of this contest success function.

success function (1). He is immediately removed from the pool of players eligible for the second prize; the recipient of the second prize is drawn from the rest of the players, and the probability of winning it—conditional on not having won the first prize—is given by the ratio of his effective output to the sum of those who remain in the pool. This process is repeated until all l prize recipients are distributed. Despite the intuitive and convenient analogy to a sequential lottery process, Fu and Lu (2012b) demonstrate that the contest is uniquely underpinned by a (simultaneous) noisy ranking system à la McFadden (1973, 1974), with the probability of being ranked in the mth place equal to that of being selected for the mth draw in the sequential lottery process.[3]

Despite its extensive applications, the literature has been scarce in illuminating the equilibrium fundamentals of this contest game. Equilibrium existence in the model was not formally established until the recent contribution of Fu, Wu, and Zhu (2021b). The majority of prior studies assume homogeneous players and solve for the symmetric effort profile that satisfies the first-order conditions. The solution is conveniently adopted for equilibrium prediction and applied to enable analysis of optimal contest design. The literature has yet to verify whether this convenient solution fulfills the requirement of Nash equilibrium and whether alternative equilibria exist.[4] More generally, an intriguing question is whether and to what extent a unique equilibrium arises in this game when the typically assumed symmetry conditions can also be relaxed. Such knowledge would lay a foundation for the numerous studies that examine contest design within this framework (e.g., Azmat and Möller, 2009; Fu and Lu, 2009, 2012a; Fu and Wu, 2021). Our paper aims to explore equilibrium uniqueness in the game, which fills the gap in the literature.

Szidarovszky and Okuguchi (1997) establish equilibrium existence and uniqueness in single-prize generalized lottery contests with concave impact functions. As demonstrated by Fu, Wu, and Zhu (2021b), however, the multi-prize nested lottery contest fundamentally differs from its single-prize variant in terms of the underlying game theoretic structure, which nullifies the various usual approaches for general equilibrium analysis in contests and discontinuous games. First, the contest, with multiple prizes, dismisses the regularity of an aggregative game possessed by its single-prize variant. Second, the multi-prize generalization compounds the payoff discontinuity inherent in lottery contests. Third, the winning probability specifications substantially complicate players' payoff functions and obscure their properties, as shown by Schweinzer and Segev (2012). These nuances have largely retarded

[3]Lu and Wang (2015) provide an axiomatic foundation for the model. Letina, Liu, and Netzer (2020) consider an optimal contest design problem, in which the designer is able to choose both the prize allocation and the contest success function; they show that the optimum can be achieved by running a nested Tullock contest.

[4]Clark and Riis (1996) examine the local concavity of players' payoff functions under this hypothetical equilibrium profile.

analysis of the contest game's equilibrium fundamentals.[5]

Snapshot of the Analysis In this paper, we first consider a symmetric contest setting with identical players, as typically assumed in the literature. We verify that the above-mentioned convenient solution—i.e., the symmetric effort profile that satisfies first-order conditions—constitutes a symmetric equilibrium, and no other equilibria exist. The uniqueness result thus lays a foundation for the numerous studies that adopt multi-prize nested lottery contest models. We then proceed to an asymmetric setting in which players may differ in their prize valuations, impact functions, and/or effort cost functions. The equilibrium remains unique when players are weakly heterogeneous. However, uniqueness may fail to hold when players are sufficiently heterogeneous, and we construct an example to demonstrate the possibility of multiple equilibria.

To be more specific, in the symmetric setting, we show that the equilibrium effort must be symmetric in a symmetric contest. The uniqueness of the symmetric equilibrium follows immediately from the concavity of players' payoff functions verified by Fu, Wu, and Zhu (2021b) under mild regularity conditions.

An asymmetric contest entails enormously more complicated strategic interactions among players. When players are equally capable and uniformly value the prizes, each simply strives to leap ahead for a higher rank. When they differ in their strengths and prize valuations, a player must plan contemplate strategically—e.g., which prize to fight for and whom to compete against. Players' strategies diverge, as do the responses, which could lead to multiple strategy profiles to form equilibrium points. Such concerns distinguish a multiple-prize contest from a single-prize one, because in the latter each player simply strives for top rank.

In the asymmetric setting, we generalize our result in the symmetric setting to show that the degree of asymmetry in equilibrium effort can be bounded by those in prize valuations, impact functions, and effort cost functions. As a result, the equilibrium effort is also weakly asymmetric in a weakly asymmetric contest. We then borrow Rosen's (1965) "*diagonal strict concavity*" condition to establish the uniqueness. Although our contest game does not meet the requirement of diagonal strict concavity, we take a detour to generalize Rosen's result and restore its relevance in our context. In particular, we show that an equilibrium is unique if (i) there exists a convex subset of the strategy space such that the diagonal strict concavity holds on the subset, and (ii) each strategy profile that does not belong to the subset cannot constitute an equilibrium. We focus on the case with weakly asymmetric impact and effort cost functions, which ensures a weakly asymmetric effort profile; we further

[5]See Fu, Wu, and Zhu (2021b) for a detailed account of the technical nuance arising from the multi-prize generalization.

verify diagonal strict concavity through a direct calculation. These allow us to establish equilibrium uniqueness.

Links to the Literature Our paper belongs to the literature that explores the equilibrium fundamentals of imperfectly discriminatory contests. A vast scholarly effort has been devoted to verifying the existence and/or uniqueness of Nash equilibrium in single-prize contests, which include Pérez-Castrillo and Verdier (1992); Szidarovszky and Okuguchi (1997); Cornes and Hartley (2005); Alcalde and Dahm (2010); Ewerhart (2015, 2017a,b); and Feng and Lu (2017). Esteban and Ray (1999) and Brookins and Ryvkin (2016) consider in-group contests, while Franke and Öztürk (2015) and Xu, Zhou, and Zenou (2020) focus on network contests. Lagerlöf (2020) applies the approach of Reny (1999) to a hybrid contest that involves both winner-pay and all-pay features.

To the best of our knowledge, our paper is the first to explore equilibrium uniqueness in noisy multi-prize contests, and the results notably contribute to this strand of the literature. First, as previously stated, the uniqueness we establish in symmetric settings lays a foundation for the results obtained in the extensive literature built on this modeling framework (e.g., Fu and Lu, 2009, 2012a; Azmat and Möller, 2009; Fu and Wu, 2021; Fu, Wang, and Wu, 2021a). Second, as stated above, the multi-prize variation nullifies the regularity of lottery contest games and conventional approaches. The technique we develop in this paper can be applied to the future analysis of contest games.

Our study also complements the broad literature on equilibrium uniqueness in discontinuous games. Our analysis builds on Rosen (1965), who establishes equilibrium uniqueness for concave continuous games. Our context deviates from his settings because the contest game lacks the key property of diagonal strict concavity, which prevents direct application of his results. Our approach restores the relevance of his uniqueness theorem in our context and broadens the scope of its applications. The technique may well be useful in future research on other forms of games that do not immediately match Rosen's requirements.

The rest of the paper is structured as follows. Section 2 describes the generalized multi-prize lottery contest. Section 3 proves the equilibrium uniqueness in symmetric contests. Section 4 introduces player heterogeneity and discusses its impact on equilibrium uniqueness, and Section 5 concludes. The appendix collects proofs that are not provided in the main text.

2 Setup and Preliminaries

In this section, we present the fundamentals of the underlying contest game.

2.1 Generalized Multi-prize Nested Lottery Contests

A contest involves $n \geq 2$ risk-neutral players who compete for n prizes. Each player's prize valuation profile is summarized by the vector $\boldsymbol{V}_i = (V_{1,i}, \dots, V_{n,i})$—which is commonly known—with $V_{m,i} \geq 0$ for all $m \in \{1, \dots, n\}$, $i \in \mathcal{N}$, and strict inequality to hold for at least one.

We define $\ell_i := \max\{m = 1, \dots, n-1 \,|\, V_{m,i} > V_{m+1,i}\}$, which can intuitively be interpreted as the actual number of prizes that could *effectively* incentivize player i's effort. Consider a contest with $V_{1,i} > V_{m,i}$ and $V_{m,i} = V_{m',i} > 0$ for all $m, m' \in \{2, \dots, n\}$, and $i \in \mathcal{N}$. The contest yields only one effective prize—$\ell = 1$—since players only strive for the top prize. Following Fu, Wu, and Zhu (2021b), we impose the following assumption throughout the paper.

Assumption 1 ***(Equal Number of Effective Prizes across Players)*** *$\ell_1 = \cdots = \ell_n =: \ell$.*

Fu, Wu, and Zhu (2021b) demonstrate that a pure-strategy equilibrium may fail to exist if Assumption 1 is violated. Assumption 1 by no means imposes a tight requirement. For instance, a standard winner-take-all contest—$V_{m,i} = 0$ for all $m \in \{2, \dots, n\}$ and $i \in \mathcal{N}$—obviously satisfies this requirement. We highlight the fact that the model only requires that the number of effective prizes be identical for all players and does not prevent them from valuing the prizes unequally.[6]

Winner-selection Mechanism The winner-selection mechanism is modeled as a popularly studied multi-prize nested lottery contest (Clark and Riis, 1996, 1998b). Players simultaneously exert their one-shot effort outlays x_is. For a given effort profile $\boldsymbol{x} := (x_1, \dots, x_n)$, player i wins the first prize with probability

$$p_{1,i}(\boldsymbol{x}) := \begin{cases} \dfrac{f_i(x_i)}{\sum_{j \in \mathcal{N}} f_j(x_j)}, & \text{if } \sum_{j \in \mathcal{N}} f_j(x_j) > 0, \\ \dfrac{1}{n}, & \text{if } \sum_{j \in \mathcal{N}} f_j(x_j) = 0. \end{cases}$$

The function $f_i(\cdot)$ converts one's effort into his effective output, with $f_i(x_i) \geq 0$ and $f_i'(x_i) > 0$ for all $x_i \geq 0$; this is conventionally called the impact function in the literature. The winner

[6] Fu, Wu, and Zhu (2021b) lay a foundation for Assumption 1: If the collection of prize valuation profiles $\boldsymbol{V}_i = (V_{1,i}, \dots, V_{n,i}) \in \mathbb{R}_+^n$ is randomly "chosen" according to some probability measure that is absolutely continuous with respect to the Lebesgue measure, then $\ell_i = n-1$ for each $i \in \mathcal{N}$, which ensures Assumption 1. To see that, note that $\boldsymbol{V}_i = (V_{1,i}, \dots, V_{n,i}) \in \mathbb{R}_+^n$, with $V_{m,i} = V_{m',i}$ for some (m, m') pair, has Hausdorff dimension $n-1$, which further implies n-dimensional Lebesgue measure zero. Further note that the set of all feasible prize valuations profiles—i.e., $\{\boldsymbol{V}_i \,|\, V_{1,i} \geq \cdots \geq V_{n,i} \geq 0\}$—has Hausdorff dimension n and nonempty interior in $\mathbb{R}_+^n$. These facts imply the claim laid out above.

of the prize is immediately removed from the pool of eligible candidates for the second prize, since each player is allowed to receive at most one. The recipient of the second prize is picked from the rest of the pool through a similar lottery. This process is repeated until all n prizes have been awarded.

To put this formally, let Ω_m, $m \in \{1, \ldots, n\}$ denote the set of $n - m + 1$ players who were not picked in the previous $m - 1$ draws and remain eligible for the mth prize, with $\Omega_1 \equiv \mathcal{N}$. We can write the probability of player i's receiving the mth prize *conditional* on his not having been picked in the previous $m - 1$ draws (i.e., $i \in \Omega_m$) as

$$p_{m,i}\left(\boldsymbol{x}; \Omega_m\right) := \begin{cases} \dfrac{f_i(x_i)}{\sum_{j \in \Omega_m} f_j(x_j)} \times \mathbb{1}\left(i \in \Omega_m\right), & \text{if } \sum_{j \in \Omega_m} f_j(x_j) > 0, \\ \dfrac{1}{n - m + 1} \times \mathbb{1}\left(i \in \Omega_m\right), & \text{if } \sum_{j \in \Omega_m} f_j(x_j) = 0, \end{cases} \tag{2}$$

where $\mathbb{1}(i \in \Omega_m)$ is an index function, with $\mathbb{1}(i \in \Omega_m) = 1$ if $i \in \Omega_m$ and $\mathbb{1}(i \in \Omega_m) = 0$ if $i \notin \Omega_m$.[7]

Winning Probabilities and Expected Payoffs Fixing an effort profile $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$, denote by $P_{m,i}(\boldsymbol{x})$ player i's ex ante probability of winning the mth prize. It can be verified that

$$P_{m,i}(\boldsymbol{x}) = \sum_{\forall\, \Omega_m} \Big[\Pr\left(\Omega_m\right) \times p_{m,i}\left(\boldsymbol{x}; \Omega_m\right) \Big],$$

where $\Pr(\Omega_m)$ is the probability that particular set Ω_m of players are up for the mth draw. A player i bears a cost $c_i(x_i)$ when he exerts effort x_i. His expected payoff can then be written as

$$\pi_i(\boldsymbol{x}) := \sum_{m=1}^{n} \left[P_{m,i}(\boldsymbol{x}) \times V_{m,i} \right] - c_i(x_i). \tag{3}$$

2.2 Regularity Condition and Equilibrium Existence

Denote by $\boldsymbol{V}_i$ the vector of player i's prize valuations $(V_{1,i}, \ldots, V_{n,i})$ for all $i \in \mathcal{N}$. Players' prize valuations $\{\boldsymbol{V}_i\}_{i=1}^n$, together with the set of impact functions $\{f_i(\cdot)\}_{i=1}^n$ and effort cost functions $\{c_i(\cdot)\}_{i=1}^n$, define a simultaneous-move generalized multi-prize nested lottery contest game, which we denote by

$$\Gamma := \left\langle \left\{ \boldsymbol{V}_i, f_i(\cdot), c_i(\cdot) \right\}_{i=1}^n \right\rangle.$$

[7] We assume in (2) that players eligible for a prize win with equal probability when all of them exert zero effort. It is useful to point out that our result concerning equilibrium uniqueness remains intact under an arbitrary tie-breaking rule. To be more specific, fixing Ω_m, $\frac{1}{n-m+1}$ in the above definition of $p_{m,i}\left(\boldsymbol{x}; \Omega_m\right)$ can be replaced by a positive constant $q_i > 0$ for each $i \in \Omega_m$, with $\sum_{i \in \Omega_m} q_i = 1$.

Further define $y_i := f_i(x_i) \geq 0$—which denotes player i's effective output—and $h_i(y_i) := c_i(f_i^{-1}(y_i))$.

The multi-prize contest $\left\langle \{\boldsymbol{V}_i, f_i(\cdot), c_i(\cdot)\}_{i=1}^n \right\rangle$ is strategically equivalent to an alternative one in which players with the same set of prize valuations $\{\boldsymbol{V}_i\}_{i=1}^n$ choose effort $y_i \geq 0$ to produce a linear output $\tilde{f}_i(y_i) = y_i$ and bear an effort cost $h_i(y_i)$. It thus suffices to analyze the transformed multi-prize contest game

$$\widetilde{\Gamma} := \left\langle \{\boldsymbol{V}_i, \tilde{f}_i(\cdot), h_i(\cdot)\}_{i=1}^n \right\rangle;$$

treating y_i as the decision variable instead of x_i proves convenient for the analysis.[8] In what follows, we call y_i player i's effort and $\boldsymbol{y} := (y_1, \dots, y_n)$ the effort profile in the transformed contest game, which is assumed to be a row vector. Further, we write $p_i(\cdot; \Omega_m)$, $P_{m,i}(\cdot)$, and $\pi_i(\cdot)$ as a function of $\boldsymbol{y} \equiv (y_1, \dots, y_n)$ instead of $\boldsymbol{x} \equiv (x_1, \dots, x_n)$ with slight abuse of notation.

The following assumption is imposed throughout the paper.

Assumption 2 ***(Regular Impact and Effort Cost Functions)*** *$h_i(\cdot) \equiv c_i(f_i^{-1}(\cdot))$ is a twice-differentiable function, with $h_i(0) = 0$, $h_i'(y_i) > 0$, and $h_i''(y_i) \geq 0$ for all $y_i > 0$ and $i \in \mathcal{N}$.*

The impact function $f_i(\cdot)$ and effort cost function $c_i(\cdot)$ are encapsulated in $h_i(\cdot)$. The conditions in the above definition automatically hold for an increasing and weakly concave impact function $f_i(\cdot)$ and an increasing and weakly convex effort cost function $c_i(\cdot)$, which are commonly assumed in the contest literature. However, it should be noted that the possibility of a convex impact function $f_i(\cdot)$ remains, since Assumption 2 can be satisfied when the effort cost function $c_i(\cdot)$ is sufficiently convex.

The following result by Fu, Wu, and Zhu (2021b) establishes the existence of a Nash equilibrium in pure strategy, which lays the foundation of our subsequent analysis regarding equilibrium uniqueness in the contest game.

Proposition 1 ***(Equilibrium Existence)*** *Suppose that Assumptions 1 and 2 are satisfied. Then every generalized multi-prize nested lottery contest $\widetilde{\Gamma} \equiv \left\langle \{\boldsymbol{V}_i, \tilde{f}_i(\cdot), h_i(\cdot)\}_{i=1}^n \right\rangle$ possesses a pure-strategy Nash equilibrium.*

3 Multi-prize Contests with Homogeneous Players

In this section, we examine the commonly assumed symmetric contests (e.g., Azmat and Möller, 2009; Fu and Lu, 2009, 2012a; Fu and Wu, 2021) and formally discuss how our

[8] A similar change of variables is invoked by Szidarovszky and Okuguchi (1997) and Cornes and Hartley (2005) to prove equilibrium existence and uniqueness in single-prize generalized lottery contests.

analysis fills the gap left by previous studies. Specifically, suppose that all players have the same increasing and (weakly) convex effort cost function $h_i(\cdot) = h(\cdot)$. Further, assume that the prizes are commonly valued by the players, i.e., $V_{m,i} = V_m$ for all $i \in \mathcal{N}$. The transformed contest is thus perfectly symmetric.

3.1 Preliminary Discussion

We first briefly introduce and discuss the traditional approach in the literature for equilibrium characterization. Let player i choose $y_i > 0$ and all other players choose $y > 0$, i.e., $\boldsymbol{y}_{-i} = (y, \ldots, y)$. Then player i's probability of obtaining the mth prize is given by

$$P_{m,i}(y_i, \boldsymbol{y}_{-i}) \equiv \frac{(N-1)!}{(N-m)!} \times \left(\prod_{j=1}^{m-1} \frac{y}{(N-j)y + y_i} \right) \times \frac{y_i}{(N-m)y + y_i}.$$

The derivation of $P_{m,i}(y_i, \boldsymbol{y}_{-i})$ would be substantially complicated if players had different impact functions or exerted different amounts of effort. It is straightforward to verify that player i's probability of winning the first prize $P_{1,i}(y_i, \boldsymbol{y}_{-i})$ is concave in y_i. However, his probability of winning other prizes—i.e., $P_{m,i}(y_i, \boldsymbol{y}_{-i})$ for $m \geq 2$—is not.

Suppose that a symmetric pure-strategy equilibrium exists, i.e., $y_i = y$. It can be verified that

$$\left. \frac{\partial P_{m,i}(y_i, \boldsymbol{y}_{-i})}{\partial y_i} \right|_{y_i = y} = \left(1 - \sum_{k=0}^{m-1} \frac{1}{n-k} \right) \times \frac{1}{ny}.$$

In addition, the first-order condition of the hypothetical symmetric equilibrium requires

$$\sum_{m=1}^{n-1} \left[\left. \frac{\partial P_{m,i}(y_i, \boldsymbol{y}_{-i})}{\partial y_i} \right|_{y_i = y} \times V_m \right] = h'(y).$$

Combining the above equations, we can obtain

$$\frac{1}{n} \sum_{m=1}^{n-1} \left[\left(1 - \sum_{k=0}^{m-1} \frac{1}{n-k} \right) \times V_m \right] = y h'(y). \tag{4}$$

With an increasing and (weakly) convex $h(\cdot)$, a unique solution to (4) ensues and has conventionally been adopted for equilibrium prediction.

This approach falls short as an equilibrium analysis. The symmetric effort profile that uniquely solves (4) has yet to be verified as an equilibrium. Specifically, it remains unclear whether the first-order condition is sufficient to pin down the equilibrium. Clark and Riis (1998b) check the local second-order condition, which precludes local deviation but not global ones.

Fu, Wu, and Zhu (2021b) develop a novel approach to verifying the concavity of a player's payoff in this model: Instead of writing the expected payoff function (3) as a linear combination of one's probabilities of winning each prize, as above, they rearrange it as a linear combination of the probabilities of not receiving one of the first several prizes. The above derivation, together with the payoff concavity the authors prove, indicates that the unique solution to (4) indeed constitutes a unique symmetric pure-strategy equilibrium of the multi-prize contest game. However, it remains elusive whether asymmetric equilibria exist. Our subsequent analysis in Section 3.2 fills this void.

3.2 Equilibrium Uniqueness in Symmetric Contests

We now formally establish equilibrium uniqueness in contests with homogeneous players. Recall that the discussion in Section 3.1 demonstrates that the derived symmetric equilibrium profile constitutes the unique symmetric pure-strategy equilibrium of the contest game. Therefore, it remains to show that there exists no asymmetric pure-strategy equilibrium in the contest game. We lay out the proof below.

Lemma 1 ***(Nonexistence of Asymmetric Equilibria in Symmetric Multi-prize Contests)*** *Suppose that Assumptions 1 and 2 are satisfied. Moreover, players are identical, i.e., $h_i(\cdot) = h(\cdot)$, $V_{m,i} = V_m$, for all $i \in \mathcal{N}$. Then the symmetric multi-prize contest $\widetilde{\Gamma} \equiv \left\langle \{\boldsymbol{V}_i, \tilde{f}_i(\cdot), h_i(\cdot)\}_{i=1}^n \right\rangle$ has no asymmetric pure-strategy equilibria.*

Proof. Suppose that the effort profile $\boldsymbol{y}^* \equiv (y_1^*, \ldots, y_n^*)$ constitutes an equilibrium. Without loss of generality, we can assume that $y_1^* \geq \cdots \geq y_n^*$. It suffices to show that $y_1^* = y_n^*$.

Clearly, $y_1^* = 0$ cannot hold in equilibrium, which implies that $y_1^* > 0$ and $y_n^* \geq 0$. Recall that $\ell \equiv \max\left\{m = 1, \ldots, n-1 \,|\, V_m > V_{m+1}\right\}$. It is straightforward to see that the number of active players should be no less than ℓ in the equilibrium. Otherwise, an inactive player has strict incentive to deviate and exert a positive amount of effort. This, together with the fact that player n exerts the smallest amount of effort of all the players, implies that the payoff function of the nth player is continuous for $y_n \geq 0$.

Define $\widetilde{P}_{m,i}(\boldsymbol{y})$ as

$$\widetilde{P}_{m,i}(\boldsymbol{y}) := \sum_{k=1}^{m} P_{k,i}(\boldsymbol{y}).$$

In words, $\widetilde{P}_{m,i}(\boldsymbol{y})$ is player i's probability of obtaining one of the first m prizes, given the effort profile $\boldsymbol{y}$. It can be verified that

$$1 - \widetilde{P}_{m,i}(\boldsymbol{y}) = \sum_{\forall\, \Omega_{m+1}} \Big[\Pr(\Omega_{m+1}) \times \mathbb{1}\left(i \in \Omega_{m+1}\right) \Big] \text{, for } m \in \{1, \ldots, n-1\}.$$

The above equation is intuitive: Player i does not receive one of the first m prizes if and only if he is eligible for the $(m+1)$th prize. Therefore, player i's expected payoff can be rewritten as

$$\begin{aligned}\pi_i(\boldsymbol{y}) &\equiv \sum_{m=1}^{n}\left[P_{m,i}(\boldsymbol{y}) \times V_{m,i}\right] - h_i(y_i)\\ &= V_{1,i} - \sum_{m=1}^{n-1}\left[\left[1-\widetilde{P}_{m,i}(\boldsymbol{y})\right] \times \left(V_{m,i} - V_{m+1,i}\right)\right] - h_i(y_i). \end{aligned} \tag{5}$$

By Equation (5), the first-order condition of $\pi_i(\boldsymbol{y})$ with respect to y_i for $i \in \{1, n\}$ in a symmetric multi-prize contest implies

$$h'(y_1^*) = \sum_{m=1}^{n-1}\left[\frac{\partial \widetilde{P}_{m,1}(\boldsymbol{y}^*)}{\partial y_1} \times (V_m - V_{m+1})\right] \tag{6}$$

and

$$h'(y_n^*) \geq \sum_{m=1}^{n-1}\left[\frac{\partial \widetilde{P}_{m,n}(\boldsymbol{y}^*)}{\partial y_n} \times (V_m - V_{m+1})\right]. \tag{7}$$

Define $\boldsymbol{y}(z) := (z, y_2^*, \ldots, y_{n-1}^*, y_1^* + y_n^* - z)$. Because the players are symmetric, we have that

$$\left.\frac{\partial \widetilde{P}_{m,1}(\boldsymbol{y})}{\partial y_1}\right|_{\boldsymbol{y}=\boldsymbol{y}(z)} = \left.\frac{\partial \widetilde{P}_{m,n}(\boldsymbol{y})}{\partial y_n}\right|_{\boldsymbol{y}=\boldsymbol{y}(y_1^*+y_n^*-z)} =: \mathcal{Q}_m(z),$$

and thus (6) and (7) can be rewritten as

$$h'(y_1^*) = \sum_{m=1}^{n-1}\left[\mathcal{Q}_m(y_1^*) \times (V_m - V_{m+1})\right] \tag{8}$$

and

$$h'(y_n^*) \geq \sum_{m=1}^{n-1}\left[\mathcal{Q}_m(y_n^*) \times (V_m - V_{m+1})\right]. \tag{9}$$

Next, by Lemma 1 in Fu, Wu, and Zhu (2021b), $\widetilde{P}_{m,i}(\boldsymbol{y})$ is concave in y_i for all $i \in \mathcal{N}$. Fix an arbitrary set of $m \in \{1, \ldots, n-1\}$ players that obtain the first m prizes. Let i_k indicate the index of the player who receives the kth prize. Denote by $\boldsymbol{I}_m$ the sequence of the index of players $(i_1, \ldots, i_m)$. In what follows, we write $i \in \boldsymbol{I}_m$ to indicate that i is an element of the sequence $\boldsymbol{I}_m$. Similarly, we write $i \notin \boldsymbol{I}_m$ to indicate that i is not an element of the sequence $\boldsymbol{I}_m$. Define $\boldsymbol{S}(m,i) := \{\boldsymbol{I}_m \,|\, i \notin \boldsymbol{I}_m\}$. Further, define $Y := \sum_{j=1}^{n} y_j$. Simple algebra would then verify

$$\widetilde{P}_{m,1}(\boldsymbol{y}) = 1 - \sum_{\boldsymbol{I}_m \in \boldsymbol{S}(m,1)} \left[\prod_{k=1}^{m} \frac{y_{i_k}}{Y - \sum_{j=1}^{k-1} y_{i_j}} \right].$$

By definition, $\boldsymbol{S}(m,1)$ refers to the set of players who win the first m prizes and does not include player 1. Therefore, the partial derivative of $\widetilde{P}_{m,1}(\boldsymbol{y})$ with respect to y_1 can be expressed as

$$\begin{aligned} \mathcal{Q}_m(z) &= \left. \frac{\partial \widetilde{P}_{m,1}(\boldsymbol{y})}{\partial y_1} \right|_{\boldsymbol{y}=\boldsymbol{y}(z)} \\ &= \sum_{\boldsymbol{I}_m \in \boldsymbol{S}(m,1)} \left\{ \left[\prod_{k=1}^{m} \frac{y_{i_k}}{Y - \sum_{j=1}^{k-1} y_{i_j}} \right] \times \left[\sum_{k=1}^{m} \frac{1}{Y - \sum_{j=1}^{k-1} y_{i_j}} \right] \right\}. \end{aligned}$$

Note that Y is independent of z by our construction of $\boldsymbol{y}(z)$. Therefore, $\prod_{k=1}^{m} \frac{y_{i_k}}{Y-\sum_{j=1}^{k-1} y_{i_j}}$ is a constant if $n \notin \boldsymbol{I}_m$ and is strictly decreasing in z if $n \in \boldsymbol{I}_m$. Similarly, $\prod_{k=1}^{m} \frac{1}{Y-\sum_{j=1}^{k-1} y_{i_j}}$ is a constant if $n \notin \boldsymbol{I}_m$ and is strictly decreasing in z if $n \in \boldsymbol{I}_m$. As a result, $\mathcal{Q}_m(z)$ is strictly decreasing in z.

The monotonicity of $\mathcal{Q}_m(\cdot)$, together with (8) and (9), implies that

$$h'(y_1^*) = \sum_{m=1}^{n-1} \left[\mathcal{Q}_m(y_1^*) \times (V_m - V_{m+1}) \right] \le \sum_{m=1}^{n-1} \left[\mathcal{Q}_m(y_n^*) \times (V_m - V_{m+1}) \right] \le h'(y_n^*) \le h'(y_1^*), \tag{10}$$

where the last inequality follows from the postulated $y_1^* \ge y_n^*$ and the weak convexity of $h(\cdot)$. As a result, we must have that $\mathcal{Q}_m(y_1^*) = \mathcal{Q}_m(y_n^*)$ for all $m \in \{1, \ldots, n-1\}$ such that $V_m > V_{m+1}$ from (10), which in turn implies that $y_1^* = y_n^*$. This concludes the proof. ■

The following result ensues.

Theorem 1 ***(Equilibrium Uniqueness with Symmetric Players)*** *Suppose that Assumptions 1 and 2 are satisfied. Moreover, players are identical, i.e., $h_i(\cdot) = h(\cdot)$, $V_{m,i} = V_m$, for all $i \in \mathcal{N}$. Then the symmetric multi-prize contest $\widetilde{\Gamma} \equiv \left\langle \{\boldsymbol{V}_i, \tilde{f}_i(\cdot), h_i(\cdot)\}_{i=1}^n \right\rangle$ possesses a unique equilibrium. In the equilibrium, players employ the same pure strategy, with each exerting effort $y^* > 0$ that uniquely solves*

$$yh'(y) = \frac{1}{n} \sum_{m=1}^{n} \left[\mu_m \times V_m \right], \tag{11}$$

where $\mu_m := 1 - \sum_{k=0}^{m-1} \frac{1}{n-k}$.

As previously stated, the vast majority of prior studies of multi-prize nested lottery contests has conveniently adopted the solution to Equation (11) as the equilibrium of the contest game (e.g., Azmat and Möller, 2009; Fu and Lu, 2009, 2012a; Fu and Wu, 2021). The uniqueness established by Theorem 1 lays a foundation for the results established in this strand of the literature.

4 Multi-prize Contests with Heterogeneous Players

In this section, we allow for heterogeneous players and investigate to what extent equilibrium uniqueness would persist in the multi-prize contest game.

4.1 Weak Player Heterogeneity and Equilibrium Uniqueness

We first show that a unique equilibrium continues to emerge in the multi-prize contest game when players are weakly heterogeneous. Note that the proof of Theorem 1 crucially relies on the assumption of homogeneous players and cannot be modified to analyze the case of heterogeneous players. More specifically, recall that Lemma 1 is central to the proof of Theorem 1, which demonstrates that all asymmetric effort profiles cannot constitute a pure-strategy equilibrium and substantially narrows the search for an equilibrium effort profile. As a result, an equilibrium, if it exists, must be symmetric. However, a result in a spirit similar to Lemma 1 can no longer be established in the presence of heterogeneous players, given that a symmetric effort profile can no longer constitute an equilibrium and an equilibrium must be asymmetric. This calls for an alternative approach to verifying equilibrium uniqueness.

We make use of Rosen (1965) to bypass this problem. In his seminal thesis on the existence and uniqueness of equilibrium for games with strictly concave and continuous payoffs, Rosen (1965), as noted previously, proposes the notion of diagonal strict concavity and shows in his Theorem 2 that a multi-player concave game that satisfies diagonal strict concavity possesses a unique pure-strategy equilibrium.

Specifically, fixing a row vector $\boldsymbol{r} := (r_1, \dots, r_n) \geq (0, \dots, 0)$ and $\boldsymbol{y} \equiv (y_1, \dots, y_n) \in \times_{i \in \mathcal{N}} \mathcal{Y}_i$, define the weighted nonnegative sum of players' payoffs for the transformed multi-prize contest game $\widetilde{\Gamma} \equiv \left\langle \{\boldsymbol{V}_i, \tilde{f}_i(\cdot), h_i(\cdot)\}_{i=1}^n \right\rangle$ as

$$\Pi(\boldsymbol{y}, \boldsymbol{r}) := \sum_{i=1}^{n} \left[r_i \pi_i(\boldsymbol{y}) \right]$$

and let

$$g(\boldsymbol{y}, \boldsymbol{r}) := \begin{pmatrix} r_1 \frac{\partial \pi_1(\boldsymbol{y})}{\partial y_1} \\ \vdots \\ r_n \frac{\partial \pi_n(\boldsymbol{y})}{\partial y_n} \end{pmatrix}.$$

The following definition is based on Rosen (1965).

Definition 1 ***(Diagonal Strict Concavity)*** *The function* $\Pi(\boldsymbol{y}, \boldsymbol{r})$ *is said to be diagonally strictly concave for* $\boldsymbol{y} \in \times_{i\in\mathcal{N}}\mathcal{Y}_i$ *if there exists some* $\boldsymbol{r} \geq (0,\ldots,0)$ *such that for every* $\boldsymbol{y}^\star, \boldsymbol{y}^{\star\star} \in \times_{i\in\mathcal{N}}\mathcal{Y}_i$*, we have* $(\boldsymbol{y}^\star - \boldsymbol{y}^{\star\star})g(\boldsymbol{y}^{\star\star}, \boldsymbol{r}) + (\boldsymbol{y}^{\star\star} - \boldsymbol{y}^\star)g(\boldsymbol{y}^\star, \boldsymbol{r}) > 0$.

Rosen (1965) further shows in his Theorem 6 that a sufficient condition for the weighted aggregate payoff function $\Pi(\boldsymbol{y}, \boldsymbol{r})$ to be diagonally strictly concave is that $J(\boldsymbol{y}, \boldsymbol{r}) + J^T(\boldsymbol{y}, \boldsymbol{r})$ is negative definite for each $\boldsymbol{y} \in \times_{i\in\mathcal{N}}\mathcal{Y}_i$, where $J(\boldsymbol{y}, \boldsymbol{r})$ is the Jacobian matrix of $g(\boldsymbol{y}, \boldsymbol{r})$ with respect to $\boldsymbol{y}$, i.e.,

$$J(\boldsymbol{y}, \boldsymbol{r}) := \begin{pmatrix} r_1 \frac{\partial^2 \pi_1(\boldsymbol{y})}{\partial y_1^2} & \cdots & r_1 \frac{\partial^2 \pi_1(\boldsymbol{y})}{\partial y_1 \partial y_n} \\ \vdots & \ddots & \vdots \\ r_n \frac{\partial^2 \pi_n(\boldsymbol{y})}{\partial y_n \partial y_1} & \cdots & r_n \frac{\partial^2 \pi_n(\boldsymbol{y})}{\partial y_n^2} \end{pmatrix}. \tag{12}$$

Rosen's results cannot be applied directly in our context because the matrix $J(\boldsymbol{y}, \boldsymbol{r}) + J^T(\boldsymbol{y}, \boldsymbol{r})$ is not negative definite in general, even in a symmetric multi-prize contest. We provide a simple example to elaborate on the nuance.

Example 1 ***(Discussion of Negative Definiteness and Diagonal Strict Concavity)*** *Consider a symmetric contest with three players and set* $\boldsymbol{V}_i = (1, 1, 0)$ *and* $h_i(y_i) = y_i$ *for* $i \in \mathcal{N} = \{1, 2, 3\}$*. Simple algebra would verify*

$$\frac{\partial^2 \pi_i(\boldsymbol{y})}{\partial y_i^2} = 2\left[\frac{Y - y_i}{Y^3} - \sum_{j\neq i} \frac{y_j}{(y_i + y_j)^3}\right], \text{ for all } i \in \mathcal{N}, \tag{13}$$

and

$$\frac{\partial^2 \pi_i(\boldsymbol{y})}{\partial y_i \partial y_j} = \frac{y_i - y_j}{(y_i + y_j)^3} - \frac{2y_i - Y}{Y^3}, \text{ for all } i, j \in \mathcal{N} \text{ and } i \neq j. \tag{14}$$

First, set $\mathcal{Y}_i^\dagger = [\epsilon, 1]$*, with* $\epsilon \in (0, 1)$*, and fix* $\bar{\boldsymbol{r}} = (1, 1, 1)$*. Combining* (12)*,* (13)*, and* (14)*, we can calculate the Jacobian matrix*

$$J(\boldsymbol{y}, \bar{\boldsymbol{r}}) = \begin{pmatrix} \frac{2(y_2+y_3)}{Y^3} - \frac{2y_2}{(y_1+y_2)^3} - \frac{2y_3}{(y_1+y_3)^3} & \frac{y_1-y_2}{(y_1+y_2)^3} - \frac{y_1-y_2-y_3}{Y^3} & \frac{y_1-y_3}{(y_1+y_3)^3} - \frac{y_1-y_2-y_3}{Y^3} \\ \frac{y_2-y_1}{(y_1+y_2)^3} - \frac{y_2-y_1-y_3}{Y^3} & \frac{2(y_1+y_3)}{Y^3} - \frac{2y_1}{(y_1+y_2)^3} - \frac{2y_3}{(y_2+y_3)^3} & \frac{y_2-y_3}{(y_2+y_3)^3} - \frac{y_2-y_1-y_3}{Y^3} \\ \frac{y_3-y_1}{(y_1+y_3)^3} - \frac{y_3-y_1-y_2}{Y^3} & \frac{y_3-y_2}{(y_2+y_3)^3} - \frac{y_3-y_1-y_2}{Y^3} & \frac{2(y_1+y_2)}{Y^3} - \frac{2y_1}{(y_1+y_3)^3} - \frac{2y_2}{(y_2+y_3)^3} \end{pmatrix},$$

from which we can obtain

$$
\begin{aligned}
&J(\boldsymbol{y},\bar{\boldsymbol{r}})+J^T(\boldsymbol{y},\bar{\boldsymbol{r}})\\
=&2\begin{pmatrix}
\frac{2(y_2+y_3)}{Y^3}-\frac{2y_2}{(y_1+y_2)^3}-\frac{2y_3}{(y_1+y_3)^3} & \frac{y_3}{Y^3} & \frac{y_2}{Y^3}\\
\frac{y_3}{Y^3} & \frac{2(y_1+y_3)}{Y^3}-\frac{2y_1}{(y_1+y_2)^3}-\frac{2y_3}{(y_2+y_3)^3} & \frac{y_1}{Y^3}\\
\frac{y_2}{Y^3} & \frac{y_1}{Y^3} & \frac{2(y_1+y_2)}{Y^3}-\frac{2y_1}{(y_1+y_3)^3}-\frac{2y_2}{(y_2+y_3)^3}
\end{pmatrix}.
\end{aligned}
$$

Consider $\boldsymbol{y}^\epsilon=(\frac{1-\epsilon}{2},\epsilon,\frac{1-\epsilon}{2})$ *and let* $\epsilon\to 0$. *If* $J(\boldsymbol{y}^\epsilon,\bar{\boldsymbol{r}})+J^T(\boldsymbol{y}^\epsilon,\bar{\boldsymbol{r}})$ *is negative definite, then its second-order leading principal minor must be positive. However, we have that*

$$
\lim_{\epsilon\to 0}\left|\begin{pmatrix} 2\left(1+\epsilon-\frac{16\epsilon}{(1+\epsilon)^3}-\frac{1}{(1-\epsilon)^3}\right) & 1-\epsilon\\ 1-\epsilon & 2\left(2-2\epsilon-16\frac{1-\epsilon}{(1+\epsilon)^3}\right)\end{pmatrix}\right| = \lim_{\epsilon\to 0}\left|\begin{pmatrix}0 & 1\\ 1 & -28\end{pmatrix}\right| = -1, \quad (15)
$$

which is a contradiction. By a similar argument, for all $\boldsymbol{r}\neq\bar{\boldsymbol{r}}$, *we can construct a strategy space such that that* $J(\boldsymbol{y},\boldsymbol{r})+J^T(\boldsymbol{y},\boldsymbol{r})$ *is not negative definite for all effort profiles* $\boldsymbol{y}$ *in it.*

Next, fix $y>0$ *and let* $\mathcal{Y}_i^{\dagger\dagger}=[y-\epsilon,y+\epsilon]$ *for all* $i\in\mathcal{N}$, *with* $\epsilon\in(0,y)$. *It can be verified that* $J(\boldsymbol{y},\bar{\boldsymbol{r}})+J^T(\boldsymbol{y},\bar{\boldsymbol{r}})$ *is negative definite for all* $\boldsymbol{y}\in\times_{i\in\mathcal{N}}\mathcal{Y}_i^{\dagger\dagger}$, *provided* ϵ *is sufficiently small. It follows immediately that* $\Pi(\boldsymbol{y},\boldsymbol{r})$ *is diagonally strictly concave for* $\boldsymbol{y}\in\times_{i\in\mathcal{N}}\mathcal{Y}_i^{\dagger\dagger}$.

Three remarks are in order. First, fixing $\bar{r}=(1,\ldots,1)$, it can be verified that $J(\boldsymbol{y},\bar{r})+J^T(\boldsymbol{y},\bar{r})$ is negative definite for all $\boldsymbol{y}\in\mathbb{R}_+^n$ in a single-prize contest with $\ell=1$, and thus diagonal strict concavity is satisfied. As a result, the failure of negative definiteness in Example 1 stems from the multi-prize nature of the contest game.

Second, according to the construction in Example 1, the failure of the negative definiteness of $J(\boldsymbol{y},\bar{r})+J^T(\boldsymbol{y},\bar{r})$ arises when some player's effort is close to 0, in which case the diagonal component of the matrix approaches 0 while the non-diagonal components remain positive [see Equation (15)].

Third, negative definiteness is satisfied in the neighborhood of a symmetric positive effort profile, which in turn implies diagonal strict concavity. In other words, the desired property of the negative definiteness of $J(\boldsymbol{y},\bar{\boldsymbol{r}})+J^T(\boldsymbol{y},\bar{\boldsymbol{r}})$ holds *locally* but not globally in our setting.

The last two remarks inspire us to take a detour and establish the following lemma—which generalizes Theorems 2 and 6 in Rosen (1965)—to circumvent the aforementioned difficulty.

Lemma 2 ***(Local Negative Definiteness and Equilibrium Uniqueness)*** *Consider a multi-player game* $\mathcal{G}$ *with a set of convex strategy space* $\{\mathcal{Y}_i\}_{i=1}^n$ *and continuous and strictly concave payoff functions* $\{\pi_i(\boldsymbol{y})\}_{i=1}^n$. *Suppose that* $\mathcal{Y}_i$ *is a subset of a finite-dimensional real vector space. Moreover, there exists a convex subset* $\mathcal{Y}'\subset\mathcal{Y}\equiv\times_{i\in\mathcal{N}}\mathcal{Y}_i$ *such that* *(i) for all $\boldsymbol{y} \in \mathcal{Y} \setminus \mathcal{Y}'$, $\boldsymbol{y}$ is not an equilibrium of the game $\mathcal{G}$;*

(ii) there exists a nonnegative vector $\boldsymbol{r}$ such that for each $\boldsymbol{y} \in \mathcal{Y}'$, $J(\boldsymbol{y}, \boldsymbol{r}) + J^T(\boldsymbol{y}, \boldsymbol{r})$ is negative definite.

Then the multi-player game $\mathcal{G}$ has at most one equilibrium.

Lemma 2 is a general result and enables us to bridge Rosen (1965) to our setting to prove equilibrium uniqueness in the presence of weak player heterogeneity. Recall that Theorem 1 shows that the unique equilibrium of a symmetric multi-prize contest is indeed symmetric. Intuitively, when players are weakly heterogeneous, a lopsided equilibrium effort profile is unlikely to constitute an equilibrium; further, players' equilibrium efforts should be close to each other and thus close to some symmetric effort profile. The former suggests that Lemma 2(i) is likely to be satisfied; the latter, together with the intuition obtained from Example 1, implies that the negative definiteness of $J(\boldsymbol{y}, \boldsymbol{r}) + J^T(\boldsymbol{y}, \boldsymbol{r})$ required in Lemma 2(ii) holds around the symmetric effort profile.

To put this formally, let

$$\delta_1 := \max_{i,j \in \mathcal{N}, m \le n-1} \frac{(V_{m,i} - V_{m+1,i})/V_{1,i}}{(V_{m,j} - V_{m+1,j})/V_{1,j}} - 1 \text{ and } \delta_2 := \sup_{i,j \in \mathcal{N}, y \in (0, \overline{y}]} \frac{h_i'(y)/V_{1,i}}{h_j'(y)/V_{1,j}} - 1,$$

where $\overline{y} := \max_{i \in \mathcal{N}} h_i^{-1}(V_{1,i})$.[9] Further, define $\delta := \max\{\delta_1, \delta_2\}$. Intuitively, δ_1 captures the degree of player heterogeneity with respect to their prize valuations $\{\boldsymbol{V}_i\}_{i=1}^n$, and δ_2 measures that in terms of their effort cost functions $\{h_i(\cdot)\}_{i=1}^n$. When players are perfectly symmetric, we have $\delta_1 = \delta_2 = 0$ and thus $\delta = 0$. The following result can then be obtained.

Theorem 2 ***(Equilibrium Uniqueness with Weakly Heterogeneous Players)*** *Suppose that Assumption 2 is satisfied. Moreover, players are weakly heterogeneous in the sense that*

$$\delta < \left(1 + \frac{1}{2n}\right)^{\frac{1}{2n+7}} - 1. \tag{16}$$

Then the asymmetric multi-prize contest $\widetilde{\Gamma} \equiv \left\langle \{\boldsymbol{V}_i, \tilde{f}_i(\cdot), h_i(\cdot)\}_{i=1}^n \right\rangle$ has a unique equilibrium.

The condition (16) provides a sufficient condition to ensure equilibrium uniqueness, which imposes an upper bound on the degree of player heterogeneity. Although Assumption 1 is not imposed directly, it follows from the condition (16). To see this, suppose, to the contrary, that (16) is satisfied and $\ell_i < \ell_j$ for some $i \neq j$. Set $m = \ell_j$. It follows that $V_{m,j} - V_{m+1,j} > 0 = V_{m,i} - V_{m+1,i}$ and $\delta_1 = +\infty$ (see Footnote 9), which implies that (16) is violated.

[9] If $V_{m,i} - V_{m+1,i} = V_{m,j} - V_{m+1,j} = 0$ for some $m \le n-1$ and $i, j \in \mathcal{N}$, we define $\frac{(V_{m,i} - V_{m+1,i})/V_{1,i}}{(V_{m,j} - V_{m+1,j})/V_{1,j}} := 1$. If $V_{m,i} - V_{m+1,i} > V_{m,j} - V_{m+1,j} = 0$, define $\frac{(V_{m,i} - V_{m+1,i})/V_{1,i}}{(V_{m,j} - V_{m+1,j})/V_{1,j}} := +\infty$.

4.2 Strong Player Heterogeneity and Multiple Equilibria

The analysis in the previous subsection relies on the negative definiteness of $J(\boldsymbol{y}, \boldsymbol{r}) + J^T(\boldsymbol{y}, \boldsymbol{r})$ that holds around a symmetric profile. This may lead to conjecture that multiple equilibria can arise if the condition established in Theorem 2 is not met, in which case players are excessively heterogeneous. Next, we provide an example to confirm this conjecture.

Example 2 ***(Multiple Equilibria under Strong Player Heterogeneity)*** *Consider a simple three-player contest with* $\boldsymbol{V}_1 = \boldsymbol{V}_3 = (1, 1, 0)$ *and* $\boldsymbol{V}_2 = (1, 0.001, 0)$*. There are two effective prizes, i.e.,* $\ell = 2$*. Moreover, players* 1 *and* 3 *equally value both prizes, whereas player* 2 *cares mostly about the first prize. Let* $\{h_i(y)\}_{i=1}^3$ *be a family of strictly increasing and convex functions, with* $c_1'(1) \approx 3.169 \times 10^{-3}$*,* $c_1'(1.009) \approx 3.311 \times 10^{-3}$*,* $c_2'(0.097) \approx 0.8175$*,* $c_2'(0.1) \approx 0.8197$*,* $c_3'(0.01) \approx 8.352$*, and* $c_3'(0.011) \approx 8.399$*. It can be verified that condition* (16) *is violated. Moreover, the two effort profiles* $\boldsymbol{y}^* = (y_1^*, y_2^*, y_3^*) = (1, 0.1, 0.01)$ *and* $\boldsymbol{y}^{**} = (y_1^{**}, y_2^{**}, y_3^{**}) = (1.009, 0.097, 0.011)$ *each constitute an equilibrium of the constructed multi-prize contest game.*

Comparison of $\boldsymbol{y}^*$ *and* $\boldsymbol{y}^{**}$ *sheds light on players' incentive structures and illuminates the source of multiple equilibria. Note that in both equilibria, player* 1 *exerts the highest amount of effort, followed by player* 2*, then player* 3*. Further, a switch from* $\boldsymbol{y}^*$ *to* $\boldsymbol{y}^{**}$ *leads to an increase in players* 1*'s and* 3*'s effort but a decrease in player* 2*'s.*

First, consider player 1*. Because she equally values the two prizes, she exerts effort simply to reduce the probability of being ranked at the bottom, which amounts to*

$$\frac{y_2}{Y} \times \frac{y_3}{Y - y_2} + \frac{y_3}{Y} \times \frac{y_2}{Y - y_3} = \frac{y_2 y_3}{Y} \times \left(\frac{1}{Y - y_2} + \frac{1}{Y - y_3} \right).$$

When players 2 *and* 3 *change their efforts from* $(y_2^*, y_3^*) = (0.1, 0.01)$ *to* $(y_2^{**}, y_3^{**}) = (0.097, 0.011)$*, player* 2*'s effort is reduced by only* 3% *but player* 3*'s significantly increases by* 10%*. This change substantially increases his chance of being ranked in the last place, which compels him to step up effort.*

Next, consider player 2*. He cares mostly about the first prize; furthermore, player* 3*'s effort is substantially smaller than his in both equilibria and can be considered negligible. As a result, he behaves as if he were competing with player* 1 *for the first prize. Given that player* 1 *is the favorite and increases his effort upon the switch from* y_1^* *to* y_1^{**}*, player* 2 *would concede in response because of the well-known strategic substitutability in single-prize lottery contests.*

We finally consider player 3*. Although player* 3 *equally values both prizes, his effort incentive is mainly shaped by the second prize due to the fact that player* 1 *is substantially stronger and his probability of receiving the first prize is almost zero. Therefore, player* 3 *strives to win the second prize and would behave as if he were competing solely with player* 2 *for that prize. In other words, his incentive is largely determined by player* 2*'s effort decision. Given that player* 2 *reduces his effort from* y_2^* *to* y_2^{**}*, player* 3 *would increase his effort, again, because of the strategic substitutability of efforts in single-prize lottery contests.*

This example illustrates the nuances described in the Introduction. When players value the (multiple) prizes differently and their competence differs, the game entails complex strategic interactions: Each has to decide which prize to fight for and whom to compete against. Such strategic consideration does not emerge in single-prize contests or the symmetric version of this game. This nuance may lead to multiple equilibrium points when players are sufficiently heterogeneous.

5 Conclusion

This paper comprehensively examines equilibrium uniqueness in a generalized multi-prize nested lottery contest model. Our paper complements Fu, Wu, and Zhu (2021b), which focuses on equilibrium existence in the same setting. We first consider a multi-prize contest with identical players. We verify that a symmetric equilibrium exists and is unique, which lays a foundation for the results obtained in the numerous studies that adopt multi-prize nested lottery contest models. We then proceed to an asymmetric setting in which players may differ in their prize valuations, impact functions, and/or effort cost functions. We show that equilibrium uniqueness persists when players are weakly heterogeneous. We also provide an example to demonstrate that uniqueness may fail—and multiple equilibria may arise—when players are sufficiently heterogeneous.

Our analysis of the setting with weakly heterogeneous players also contributes to the literature on equilibrium uniqueness. The technique we develop in the paper bridges Rosen (1965) to our setting and revives the relevance of his results, despite the missing key property of (global) diagonal strict concavity. Our approach can be useful for future analysis of equilibrium uniqueness in other forms of games that do not immediately meet his requirement.

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Appendix: Additional Proofs

Proof of Lemma 2

Proof. For expositional convenience, we assume that y_i is unidimensional. The proof can easily be extended to allow y_i to be multidimensional. Suppose, to the contrary, that there exist two equilibria $\boldsymbol{y}^*$ and $\boldsymbol{y}^{**}$ of the multi-player game $\mathcal{G}$, with $\boldsymbol{y}^* \neq \boldsymbol{y}^{**}$. By assumption, we have $\boldsymbol{y}^* \in \mathcal{Y}'$ and $\boldsymbol{y}^{**} \in \mathcal{Y}'$.

From player i's first-order condition at the equilibrium $\boldsymbol{y}^*$, we have $\frac{\partial \pi_i(\boldsymbol{y}^*)}{\partial y_i} = 0$ for $y_i^* > 0$ and $\frac{\partial \pi_i(\boldsymbol{y}^*)}{\partial y_i} \leq 0$ for $y_i^* = 0$, which in turn implies that

$$\sum_{i \in \mathcal{N}} \left[r_i(y_i^* - y_i^{**}) \times \frac{\partial \pi_i(\boldsymbol{y}^*)}{\partial y_i} \right] \geq 0. \tag{17}$$

By the same argument, we can obtain that

$$\sum_{i \in \mathcal{N}} \left[r_i(y_i^{**} - y_i^*) \times \frac{\partial \pi_i(\boldsymbol{y}^{**})}{\partial y_i} \right] \geq 0. \tag{18}$$

Combining (17) and (18) yields

$$\sum_{i \in \mathcal{N}} \left[r_i(y_i^* - y_i^{**}) \times \left(\frac{\partial \pi_i(\boldsymbol{y}^*)}{\partial y_i} - \frac{\partial \pi_i(\boldsymbol{y}^{**})}{\partial y_i} \right) \right] \geq 0. \tag{19}$$

Define $\phi : [0,1] \to \mathcal{Y}'$ such that $\phi(\lambda) = (1-\lambda)\boldsymbol{y}^* + \lambda \boldsymbol{y}^{**}$ for each $\lambda \in [0,1]$. It follows from the convexity of $\mathcal{Y}'$ that $\phi(\lambda) \in \mathcal{Y}'$ for all $\lambda \in [0,1]$. By assumption, there exists a nonnegative vector $\boldsymbol{r}$ such that $J(\boldsymbol{y}, \boldsymbol{r}) + J^T(\boldsymbol{y}, \boldsymbol{r})$ is negative definite for all $\boldsymbol{y} \in \mathcal{Y}'$. Therefore, for each $\lambda \in (0,1)$, we have that

$$\begin{aligned} &\left(\frac{d\phi(\lambda)}{d\lambda} \right) J\left(\phi(\lambda), \boldsymbol{r}\right) \left(\frac{d\phi(\lambda)}{d\lambda} \right)^T \\ =&\frac{1}{2} \left(\frac{d\phi(\lambda)}{d\lambda} \right) \left[J\left(\phi(\lambda), \boldsymbol{r}\right) + J^T\left(\phi(\lambda), \boldsymbol{r}\right) \right] \left(\frac{d\phi(\lambda)}{d\lambda} \right)^T < 0, \end{aligned} \tag{20}$$

where the strict inequality follows from the fact that $J\big(\phi(\lambda), \boldsymbol{r}\big) + J^T\big(\phi(\lambda), \boldsymbol{r}\big)$ is negative definite and $\frac{d\phi(\lambda)}{d\lambda} = \boldsymbol{y}^{**} - \boldsymbol{y}^* \neq \boldsymbol{0}$.

Combining (19) and (20) yields

$$
\begin{aligned}
0 \leq & \sum_{i \in \mathcal{N}} \left[r_i \left(y_i^* - y_i^{**} \right) \times \left(\frac{\partial \pi_i(\boldsymbol{y}^*)}{\partial y_i} - \frac{\partial \pi_i(\boldsymbol{y}^{**})}{\partial y_i} \right) \right] \\
= & \int_0^1 \left(\frac{d\phi(\lambda)}{d\lambda} \right) J\left(\phi(\lambda), \boldsymbol{r}\right) \left(\frac{d\phi(\lambda)}{d\lambda} \right)^T d\lambda < 0,
\end{aligned}
$$

which is a contradiction. This concludes the proof. ■

Proof of Theorem 2

Proof. As stated in the main text, (16) implies Assumption 1 and thus equilibrium existence follows immediately from Proposition 1; it remains to prove uniqueness. In what follows, we normalize $V_{1,i}$ to 1 for all $i \in \mathcal{N}$ without loss of generality.

To proceed, we state several useful intermediary results.

Lemma 3 *Suppose that Assumption 2 is satisfied and a pure-strategy equilibrium* $\boldsymbol{y}^* \equiv (y_1^*, \ldots, y_n^*)$ *exists. Then* $\max_{i \in \mathcal{N}} y_i^* \leq (1+\delta)^2 \min_{i \in \mathcal{N}} y_i^*$.

Proof. Without loss of generality, suppose that $y_1^* \geq \cdots \geq y_n^* \geq 0$. Clearly, $y_1^* = \cdots = y_n^* = 0$ cannot constitute an equilibrium, which implies that $y_1^* > 0$. The first-order condition of $\pi_i(\boldsymbol{y})$ with respect to y_i implies that

$$
h_1'(y_1^*) = \sum_{m=1}^{n-1} \left[\frac{\partial \widetilde{P}_{m,1}(\boldsymbol{y}^*)}{\partial y_1} \times (V_{m,1} - V_{m+1,1}) \right]
$$

and

$$
h_n'(y_n^*) \geq \sum_{m=1}^{n-1} \left[\frac{\partial \widetilde{P}_{m,n}(\boldsymbol{y}^*)}{\partial y_n} \times (V_{m,n} - V_{m+1,n}) \right].
$$

Further, it follows from the definition of δ and Assumption 2 that $V_{m,1} - V_{m+1,1} \leq (1+\delta)(V_{m,n} - V_{m+1,n})$ for all $m \in \{1, \ldots, n-1\}$ and $h_1'(y_1^*) \geq h_1'(y_n^*) \geq h_n'(y_n^*)/(1+\delta)$. Therefore, we have

$$
\begin{aligned}
(1+\delta)^2 \times \sum_{m=1}^{n-1} \left[\frac{\partial \widetilde{P}_{m,1}(\boldsymbol{y}^*)}{\partial y_1} \times (V_{m,n} - V_{m+1,n}) \right] \geq & (1+\delta) \times \sum_{m=1}^{n-1} \left[\frac{\partial \widetilde{P}_{m,1}(\boldsymbol{y}^*)}{\partial y_1} \times (V_{m,1} - V_{m+1,1}) \right] \\
= & (1+\delta) \times h_1'(y_1^*) \\
\geq & h_n'(y_n^*) \\
\geq & \sum_{m=1}^{n-1} \left[\frac{\partial \widetilde{P}_{m,n}(\boldsymbol{y}^*)}{\partial y_n} \times (V_{m,n} - V_{m+1,n}) \right].
\end{aligned}
$$

From the above condition, there exists $\widehat{m} \in \{1, 2, \ldots, n-1\}$ such that

$$(1+\delta)^2 \times \frac{\partial \widetilde{P}_{\widehat{m},1}(\boldsymbol{y}^*)}{\partial y_1} \geq \frac{\partial \widetilde{P}_{\widehat{m},n}(\boldsymbol{y}^*)}{\partial y_n}. \tag{21}$$

Let $Y^* = \sum_{j=1}^{n} y_j^*$. From the proof of Lemma 1, we have that

$$\frac{\partial \widetilde{P}_{\widehat{m},1}(\boldsymbol{y}^*)}{\partial y_1} = \sum_{\boldsymbol{I}_{\widehat{m}} \in \boldsymbol{S}(\widehat{m},1)} \left[\left(\prod_{k=1}^{\widehat{m}} \frac{y_{i_k}^*}{Y^* - \sum_{j=1}^{k-1} y_{i_j}^*} \right) \times \left(\sum_{k=1}^{\widehat{m}} \frac{1}{Y^* - \sum_{j=1}^{k-1} y_{i_j}^*} \right) \right] \tag{22}$$

and

$$\frac{\partial \widetilde{P}_{\widehat{m},n}(\boldsymbol{y}^*)}{\partial y_n} = \sum_{\boldsymbol{I}_{\widehat{m}} \in \boldsymbol{S}(\widehat{m},n)} \left[\left(\prod_{k=1}^{\widehat{m}} \frac{y_{i_k}^*}{Y^* - \sum_{j=1}^{k-1} y_{i_j}^*} \right) \times \left(\sum_{k=1}^{\widehat{m}} \frac{1}{Y^* - \sum_{j=1}^{k-1} y_{i_j}^*} \right) \right], \tag{23}$$

where $\boldsymbol{S}(\widehat{m}, i) \equiv \{\boldsymbol{I}_{\widehat{m}} \,|\, i \notin \boldsymbol{I}_{\widehat{m}}\}$.

Note that there exists a bijection between $\boldsymbol{S}(\widehat{m}, 1)$ and $\boldsymbol{S}(\widehat{m}, n)$: Given an arbitrary sequence $\boldsymbol{I}_{\widehat{m}} \in \boldsymbol{S}(\widehat{m}, 1)$, we can replace the element n in the sequence with 1 to obtain an alternative sequence $\boldsymbol{I}'_{\widehat{m}} \in \boldsymbol{S}(\widehat{m}, n)$, and vice versa. Fix the sequence pair $(\boldsymbol{I}_{\widehat{m}}, \boldsymbol{I}'_{\widehat{m}})$. Combining (21), (22), and (23) yields

$$\begin{aligned} &(1+\delta)^2 \times \left[\left(\prod_{k=1}^{\widehat{m}} \frac{y_{i_k}^*}{Y^* - \sum_{j=1}^{k-1} y_{i_j}^*} \right) \times \left(\sum_{k=1}^{\widehat{m}} \frac{1}{Y^* - \sum_{j=1}^{k-1} y_{i_j}^*} \right) \right] \Bigg|_{\boldsymbol{I}_{\widehat{m}}} \\ \geq &\left[\left(\prod_{k=1}^{\widehat{m}} \frac{y_{i_k}^*}{Y^* - \sum_{j=1}^{k-1} y_{i_j}^*} \right) \times \left(\sum_{k=1}^{\widehat{m}} \frac{1}{Y^* - \sum_{j=1}^{k-1} y_{i_j}^*} \right) \right] \Bigg|_{\boldsymbol{I}'_{\widehat{m}}}. \end{aligned} \tag{24}$$

Moreover, it follows from the postulated $y_1^* \geq y_n^*$ that

$$\begin{aligned} &\left[\left(\prod_{k=1}^{\widehat{m}} \frac{1}{Y^* - \sum_{j=1}^{k-1} y_{i_j}^*} \right) \times \left(\sum_{k=1}^{\widehat{m}} \frac{1}{Y^* - \sum_{j=1}^{k-1} y_{i_j}^*} \right) \right] \Bigg|_{\boldsymbol{I}_{\widehat{m}}} \\ \leq &\left[\left(\prod_{k=1}^{\widehat{m}} \frac{1}{Y^* - \sum_{j=1}^{k-1} y_{i_j}^*} \right) \times \left(\sum_{k=1}^{\widehat{m}} \frac{1}{Y^* - \sum_{j=1}^{k-1} y_{i_j}^*} \right) \right] \Bigg|_{\boldsymbol{I}'_{\widehat{m}}}. \end{aligned} \tag{25}$$

Combining (24) and (25) yields $y_1^* \leq (1+\delta)^2 y_n^*$, which concludes the proof. ∎

Lemma 4 *Let $A \in \mathbb{R}^{n\times n}$ be a negative definite matrix and denote the largest eigenvalue of A by $\lambda_{max}(A)$. Consider an alternative matrix $E \in \mathbb{R}^{n\times n}$ such that $\|E\|_1 + \|E\|_\infty < -2\lambda_{max}(A)$. Then $(A+E)+(A+E)^T$ is negative definite.*

Proof. Note that

$$(A+E)+(A+E)^T = 2\left[A-\lambda_{\max}(A)I\right] + \left[E+E^T+2\lambda_{\max}(A)I\right],$$

where I is the identity matrix of size n. It is straightforward to verify that $A-\lambda_{\max}(A)I$ is semi-negative definite. Moreover, the symmetric matrix $E+E^T+2\lambda_{\max}(A)I$ is diagonally dominant with negative diagonal entries and is thus negative definite. These indicate the negative definiteness of $(A+E)+(A+E)^T$ and conclude the proof. ■

Lemma 5 *Suppose that Assumption 2 is satisfied and consider an effort profile $\boldsymbol{y} = (y_1,\ldots,y_n) > (0,\ldots,0)$ such that $\frac{\max_{i\in\mathcal{N}} y_i}{\min_{i\in\mathcal{N}} y_i} \leq (1+\delta)^2$, where δ satisfies (16), i.e., $\delta < \left(1+\frac{1}{2n}\right)^{\frac{1}{2n+7}} - 1$. Then $J(\boldsymbol{y},\bar{\boldsymbol{r}}) + J^T(\boldsymbol{y},\bar{\boldsymbol{r}})$ is negative definite, where $\bar{\boldsymbol{r}} \equiv (1,\ldots,1)$.*

Proof. For notational convenience, define

$$J_m(\boldsymbol{y}) := \begin{pmatrix} (V_{m,1}-V_{m+1,1})\times\frac{\partial^2\widetilde{P}_{m,1}(\boldsymbol{y})}{\partial y_1^2} & \cdots & (V_{m,1}-V_{m+1,1})\times\frac{\partial^2\widetilde{P}_{m,1}(\boldsymbol{y})}{\partial y_1\partial y_n} \\ \vdots & \ddots & \vdots \\ (V_{m,n}-V_{m+1,n})\times\frac{\partial^2\widetilde{P}_{m,n}(\boldsymbol{y})}{\partial y_n\partial y_1} & \cdots & (V_{m,n}-V_{m+1,n})\times\frac{\partial^2\widetilde{P}_{m,n}(\boldsymbol{y})}{\partial y_n^2} \end{pmatrix}$$

for $m \in \{1,\ldots,n-1\}$ and

$$H(\boldsymbol{y}) := \begin{pmatrix} h_1''(y_1) & & \\ & \ddots & \\ & & h_n''(y_n) \end{pmatrix}.$$

It follows immediately that

$$J(\boldsymbol{y},\bar{\boldsymbol{r}}) = \sum_{m=1}^{n-1}\left[J_m(\boldsymbol{y})\right] - H(\boldsymbol{y}).$$

The weak convexity of $h_i(\cdot)$ required in Assumption 2 implies that the diagonal matrix $H(\boldsymbol{y})$ is semi-positive definite, which in turn indicates the semi-positive definiteness of $H(\boldsymbol{y}) + H^T(\boldsymbol{y})$. Therefore, it remains to show that $J_m(\boldsymbol{y}) + J_m^T(\boldsymbol{y})$ is negative definite for each $m \in \{1,\ldots,n-1\}$.

Let $\overline{V}_m := \frac{1}{n} \times \sum_{i\in\mathcal{N}} V_{m,i}$, and $\boldsymbol{y}^{\text{sym}} := \frac{1}{n} \times (Y, \ldots, Y)$. Then $J_m(\boldsymbol{y})$ can be written as

$$J_m(\boldsymbol{y}) = J_m^{\text{sym}}(\boldsymbol{y}) + \mathcal{E}_m(\boldsymbol{y}),$$

where $J_m^{\text{sym}}(\boldsymbol{y})$ is defined as

$$J_m^{\text{sym}}(\boldsymbol{y}) := \begin{pmatrix} \left(\overline{V}_m - \overline{V}_{m+1}\right) \times \frac{\partial^2 \widetilde{P}_{m,1}(\boldsymbol{y}^{\text{sym}})}{\partial y_1^2} & \cdots & \left(\overline{V}_m - \overline{V}_{m+1}\right) \times \frac{\partial^2 \widetilde{P}_{m,1}(\boldsymbol{y}^{\text{sym}})}{\partial y_1 \partial y_n} \\ \vdots & \ddots & \vdots \\ \left(\overline{V}_m - \overline{V}_{m+1}\right) \times \frac{\partial^2 \widetilde{P}_{m,n}(\boldsymbol{y}^{\text{sym}})}{\partial y_n \partial y_1} & \cdots & \left(\overline{V}_m - \overline{V}_{m+1}\right) \times \frac{\partial^2 \widetilde{P}_{m,n}(\boldsymbol{y}^{\text{sym}})}{\partial y_n^2} \end{pmatrix}$$

and $\mathcal{E}_m(\boldsymbol{y}) := J_m(\boldsymbol{y}) - J_m^{\text{sym}}(\boldsymbol{y})$.

Simple algebra would verify that

$$\begin{aligned} &\frac{\partial^2 \widetilde{P}_{m,i}(\boldsymbol{y})}{\partial y_i^2} \\ =& -\sum_{\boldsymbol{I}_m \in \boldsymbol{S}(m,i)} \left\{ \left(\prod_{k=1}^{m} \frac{y_{i_k}}{Y - \sum_{j=1}^{k-1} y_{i_j}} \right) \times \left[\left(\sum_{k=1}^{m} \frac{1}{Y - \sum_{j=1}^{k-1} y_{i_j}} \right)^2 + \sum_{k=1}^{m} \frac{1}{(Y - \sum_{j=1}^{k-1} y_{i_j})^2} \right] \right\}. \end{aligned}$$

The above condition, together with $\frac{y_i}{y_j} \leq (1+\delta)^2$ for all $i, j \in \mathcal{N}$, implies that

$$\begin{aligned} &(V_{m,i} - V_{m+1,i}) \frac{\partial^2 \widetilde{P}_{m,i}(\boldsymbol{y})}{\partial y_i^2} \\ \geq& -(1+\delta)^{2n+5} \times (\overline{V}_m - \overline{V}_{m+1}) \times \frac{n(n-m)}{Y^2} \times \left[\left(\sum_{k=1}^{m} \frac{1}{n-k+1} \right)^2 + \sum_{k=1}^{m} \frac{1}{(n-k+1)^2} \right] \\ =& -(1+\delta)^{2n+5} \times (\overline{V}_m - \overline{V}_{m+1}) \times \frac{n(n-m)}{Y^2} \times \left[\gamma_m^2 + \zeta_m \right] \end{aligned}$$

and

$$\begin{aligned} &(V_{m,i} - V_{m+1,i}) \frac{\partial^2 \widetilde{P}_{m,i}(\boldsymbol{y})}{\partial y_i^2} \\ \leq& -(1+\delta)^{-(2n+5)} \times (\overline{V}_m - \overline{V}_{m+1}) \times \frac{n(n-m)}{Y^2} \times \left[\left(\sum_{k=1}^{m} \frac{1}{n-k+1} \right)^2 + \sum_{k=1}^{m} \frac{1}{(n-k+1)^2} \right] \\ =& -(1+\delta)^{-2n-5} \times (\overline{V}_m - \overline{V}_{m+1}) \times \frac{n(n-m)}{Y^2} \times \left[\gamma_m^2 + \zeta_m \right], \end{aligned}$$

where $\gamma_m := \sum_{k=1}^{m} \frac{1}{n-k+1}$ and $\zeta_m := \sum_{k=1}^{m} \frac{1}{(n-k+1)^2}$.

Similarly, we have that

$$
\begin{aligned}
&\frac{\partial^2 \widetilde{P}_{m,i}(\boldsymbol{y})}{\partial y_i \partial y_j} \\
= &- \sum_{\boldsymbol{I}_m \in \boldsymbol{S}(m,i), \boldsymbol{I}_m \not\ni j} \left\{ \left(\prod_{k=1}^{m} \frac{y_{i_k}}{Y - \sum_{t=1}^{k-1} y_{i_t}} \right) \times \left[\left(\sum_{k=1}^{m} \frac{1}{Y - \sum_{t=1}^{k-1} y_{i_t}} \right)^2 + \sum_{k=1}^{m} \frac{1}{(Y - \sum_{t=1}^{k-1} y_{i_t})^2} \right] \right\} \\
&- \sum_{k_0=1}^{m} \sum_{\boldsymbol{I}_m \in \boldsymbol{S}(m,i), j = i_{k_0}} \left\{ \left(\prod_{k=1}^{m} \frac{y_{i_k}}{Y - \sum_{t=1}^{k-1} y_{i_t}} \right) \times \left[\sum_{k=1}^{k_0} \frac{1}{(Y - \sum_{t=1}^{k-1} y_{i_t})^2} \right.\right. \\
&\qquad \left.\left. + \left(\sum_{k=1}^{m} \frac{1}{Y - \sum_{t=1}^{k-1} y_{i_t}} \right) \times \left(-\frac{1}{y_j} + \sum\nolimits_{k=1}^{k_0} \frac{1}{Y - \sum_{t=1}^{k-1} y_{i_t}} \right) \right] \right\}.
\end{aligned}
$$

The lower bound and the upper bound of $(V_{m,i} - V_{m+1,i}) \frac{\partial^2 \widetilde{P}_{m,i}(\boldsymbol{y})}{\partial y_i \partial y_j}$ can be established as follows:

$$
\begin{aligned}
&(V_{m,i} - V_{m+1,i}) \frac{\partial^2 \widetilde{P}_{m,i}(\boldsymbol{y})}{\partial y_i \partial y_j} \\
\geq &- (1+\delta)^{2n+5} \times (\overline{V}_m - \overline{V}_{m+1}) \times \frac{n(n-m)}{Y^2} \times \frac{1}{n-1} \times \left(\gamma_m - \gamma_m^2 - \zeta_m \right) \\
&- \left[(1+\delta)^4 - 1 \right] \times (1+\delta)^{2n+3} \times (\overline{V}_m - \overline{V}_{m+1}) \times \frac{n(n-m)}{Y^2} \times \frac{m}{n-1} \times \gamma_m
\end{aligned}
$$

and

$$
\begin{aligned}
&(V_{m,i} - V_{m+1,i}) \frac{\partial^2 \widetilde{P}_{m,i}(\boldsymbol{y})}{\partial y_i \partial y_j} \\
\leq &- (1+\delta)^{-(2n+5)} \times (\overline{V}_m - \overline{V}_{m+1}) \times \frac{n(n-m)}{Y^2} \times \frac{1}{n-1} \times \left(\gamma_m - \gamma_m^2 - \zeta_m \right) \\
&+ \left[(1+\delta)^4 - 1 \right] \times (1+\delta)^{2n+3} \times (\overline{V}_m - \overline{V}_{m+1}) \times \frac{n(n-m)}{Y^2} \times \frac{m}{n-1} \times \gamma_m.
\end{aligned}
$$

Moreover, the elements of $J_m^{\mathrm{sym}}(\boldsymbol{y})$ can be derived as

$$
\left[J_m^{\mathrm{sym}}(\boldsymbol{y}) \right]_{ii} := -(\overline{V}_m - \overline{V}_{m+1}) \times \frac{n(n-m)}{Y^2} \times \left[\gamma_m^2 + \zeta_m \right]
$$

and

$$
\left[J_m^{\mathrm{sym}}(\boldsymbol{y}) \right]_{ij} := -(\overline{V}_m - \overline{V}_{m+1}) \times \frac{n(n-m)}{Y^2} \times \frac{1}{n-1} \times \left(\gamma_m - \gamma_m^2 - \zeta_m \right).
$$

Therefore, $\left|[\mathcal{E}_m(\boldsymbol{y})]_{ii}\right|$ and $\left|[\mathcal{E}_m(\boldsymbol{y})]_{ij}\right|$ can be bounded from above by

$$\left|\left[\mathcal{E}_m(\boldsymbol{y})\right]_{ii}\right| \leq \left[(1+\delta)^{2n+5}-1\right] \times (\overline{V}_m - \overline{V}_{m+1}) \times \frac{n(n-m)}{Y^2} \times \left[\gamma_m^2 + \zeta_m\right],$$

and

$$\left|\left[\mathcal{E}_m(\boldsymbol{y})\right]_{ij}\right| \leq \left[(1+\delta)^{2n+7}-1\right] \times (\overline{V}_m - \overline{V}_{m+1}) \times \frac{n(n-m)}{Y^2} \times \frac{1}{n-1} \times \left[(m+1)\gamma_m - \gamma_m^2 - \zeta_m\right],$$

from which we can conclude that

$$\max\left\{\|\mathcal{E}_m(\boldsymbol{y})\|_1, \|\mathcal{E}_m(\boldsymbol{y})\|_\infty\right\} \leq \left[(1+\delta)^{2n+7}-1\right] \times (\overline{V}_m - \overline{V}_{m+1}) \times \frac{n(n-m)}{Y^2} \times (m+1)\gamma_m. \tag{26}$$

On the other hand, the maximal eigenvalue of $J_m^{\text{sym}}(\boldsymbol{y})$ is

$$\begin{aligned} &\lambda_{\max}\left(J_m^{\text{sym}}(\boldsymbol{y})\right) \\ =&J_m^{\text{sym}}(\boldsymbol{y})_{ii} - J_m^{\text{sym}}(\boldsymbol{y})_{ij} \\ =&-(\overline{V}_m - \overline{V}_{m+1}) \times \frac{n(n-m)}{Y^2} \times \frac{1}{n-1} \times \left(n\gamma_m^2 + n\zeta_m - \gamma_m\right). \end{aligned} \tag{27}$$

Next, note that

$$\begin{aligned} \frac{1}{n-1}\left(n\gamma_m^2 + n\zeta_m - \gamma_m\right) \geq& \frac{1}{n-1}\left[\frac{n(m+1)}{m}\gamma_m^2 - \gamma_m\right] \\ \geq& \frac{m}{n-1}\gamma_m \geq \frac{m+1}{2n}\gamma_m, \end{aligned}$$

where the first inequality follows from the Cauchy-Schwarz inequality and the second inequality from $\gamma_m \equiv \sum_{k=1}^m \frac{1}{n-k+1} \geq \frac{m}{n}$. The above inequality, together with the condition $(1+\delta)^{2n+7} - 1 < \frac{1}{2n}$ from (16), implies that

$$\left[(1+\delta)^{2n+7}-1\right] \times (m+1)\gamma_m < \frac{1}{n-1} \times \left[n\gamma_m^2 + n\zeta_m - \gamma_m\right]. \tag{28}$$

Combining (26), (27), and (28), we can obtain

$$\|\mathcal{E}_m(\boldsymbol{y})\|_1 + \|\mathcal{E}_m(\boldsymbol{y})\|_\infty \leq 2\max\left\{\|\mathcal{E}_m(\boldsymbol{y})\|_1, \|\mathcal{E}_m(\boldsymbol{y})\|_\infty\right\} < -2\lambda_{\max}\left(J_m^{\text{sym}}(\boldsymbol{y})\right). \tag{29}$$

It follows immediately from (29) and Lemma 4 that $J_m(\boldsymbol{y}) + J_m^T(\boldsymbol{y})$ is negative definite, which in turn implies the negative definiteness of $J(\boldsymbol{y}, \bar{\boldsymbol{r}}) + J^T(\boldsymbol{y}, \bar{\boldsymbol{r}})$ and concludes the proof. ■

Now we can prove Theorem 2. Define $\mathcal{Y}^{\text{sym}} := \big\{\boldsymbol{y} \in \mathbb{R}_+^n : 0 < \max_{i\in\mathcal{N}} y_i \leq (1+$

$\delta)^2 \min_{i\in\mathcal{N}} y_i\big\}$. Evidently, $\mathcal{Y}^{\text{sym}}$ is a convex set. Consider an equilibrium effort profile $\boldsymbol{y}^*$. By Lemma 3, $\boldsymbol{y}^* \in \mathcal{Y}^{\text{sym}}$. Fix $\bar{\boldsymbol{r}} = (1, \ldots, 1)$. By Lemma 5, $J(\boldsymbol{y}, \bar{\boldsymbol{r}}) + J^T(\boldsymbol{y}, \bar{\boldsymbol{r}})$ is negative definite for all $\boldsymbol{y} \in \mathcal{Y}^{\text{sym}}$. Next, set $\mathcal{G} = \widetilde{\Gamma}$, $\mathcal{Y} = \times_{i\in\mathcal{N}}\big[0, h_i^{-1}(V_{1,i})\big]$, and $\mathcal{Y}' = \mathcal{Y}^{\text{sym}}$. Equilibrium uniqueness can be established by invoking Lemma 2. This concludes the proof.
■