Bid Caps in Noisy Contests*

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Abstract

This paper studies optimal bid caps in a multi-player noisy contest, in which a higher bid does not guarantee a sure win. The bid cap can be either rigid or flexible. The former imposes outright bidding restrictions on players' bids, while the latter taxes bids. A designer structures the bid cap to maximize a weighted sum between aggregate bid and tax revenue. Our analysis characterizes the optimum. A rigid bid is always outperformed by flexible ones, and a laissez-faire policy—i.e., no cap—is optimal when the designer maximizes the aggregate bid. The results also generate novel practical implications.

Keywords: Bid Caps; Noisy Contests; Tax Revenue; Contest Design

JEL Classification Codes: C72; D72

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1 Introduction

A broad spectrum of competitive activities resemble a contest, ranging from influence politics and legal disputes to sporting events: Players sink costly bids to strive for limited prizes—e.g., the patronage awarded by a politician or a trophy in a sports league—while their outlays are nonrefundable regardless of the outcome. Players' bidding activities are often restricted by various forms of institutional constraints that literally limit excessive spending. For instance, federal contribution limits in the United States cap individuals' or organizations' "hard money" contributions to a candidate's campaign (Che and Gale, 1998). Salary caps are widely employed by professional sports leagues—e.g., the National Football League (NFL) and National Hockey League (NHL)—to limit teams' salary budgets. To combat the private tutoring craze, the Korean government imposed a 10 pm closing time for tuition centres, which limits the amount of tutoring a primary or secondary school student can receive. We follow the literature and call these regulations *bid caps* (e.g., Olszewski and Siegel, 2019).

Bid caps can take different forms. A *rigid cap* places an outright limit on the bid and a breach triggers severe sanctions, which are usually subject to the judgment of the ruling authority. For instance, after the discovery of Melbourne Storm's violation of the salary cap enforced by Australia's National Rugby League (NRL) in 2011, the league revoked all of the trophies awarded to the club during the previous 5 years. In contrast, a *flexible cap* imposes an additional cost on one's bid at a prespecified rate, which resembles a taxation scheme and can be exemplified by the practices of the National Basketball Association (NBA) and Major League Baseball (MLB), among many others: A team pays a luxury tax for spending that exceeds a figure. Similarly, there has been extensive and vehement advocacy—by politicians, political pundits, and academics—for a progressive tax imposed on corporate special interest lobbying. U.S. Senator Elizabeth Warren, for instance, famously proposed that corporations and trade organizations be taxed for spending in excess of $500,000. Three goals are often cited in support of a lobbying tax: (i) to curb wasteful spending, (ii) to level the playing field for public interest groups, and (iii) to raise federal revenues.[1]

How does a bid cap, rigid or flexible, reshape contenders' bidding incentives? Will the salary caps in professional sports leagues help level the playing field and maintain their competitiveness? Will the proposed regulations curb lobbying activities? More generally, how could an administrator strategically exploit a bid cap scheme as a design instrument to advance her interests? This paper conducts a formal analysis of optimal bid caps in a multi-player *noisy* contest with heterogeneous players to address these questions.

The economics literature has yielded mixed answers. For instance, Che and Gale (1998)

[1] See `https://evonomics.com/tax-shame-excessive-corporate-lobbying/`.

show, in their seminal study, that a rigid bid cap on campaign contributions may, paradoxically, benefit a rent-seeking politician, which conflicts with the goal the policy is intended to achieve. Kaplan and Wettstein (2006) establish that an unenforceable cap—i.e., players can bear additional cost to bypass enforcement—decreases the aggregate bid, while total bidding costs and the stochastic outcome of the contest—i.e., contenders' winning odds—remain neutral, which implicitly refutes the argument for a lobbying tax to limit lobbying activities and level the playing field.[2,3] Gavious, Moldovanu, and Sela (2002) show that with ex ante symmetric players, a rigid cap decreases the aggregate bid for (weakly) concave bidding costs and may increase it for convex costs, while Olszewski and Siegel (2019) conclude the same in this regard, allowing for ex ante heterogeneous players in large contests.

This strand of the literature has conventionally modeled contest-like competitions as all-pay auctions, in which a slightly higher bid ensures one's victory. In contrast, we focus on the practically more plausible scenario of noisy contests in which a higher bid only improves one's winning odds. One's win or loss in real-world competitions often depends not only on players' competitive bids but also on random factors. Kang (2016), for instance, quantifies the efficacy of firms' lobbying efforts in influencing policy enactment and finds that the magnitude of its effect is small. Analogously, suspense and surprises are pervasive in sports.[4]

The roles played by bid caps, either rigid or flexible, in a multi-player noisy contest may substantially differ from those in all-pay auctions. Conventional wisdom has held that a more level playing field creates more competition, and bid caps may play such a role. This catalyzes the counterintuitive finding of Che and Gale (1998), since a bid cap handicaps the favorite and encourages the underdog. A bid cap may nevertheless lose its appeal in noisy contests as an equalizing device. A natural trade-off arises when a bid cap is imposed. On the one hand, there is a direct *cost effect*: A bid cap is counterproductive by nature, since the elevated cost discourages bidding. On the other hand, there is an indirect *competition effect*: When the bid cap diminishes the favorite's advantage, the handicap may revive the underdog's incentive, which fuels more competition, as shown in all-pay auction models (e.g., Che and Gale, 1998, 2006). However, the randomness inherent in a noisy contest casts

[2]Pastine and Pastine (2013) allow firms to contribute "soft money"—e.g., buying advertisements for a candidate—to circumvent a cap for a hard money contribution, albeit at a higher cost. They show that the neutrality established by Kaplan and Wettstein (2006) would not hold when the politician has a bias for policy preference. Policy distortion and lobbying expenditure thus depend on the prevailing limit for hard money contributions.

[3]It should be noted that the flexible cap in our paper conceptually differs from those in Kaplan and Wettstein (2006); Che and Gale (2006); and Pastine and Pastine (2013). They consider a scenario in which a rigid cap is imposed but players can bear extra costs to bypass the cap. The cost incurred is exogenous and only dissipates the rent. The additional bidding costs resemble taxation, and the revenue can accrue to the benefit of the designer.

[4]Ben-Naim, Vazquez, and Redner (2007) and Anderson and Sally (2013), among others, statistically measure the level of noise involved in various sports.

doubt on the procompetitive role of a bid cap. First, the noise can play a similar equalizing role: It undermines the efficacy of a higher bid in determining a win, thereby eroding the favorite's advantage and leveling the playing field. The presence of noise could eliminate the need for additional costly intervention such as a bid cap. Second, the noise weakens the competition effect: An equalizing device can less effectively encourage an underdog to step up his bid when winner selection depends more on luck. It is thus a priori unclear whether the conclusions drawn from all-pay auctions extend to noisy contests.

We present an analysis of optimal bid cap in a noisy contest to fill this void. The gap in the literature is partly caused by the technical challenge posed by such analysis. It is well known that a noisy contest with multiple heterogeneous players typically does not yield a closed-form solution, even without a bid cap; this precludes the usual approach for optimal contest design as a mathematical programming with equilibrium constraints (MPEC).[5] We develop a technique in a vein similar to that of Fu and Wu (2020) that allows us to bypass this difficulty by characterizing the optimum without explicitly solving for the equilibrium.

Model, Findings, and Implications: Snapshot We model the noisy contest as a generalized lottery contest with a concave impact function, which implies a sufficiently noisy winner-selection mechanism. Players differ in their prize valuations. Our framework defines bid cap generally, which resembles a taxation scheme set by a designer: A marginal tax rate is imposed for every level of bid, which elevates players' bidding costs. A rigid cap is a special case because a prohibitively high tax rate can effectively discourage bidding above a certain threshold. The same holds for a laissez-faire scheme—i.e., without a cap—which simply boils down to a zero tax rate at relevant bid levels. The designer sets the tax schedule to maximize a weighted sum between players' aggregate bid and tax revenues. Our setting can be exemplified by MLB: Teams' payrolls help recruit talent, which, as their competitive bids, determine their performance in the league; the luxury tax paid to the league is put into its Industry Growth Fund, which adds to its budget for promoting and developing baseball. Alternatively, as pointed out by Cotton (2009), the revenue collected through taxing political contributions may benefit both the politician and the constituent.

We highlight several key properties of the optimum in noisy contests. First, we show that a rigid cap—which sets a prohibitively high tax rate to discourage players from placing bids above a certain threshold—is always suboptimal: Any rigid cap can be strictly dominated by a flexible one regardless of the designer's preference (Theorem 1). As a result, the optimum requires either a flexible cap or a laissez-faire scheme, i.e., no cap.

Second, although the model allows the designer to impose a general bidding cost struc-

[5] This approach requires that one solve for the equilibrium for every possible contest structure and search for the optimal contest structure that, in turn, induces the most preferred equilibrium.

ture, we demonstrate that it is without loss of generality to focus the search on the class of piecewise constant functions (Theorem 2). The candidate's optimal cap can thus be parameterized as a finite set of tax rates applied to different bid brackets.

Third, when the designer cares only about players' aggregate bid and not tax revenue, she always imposes no cap in the optimum (Theorem 3). In other words, when the contest is sufficiently noisy in the sense that the impact function is concave, a bid cap always results in a lower aggregate bid and can be optimal only when the designer benefits nontrivially from the tax revenue collected.

However, a flexible cap can outperform a laissez-faire scheme when the designer cares about tax revenue, and we demonstrate the environment in which this occurs (e.g., Figure 1). Our observations illuminate the nature of bid caps in noisy contests and allow us to develop an intuitive account of how the optimum depends on the various environmental factors, such as the noisiness of the winner-selection mechanism and player heterogeneity. More specifically, they testify to the above-mentioned rationales: (i) the tension between cost and competition effects for bid cap and (ii) the erosive effect of noise on the efficacy of a bid cap as an equalizing device. A less even race may call for an equalizing bid cap; conversely, it loses its appeal when substantial noise is present in the contest.

Our results can be interpreted broadly to generate novel practical implications. For instance, our results shed light on the policy debate regarding an excessive lobbying tax. A progressive lobbying tax, as a flexible cap, can achieve all three of the aforementioned frequently cited goals: (i) curb wasteful lobbying; (ii) level the playing field for public interest groups; and (iii) raise federal revenue. Theorem 3 demonstrates that no cap is optimal when maximizing the aggregate bid in a noisy contest. This implies that lobbying activities can effectively be reduced by imposing a tax (flexible cap) or even an outright spending limit (rigid cap), regardless of its specific form. This stands in sharp contrast to the implications of Che and Gale (1998) and Kaplan and Wettstein (2006). Further, a progressive tax imposes a higher marginal tax rate on a stronger contender—which, in the context of influence politics, effectively evens the race for "poorly coordinated and poorly financed" public interest lobbyists (Zingales, 2020). This conflicts with the prediction of Kaplan and Wettstein (2006), who expound a neutrality of equilibrium outcome for flexible caps.[6]

To explore the linkage between our analysis and previous literature, we extend the model to allow for moderately convex impact functions and intermediate discriminatory power (Propositions 2 and 3). The results further elucidate the role played by a bid cap and allow us to develop a rationale that reconciles the contrasting observations obtained from the two extreme settings, i.e., noisy contests with concave impact functions and all-pay auctions.

[6]By neutrality, the total expenditure and players' winning probabilities remain constant regardless of the bidding cost functions (flexible cap of any form or no cap).

Links to the Literature To the best of our knowledge, our paper is the first to study a generally defined bid cap in asymmetric multi-player noisy contests.

Che and Gale (1998, 2006) and Kaplan and Wettstein (2006) model lobbying competitions as two-player complete-information all-pay auctions.[7] Pastine and Pastine (2013) allow one player to have a headstart in an all-pay auction, which abstracts the politician's biased policy preference and dissolves the neutrality result established by Kaplan and Wettstein (2006). Szech (2015) revisits two-player complete-information all-pay auctions with caps and introduces an alternative tie-breaking rule. Sahuguet (2006) considers rigid bid caps in two-player asymmetric all-pay auctions with incomplete information, and shows that the result of Che and Gale (1998) extends to an incomplete-information setting.

Although these studies do not explicitly address contest design, a closer look at Che and Gale (1998) and Kaplan and Wettstein (2006) allows us to infer an unambiguous ranking of different bid cap schemes when a designer maximizes a weighted sum between the aggregate bid and tax revenue in a two-player complete-information all-pay auction. Our result stands in sharp contrast to their implications. We elaborate on how our predictions depart from the implications obtained from complete-information all-pay auctions after our analysis unfolds (Section 4.2).

Gavious, Moldovanu, and Sela (2002) were the first to examine the effect of rigid bid caps beyond the setting of two-player contests. They consider an incomplete-information all-pay auction in which players' private valuations of the prize are privately known and independently and identically distributed. As stated above, they show that a bid cap reduces the aggregate bid for (weakly) concave biding costs—which differs from Che and Gale (1998)—and may increase it for convex costs. Olszewski and Siegel (2019) consider caps in an all-pay auction with a sufficiently large number of heterogeneous players, which is enabled by the approximation approach of Olszewski and Siegel (2016). They echo Gavious, Moldovanu, and Sela (2002) in terms of the effect of a rigid cap on the aggregate bid; moreover, a flexible cap has negligible effect on the aggregate cost, but also reduces the aggregate bid.[8]

We further discuss our paper in relation to the literature on bid caps in Section 4.2. As stated above, our study explicitly treats the cap as a taxation scheme, and the extra costs levied on players (at least partly) accrue to the benefit of the designer. Our paper is thus naturally linked to the literature on taxation in rent-seeking competitions. Glazer and Konrad (1999) pioneered modeling taxation in contests based on either rent-seeking activities or the profits that result from rent-seeking activities. They consider a two-player setting and

[7]Relatedly, Lagziel (2019) considers bid caps in credit auctions in which the winning bidder may default after the price is determined.

[8]Analogous to Olszewski and Siegel (2019), we also show that bid caps reduce the aggregate bid. However, they conclude that a flexible cap's impact on total bidding costs is negligible, which stands in contrast to our findings. The diverging implications can largely be attributed to the difference between all-pay auctions and noisy contests.

examine the impact of a *proportional tax*. In particular, they show that in a Tullock lottery contest, a tax on rent-seeking effort does not change firms' outlays, the stochastic outcome of the competitions, or firms' expected profits; they nevertheless demonstrate that a tax on profits may either increase or decrease firms' rent-seeking efforts. Our paper complements that of Glazer and Konrad (1999) by allowing for multiple players, nonlinear impact functions, and nonlinear taxes.

Relatedly, Cotton (2009) focuses on a politician's choice between selling policy favor and selling political access under different regulations for contributions: Policy favor allows the recipient to choose his preferred policy, and political access allows him to present information in favor of his preferred policy option. The competition between two interest groups is modeled as an all-pay auction. Cotton (2009) shows that a tax on contributions (i.e., a flexible cap) outperforms a binding contribution limit (i.e., a rigid cap) or no regulation (i.e., no cap), because it compels the politician to sell access without distorting his information.

The rest of the paper is structured as follows. Section 2 sets up the model and design problem. Section 3 characterizes the properties of the optimal contests and elaborates on the key insights of our results. Section 4 considers an extension of contests with convex impact functions and discusses our results in relation to the literature. Section 5 concludes.

2 Model

There are $n \geq 2$ risk-neutral players competing for a prize, e.g., a policy favor in a lobbying context or a trophy in a sporting event. The prize bears a value $v_i > 0$ for each player $i \in \mathcal{N} \equiv \{1, \ldots, n\}$, with $v_1 \geq \cdots \geq v_n > 0$, which is common knowledge.

Players simultaneously commit to their bids $x_i \geq 0$. For an effort profile $\boldsymbol{x} := (x_1, \ldots, x_n)$, a player i wins with probability

$$p_i(\boldsymbol{x}) := \begin{cases} \dfrac{f(x_i)}{\sum_{j\in\mathcal{N}} f(x_j)}, & \text{if } \sum_{j\in\mathcal{N}} f(x_j) \neq 0, \\ \dfrac{1}{n}, & \text{if } \sum_{j\in\mathcal{N}} f(x_j) = 0; \end{cases} \tag{1}$$

the impact function $f(\cdot)$ converts one's bid into his effective output and satisfies $f(x_i) \geq 0$ for all $x_i \geq 0$. We impose the following condition on $f(\cdot)$.

Assumption 1 *$f(\cdot)$ is twice differentiable, with $f'(x_i) > 0$, $f''(x_i) \leq 0$, and $f(0) = 0$.*

A concave impact function in Assumption 1 implies that a larger bid adds less to one's winning odds; this conventionally abstracts a sufficiently noisy winner-selection mechanism under which randomness plays a nontrivial role in determining the outcome. For instance, consider the popularly adopted Tullock contest model and parameterize the impact function as $f(x_i) = x_i^r$, with $r > 0$. Assumption 1 requires $r \leq 1$. A smaller r implies a less precise winner-selection mechanism, since the win or loss depends less on the difference in bids; that is, a higher effort can less effectively be converted into larger winning odds, and one's win depends more on luck. In the extreme case in which r approaches zero, the probability of a player's winning the contest is independent of his bid and converges to $1/n$. As r approaches infinity, the contest boils down to an all-pay auction, as assumed by Che and Gale (1998), in which case a higher bid ensures a win.[9]

2.1 Bid Cap Scheme and Bidding Costs

We define a bid cap scheme as a nondecreasing tax schedule $t(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$, which specifies the marginal tax rate imposed on each player's bid. Define the set of all eligible tax schedules to be

$$\mathcal{T} := \left\{ t(\cdot) \,\middle|\, t(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+,\, t(x) \geq 0 \text{ for all } x \geq 0, \text{ and is nondecreasing in } x \right\}.$$

We implicitly assume a (weakly) progressive tax schedule, which is common in practice; e.g., the luxury tax schedule in MLB.

A player's bidding cost consists of two parts: (i) the direct cost incurred by the bid, with a unity marginal cost; and (ii) the tax imposed on his bid. This gives the following bidding cost function:

$$c(x_i) = \underbrace{x_i}_{\text{bid}} + \underbrace{\int_0^{x_i} t(s)ds}_{\text{tax}}. \tag{2}$$

It is straightforward to verify that the bidding cost function $c(\cdot)$ is weakly convex.

Given the contest success function (1), the bidding cost function (2), and the profile of players' bids $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$, player i's expected payoff is given by

$$\pi_i(\boldsymbol{x}) := p_i(\boldsymbol{x})v_i - c(x_i), \forall\, i \in \mathcal{N}.$$

2.2 Contest Objective and Design Problem

With an increasing and weakly concave impact function $f(\cdot)$ and an increasing and weakly convex bidding cost function $c(\cdot)$, the contest, by Szidarovszky and Okuguchi (1997) and Fu and Wu (2020), yields concave payoff functions, well-behaved best-response correspondence,

[9] See also Kaplan and Wettstein (2006); Gavious, Moldovanu, and Sela (2002); Szech (2015); and Olszewski and Siegel (2019).

and therefore a unique pure-strategy equilibrium for an arbitrary cap scheme $t \in \mathcal{T}$.[10] Denote by $\boldsymbol{x}^*(t) := \big(x_1^*(t), \ldots, x_n^*(t)\big)$ the unique equilibrium strategy profile. The corresponding aggregate bid and total tax revenue are $\sum_{i=1}^n x_i^*(t)$ and $\sum_{i=1}^n \int_0^{x_i^*(t)} t(s)ds$, respectively.

Anticipating players' equilibrium bidding strategies, the designer sets $t(\cdot) \in \mathcal{T}$ to maximize her objective function. The contest design problem thus entails a typical mathematical programming with equilibrium constraints (MPEC). The designer values both the players' performance—i.e., the equilibrium aggregate bid—and the tax revenue. More formally, we assume that the designer's objective function is

$$\mathcal{F}(t) := \underbrace{\sum\nolimits_{i=1}^n x_i^*(t)}_{\text{aggregate bid}} + \lambda \times \underbrace{\sum\nolimits_{i=1}^n \int_0^{x_i^*(t)} t(s)ds}_{\text{tax revenue}}, \text{ with } 0 \leq \lambda \leq 1. \tag{3}$$

We assume that $\lambda \leq 1$, i.e., the designer values players' performance in the contest (weakly) more than her tax revenue. Consider, for instance, the case of MLB, in which teams' performance contributes the most to the league's core interests. The design problem degenerates to the conventional aggregate bid (effort) maximization problem for $\lambda = 0$.

2.3 Rigid Cap and Flexible Cap

Let $\boldsymbol{x}^\dagger := (x_1^\dagger, \ldots, x_n^\dagger)$ be the equilibrium bid profile in the original *unconstrained* contest. We introduce the following terminologies.

Definition 1 ***(Rigid Cap and Flexible Cap)*** *A bid cap scheme $t \in \mathcal{T}$ is called a rigid cap if $\boldsymbol{x}^*(t) \neq \boldsymbol{x}^\dagger$ and $\sum_{i=1}^n \int_0^{x_i^*(t)} t(s)ds = 0$. Similarly, a scheme $t \in \mathcal{T}$ is called a flexible cap if $\boldsymbol{x}^*(t) \neq \boldsymbol{x}^\dagger$ and $\sum_{i=1}^n \int_0^{x_i^*(t)} t(s)ds > 0$.*

In words, a rigid cap varies players' bidding strategy and does not generate tax revenue, which is equivalent to the outright bid limit assumed in Che and Gale (1998) and Olszewski and Siegel (2019), whereas a flexible cap scheme effectively taxes players and thus affects their equilibrium bidding behavior.

In what follows, we say that a bid cap is *binding* if it is either rigid or flexible, as defined above. In contrast, we say that a scheme $t \in \mathcal{T}$ is *nonbinding*, or can interchangeably be called *no cap*, if $\boldsymbol{x}^*(t) = \boldsymbol{x}^\dagger$ and $\sum_{i=1}^n \int_0^{x_i^*(t)} t(s)ds = 0$: It neither elicits tax revenue nor influences players' bidding behavior.[11] For example, a tax schedule with $t(x) > 0$ only for

[10] It is worth noting that Szidarovszky and Okuguchi (1997) and Fu and Wu (2020) require a twice-differentiable bidding cost function to establish equilibrium uniqueness, which may not be satisfied in our setting given that that the bidding cost function can be manipulated by the designer. Still, we can adapt their proofs to our setting.

[11] It can be verified that $\boldsymbol{x}^*(t) = \boldsymbol{x}^\dagger$ and $\sum_{i=1}^n \int_0^{x_i^*(t)} t(s)ds > 0$ cannot occur.

$x > \max\{x_1^\dagger, \ldots, x_n^\dagger\}$ does not impose effective constraints on the contest and is a nonbinding cap: It leads to the equilibrium bid profile $\boldsymbol{x}^\dagger \equiv (x_1^\dagger, \ldots, x_n^\dagger)$ and generates zero tax revenue.

3 Analysis

We now present the results that delineate the general properties of the optimal contest.

3.1 Suboptimality of Rigid Caps

We first demonstrate that flexible caps can more effectively advance the designer's interests than rigid ones.

Theorem 1 ***(Suboptimality of a Rigid Cap)*** *Suppose that Assumption 1 is satisfied. For every rigid cap, there exists a flexible cap $t(\cdot) \in \mathcal{T}$ that generates a strictly higher aggregate bid in the equilibrium than the rigid cap.*

Theorem 1 states that in noisy contests with concave contest success functions, a flexible cap outperforms rigid cap regardless of the designer's preference: For every given rigid cap, a properly set flexible cap always exists that not only generates a positive tax revenue but also elicits a larger aggregate bid than the rigid cap. A rigid cap explicitly constrains a stronger player's ability to outperform his weaker opponents. This outright bidding restriction evens the race, but entirely precludes the stronger players' contributions above the imposed maximum. In contrast, a flexible cap manipulates the competition in a more nuanced manner. Note that bidders' prize valuations are ordered in descending order with $v_1 \geq \cdots \geq v_n$, as are their equilibrium bids under a uniform taxation schedule. Higher bids trigger heightened marginal tax rates, which penalize stronger players but in a gentler way relative to a rigid cap. By Theorem 1, flexible caps preserve incentives and promote competitions more effectively than rigid caps.

It is useful to point out that Theorem 1 does not require a specific form of the impact function $f(\cdot)$, as long as it is weakly concave; this reflects the *noisiness* of the contest.[12] We use a simple two-player setting with $v_1 > v_2$ and a strictly concave $f(\cdot)$ to elaborate on the role of the concavity of the impact function in the proof for Theorem 1. Recall that the equilibrium bid profile in the unconstrained contest is denoted by $\boldsymbol{x}^\dagger := (x_1^\dagger, x_2^\dagger)$. Consider a rigid cap on players' bid that specifies the maximal admissible bid of ℓ, with $x_2^\dagger < \ell < x_1^\dagger$.[13]

[12] As we will show later in Propositions 2 and 3, a rigid cap can arise in the optimal contest as the impact function turns convex in the case of $\lambda = 0$.

[13] The bid cap scheme is nonbinding if $\ell \geq x_1^\dagger$ and the equilibrium bid profile is (ℓ, ℓ) if $\ell \leq x_2^\dagger$.

It can be verified that player 1's equilibrium bid is $x_1^* = \ell$ upon imposition of the rigid cap, and the following inequality holds:

$$\left.\frac{\partial \pi_1}{\partial x_1}\right|_{x_1=x_1^*} = \frac{f'\left(x_1^*\right) p_1\left(\boldsymbol{x}^*\right)\left[1-p_1\left(\boldsymbol{x}^*\right)\right]}{f\left(x_1^*\right)} v_1 - 1 > 0. \tag{4}$$

Further, player 2's equilibrium bid x_2^* is governed by the following first-order condition:

$$\left.\frac{\partial \pi_2}{\partial x_2}\right|_{x_2=x_2^*} = \frac{f'\left(x_2^*\right) p_2\left(\boldsymbol{x}^*\right)\left[1-p_2\left(\boldsymbol{x}^*\right)\right]}{f\left(x_2^*\right)} v_2 - 1 = 0. \tag{5}$$

Consider a bid pair $\check{\boldsymbol{x}}^* \equiv (\check{x}_1^*, \check{x}_2^*) = (x_1^* + \epsilon_1, x_2^* - \epsilon_2)$, with $\epsilon_1, \epsilon_2 > 0$, that satisfies

$$f(x_1^*) + f(x_2^*) = f(\check{x}_1^*) + f(\check{x}_2^*). \tag{6}$$

Compared with $\boldsymbol{x}^* \equiv (x_1^*, x_2^*)$, the favorite's bid increases and the underdog's bid decreases, resulting in a more dispersed bid profile. The above conditions, together with the strict concavity of $f(\cdot)$, imply immediately that $x_1^* + x_2^* < \check{x}_1^* + \check{x}_2^*$. Next, consider the following tax schedule:

$$\check{t}(x) := \begin{cases} \frac{f'\left(\check{x}_2^*\right)p_2(\check{\boldsymbol{x}}^*)\left[1-p_2(\check{\boldsymbol{x}}^*)\right]}{f\left(\check{x}_2^*\right)} v_2 - 1, & x \le \check{x}_2^*, \\ \frac{f'\left(\check{x}_1^*\right)p_1(\check{\boldsymbol{x}}^*)\left[1-p_1(\check{\boldsymbol{x}}^*)\right]}{f\left(\check{x}_1^*\right)} v_1 - 1, & x > \check{x}_2^*. \end{cases}$$

By (4), (5), (6), and, again, the strict concavity of $f(\cdot)$, the above tax scheme $\check{t}(x)$ is progressive and well defined for sufficiently small ϵ_1 and ϵ_2. Moreover, it is straightforward to verify that $\check{\boldsymbol{x}}^* \equiv (\check{x}_1^*, \check{x}_2^*)$ constitutes an equilibrium under the constructed bid cap $\check{t}(\cdot)$ and generates positive tax revenue. In summary, the constructed tax schedule $\check{t}(\cdot)$ strictly outperforms the rigid cap ℓ regardless of the designer's preference.

3.2 Optimality of Piecewise Constant Tax Schedules

By Theorem 1, it suffices to compare the performance under no cap and that under an optimally set flexible cap in search of the optimum. Next, we provide more details about the optimal tax schedules. Define

$$\overline{\mathcal{T}} := \left\{ t \in \mathcal{T} \;\middle|\; \begin{array}{l} t(x) = \mathbb{1}\left(x > \overline{x}_1\right)\tau_1 + \sum_{i=1}^{n-1} \mathbb{1}\left(\overline{x}_{i+1} < x \le \overline{x}_i\right)\tau_i + \mathbb{1}\left(0 \le x \le \overline{x}_n\right)\tau_n, \\ \text{with } \overline{x}_1 \ge \cdots \ge \overline{x}_n \ge 0 \text{ and } \tau_1 \ge \cdots \ge \tau_n \ge 0, \forall\, i \in \mathcal{N}. \end{array} \right\}.$$

In words, the set $\overline{\mathcal{T}}$ collects all eligible progressive piecewise constant tax schedules. Such a tax schedule is fully characterized by a profile of marginal tax rate $\boldsymbol{\tau} := (\tau_1, \ldots, \tau_n)$ and a profile of cutoffs $\overline{\boldsymbol{x}} := (\overline{x}_1, \ldots, \overline{x}_n)$. Clearly, $\overline{\mathcal{T}} \subset \mathcal{T}$. The following result further narrows the search for the optimal tax schedule.

Theorem 2 ***(Optimality of Piecewise Constant Tax Schedules)*** *Suppose that Assumption 1 is satisfied. For every cap scheme $t \in \mathcal{T}$, there exists $\tilde{t} \in \overline{\mathcal{T}}$ such that $\mathcal{F}(\tilde{t}) \geq \mathcal{F}(t)$.*

Theorem 2 allows us to focus on piecewise constant tax schedule $t(\cdot)$ without loss of generality, i.e., within the set $\overline{\mathcal{T}}$. A candidate optimal flexible cap can take the form of

$$t(x) = \begin{cases} \tau_n, & 0 \leq x \leq \overline{x}_n, \\ \tau_i, & \overline{x}_{i+1} < x \leq \overline{x}_i, i \in \{1, \ldots, n-1\}, \\ \tau_1, & x > \overline{x}_1. \end{cases}$$

A constant tax rate τ_i is imposed on each bracket, with the bracket capped by $\overline{x}_i$; the tax rate τ_i decreases with i for $i \in \{1, \ldots, n\}$ and takes no more than n nonnegative values.

Recall that the profile of equilibrium bidding strategies under $t(\cdot) \in \mathcal{T}$ is denoted by $\boldsymbol{x}^*(t) = \big(x_1^*(t), \ldots, x_n^*(t)\big)$. To elaborate on the logic underlying the proof of Theorem 2, again we consider a two-player case with $v_1 > v_2$ and begin with a continuous and strictly increasing tax schedule $t(\cdot) \in \mathcal{T}$. It is straightforward to verify that both players would remain active in equilibrium, and their equilibrium bids are governed by the following first-order conditions:

$$\left.\frac{\partial \pi_i}{\partial x_i}\right|_{x_i = x_i^*(t)} = \frac{f'\left(x_i^*(t)\right) p_i\left(\boldsymbol{x}^*(t)\right)\left[1 - p_i\left(\boldsymbol{x}^*(t)\right)\right]}{f\left(x_i^*(t)\right)} v_i - \left[1 + t\left(x_i^*(t)\right)\right] = 0,\ i \in \{1, 2\}. \quad (7)$$

Now consider the following piecewise constant tax schedule $\tilde{t}(\cdot)$:

$$\tilde{t}(x) = \begin{cases} t\left(x_2^*(t)\right), & x \leq x_2^*(t), \\ t\left(x_1^*(t)\right), & x > x_2^*(t). \end{cases} \quad (8)$$

Note that Equation (7) holds for $\boldsymbol{x}^*(t)$ and $\tilde{t}(\cdot)$, which indicates that $\boldsymbol{x}^*(t)$ also constitutes an equilibrium under the constructed piecewise constant tax schedule $\tilde{t}(\cdot) \in \overline{\mathcal{T}}$, i.e., $\boldsymbol{x}^*(\tilde{t}) = \boldsymbol{x}^*(t)$. As a result, the aggregate bid remains unchanged when the prevailing bid cap scheme is switched from $t(\cdot)$ to $\tilde{t}(\cdot)$. Further, $\tilde{t}(x) \geq t(x)$ for all $x \in \left[0, x_1^*(t)\right]$ by our construction, which in turn implies that $\sum_{i=1}^2 \int_0^{x_i^*(\tilde{t})} \tilde{t}(s)ds \geq \sum_{i=1}^2 \int_0^{x_i^*(t)} t(s)ds$—i.e., a higher tax revenue under $\tilde{t}(x)$—and consequently $\mathcal{F}(\tilde{t}) \geq \mathcal{F}(t)$.

The constructed piecewise tax schedule (8) sets the caps $\overline{x}_2$ for the minimum bracket precisely at player 2's equilibrium bid $x_2^*(\tilde{t})$. Indeed, this construction can easily be adapted to a multi-player setting, and we can restrict our attention to the piecewise constant tax schedules under which the cap imposed on each tax bracket coincides with one player's equilibrium bid. This property greatly simplifies our analysis.

Corollary 1 *Suppose that the optimal tax schedule exists. Then the optimum can be achieved by some piecewise constant tax schedule* $\langle \boldsymbol{\tau}^\star, \overline{\boldsymbol{x}}^\star \rangle \equiv \langle (\tau_1^\star, \ldots, \tau_n^\star), (\overline{x}_1^\star, \ldots, \overline{x}_n^\star) \rangle$ *under which the equilibrium bidding profile is* $\overline{\boldsymbol{x}}^\star$*.*

3.3 Aggregate Bid Maximization

Theorems 1 and 2 set a clear path for our search for the optimum: It depends on the comparison between an optimally set flexible cap with a piecewise constant tax schedule and a laissez-faire policy, i.e., no cap. We now consider the commonly studied case of aggregate bid maximization—i.e., $\lambda = 0$—which allows us to focus on a bid cap's effect on players' bidding behavior, while abstracting away the concern about tax revenue.

As stated in the Introduction, a flexible cap alters players' incentives and triggers two competing effects with regard to players' bidding. On the one hand, a flexible cap elevates bidding costs and discourages bidding (*cost effect*); on the other hand, such a bid cap leads to an "equalizing shift" in the spirit of Che and Gale (2006): A progressive tax schedule automatically handicaps a stronger player because of his higher bid, which levels the playing field and fuels competition (*competition effect*). Our next result nevertheless shows that a flexible bid cap's (indirect) competition effect cannot offset the (direct) discouragement caused by higher costs in a noisy contest.

Theorem 3 ***(Optimality of a Laissez-faire Scheme under Aggregate Bid Maximization)*** *Suppose that Assumption 1 is satisfied and* $\lambda = 0$*. The optimum imposes no bid cap, i.e.,* $\tau_1^\star = \cdots = \tau_n^\star = 0$*.*

Theorem 3 states that a bid cap is always suboptimal when the designer does not benefit from tax revenue. We highlight the fact that the prediction, together with Theorem 1, overturns the implications obtained from all-pay auction models (Che and Gale, 1998; Kaplan and Wettstein, 2006): A binding bid cap, regardless of its form, always decreases the level of the aggregate bid in the contest.

Theorem 3 provides not only a theoretical contribution but also policy relevance. Recall the aforementioned policy debate regarding excessive lobbying tax. By the theorem, a binding bid cap of any form leads to a lower aggregate bid. As a result, a lobbying tax, as a revenue-generating flexible cap, would unambiguously curb lobbying activities, which stands in contrast to the prediction of Che and Gale (1998). Further, a progressive lobbying tax limits the advantage of stronger contenders—e.g., corporate lobbyists—which in turn levels the playing field for public interests groups; this contrasts with the neutrality result of Kaplan and Wettstein (2006), which will be discussed more extensively in Section 4.2.

In what follows, we interpret the economic logic underlying this result. The rationale provides an intuitive account of the roles played by a bid cap in noisy contests.

Intuition We consider a simple two-player example to interpret the logic underlying Theorem 3. Assuming $v_1 > v_2$, we let $f(x_i)$ take the functional form of $f(x_i) = x_i^r$, with $r \in (0, 1]$, which yields the frequently studied Tullock contest and enables an equilibrium solution in closed form. The parameter r is conventionally interpreted as a measure of the noisiness of the winner-selection mechanism. A larger r implies that additional effort can more effectively be converted into higher winning odds.

Fix a piecewise constant tax schedule $\langle \boldsymbol{\tau}^\star, \overline{\boldsymbol{x}}^\star \rangle \equiv \langle (\tau_1^\star, \tau_2^\star), (\overline{x}_1^\star, \overline{x}_2^\star) \rangle$, with $\tau_1^\star > \tau_2^\star > 0$ and $\overline{x}_1^\star > \overline{x}_2^\star > 0$ as described in Corollary 1. The tax schedule $\langle \boldsymbol{\tau}^\star, \overline{\boldsymbol{x}}^\star \rangle$ induces the equilibrium bidding profile $\overline{\boldsymbol{x}}^\star$, and the equilibrium bid pair $(\overline{x}_1^\star, \overline{x}_2^\star)$ can be solved for by the following first-order conditions:

$$\frac{v_i}{1+\tau_i^\star} \times \left[1 - p_i(\overline{\boldsymbol{x}}^\star)\right] \times \frac{f'(\overline{x}_i^\star)}{\sum_{j=1}^2 f(\overline{x}_j^\star)} = 1, i \in \{1, 2\}.$$

The conditions imply that the equilibrium bid pair $(\overline{x}_1^\star, \overline{x}_2^\star)$ under $\langle \boldsymbol{\tau}^\star, \overline{\boldsymbol{x}}^\star \rangle$ also constitutes the unique equilibrium in an *unconstrained* contest with prize valuations $\big(v_1/(1+\tau_1^\star), v_2/(1+\tau_2^\star)\big)$.[14] The equilibrium bid can be obtained as

$$\overline{x}_i^\star = r\frac{\left(\frac{v_1}{1+\tau_1^\star}\right)^r \left(\frac{v_2}{1+\tau_2^\star}\right)^r}{\left[\left(\frac{v_1}{1+\tau_1^\star}\right)^r + \left(\frac{v_2}{1+\tau_2^\star}\right)^r\right]^2} \times \frac{v_i}{1+\tau_i^\star}, i \in \{1, 2\}.$$

The above equilibrium, together with the fact that $\overline{x}_1^\star > \overline{x}_2^\star$, implies that $v_1/v_2 > (1+\tau_1^\star)/(1+\tau_2^\star)$.

We take two steps to show how a flexible cap affects the equilibrium aggregate bid in a noisy contest.[15] First, we set the marginal tax rate for the favorite to that for the underdog: The designer imposes a *proportional* tax schedule with a flat marginal tax rate of $\tau_2^\star > 0$. The resultant equilibrium bid pair, denoted by $(\overline{x}_1^{\star\star}, \overline{x}_2^{\star\star})$, can be obtained as

$$\overline{x}_i^{\star\star} = r\frac{(v_1)^r (v_2)^r}{\left[(v_1)^r + (v_2)^r\right]^2} \times \frac{v_i}{1+\tau_2^\star}, i \in \{1, 2\}.$$

Intuitively, on the one hand, such a tax reduction reduces the favorite's marginal bidding cost and directly boosts his incentive. This, on the other hand, restores the favorite's advantage

[14] It is noteworthy that the tax revenue collected through the two scenarios would be different.

[15] A similar procedure can be applied when the contest consists of three or more players.

in the competition and, in turn, discourages the underdog. The tension is evidenced by $\overline{x}_1^{\star\star} > \overline{x}_1^{\star}$ and $\overline{x}_2^{\star\star} < \overline{x}_2^{\star}$ from the condition $v_1/v_2 > (1+\tau_1^\star)/(1+\tau_2^\star)$. The former direct positive effect outweighs the indirect negative competition effect for the case of small r—i.e., $r \leq 1$—which causes the aggregate bid to increase in response as $\tau_1^\star$ reduces to $\tau_2^\star$.

Next, we remove the constant marginal tax rate $\tau_2^\star > 0$. By the argument laid out above, this is equivalent to a uniform increase in players' prize valuations—i.e., from $(v_1/(1 + \tau_2^\star), v_2/(1+\tau_2^\star))$ to (v_1, v_2)—without disturbing the balance of the playing field. Equilibrium bids would increase proportionally, which leads to further gains in the aggregate bid.

Recall the tension described in the Introduction between the cost and competition effects. The first step in this argument provides an intuitive account of such a trade-off, since reducing $\tau_1^\star$ to $\tau_2^\star$ unwinds both effects: This lowers bidding cost for the favorite, thereby incentivizing him; however, it restores the favorite's advantage and unevens the contest, which discourages the underdog. The cost effect outweighs the competition effect in this noisy contest. It is important to note that noise erodes the efficacy of a bid cap as an equalizing device in the contest. Setting $\tau_1^\star > \tau_2^\star$ is less effective for incentivizing the underdog, since a win or loss is largely random—i.e., with $r \leq 1$. As a result, when $\tau_1^\star$ decreases to $\tau_2^\star$, the increase in player 1's bid—due to the unwinding of the cost effect—more than offsets the decrease in player 2's bid when the competition effect is muted. The second step further testifies to the costly nature of a bid cap, as it directly discounts players' incentives.

This rationale can well extend to multi-player settings. The tension between the cost and competition effects will be further tilted toward the former as the number of players increases. The addition of players plays a role analogous to that of a noisier winner-selection mechanism: It is more difficult to inspire an underdog to greater efforts when he has to outperform more opponents for a win; thus a costly equalizing device—i.e., a bid cap—tends to further lose its appeal when the contest involves more players.

3.4 Discussion of Key Insights

Theorems 1 and 3 do not provide a ranking of no cap vis-à-vis flexible cap if the designer values the tax revenue, i.e., if $\lambda > 0$. We subsequently demonstrate that either can prevail, and the aforementioned trade-off between the cost and competition effects determines the optimum. The observations shed further light on the nature of the bid cap.

When the Designer Values Tax Revenue We continue with the two-player Tullock contest setting. Define $v := v_2/v_1 \in (0, 1]$, which measures the evenness of the contest: A smaller v implies a more lopsided race. In Online Appendix A, we provide the sufficient condition under which a flexible cap or no cap can be optimal when $\lambda > 0$. A numerical exercise fully characterizes the optimum, which is illustrated in Figure 1.

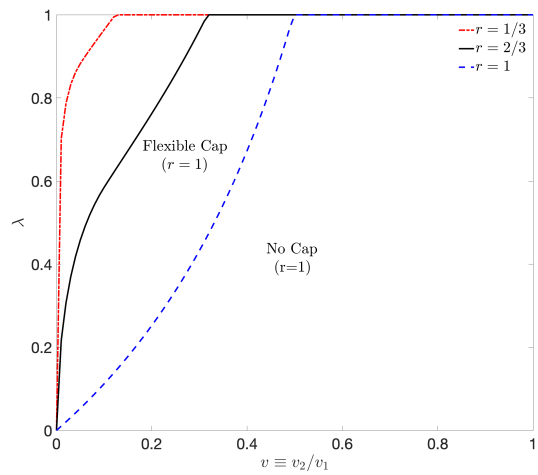

Figure 1: Optimal bid cap in two-player Tullock contests: $r \in \{1/3, 2/3, 1\}$.

The horizontal axis measures $v \equiv v_2/v_1$ and the vertical axis traces the value of λ, with both ranging from 0 to 1. The three positively sloped curves delineate the optimal cap schemes for different r, which take the value 1/3, 2/3, and 1, respectively, from left to right. For a given r, a flexible cap is optimal when (v, λ) falls in the region to the left of the corresponding curve, while no cap prevails for (v, λ) to the right of the curve. Obviously, as Figure 1 shows, a flexible cap is more likely to prevail when λ increases, i.e., when the designer values tax revenue more.

Two observations are noteworthy. First, all curves are positively sloped, which implies that the optimum is more likely to involve no cap when players are closer in terms of their strengths, i.e., with a larger v. Second, the curve is shifted rightward when the contest becomes less noisy—i.e., when r increases—which implies that a flexible cap is more likely to emerge in the optimum when the winner-selection mechanism is more precise.

These observations further demonstrate the trade-off between the cost and competition effects. The tension fades away when v increases: An even race does not require costly intervention that compromises the stronger player's bidding incentive. Further, substantial noise—i.e., a smaller r—evens the race because it diminishes the advantage of the stronger player, which also eliminates the need for a costly bid cap. More formally, we state the following.

Remark 1 *Suppose that $n = 2$, $\lambda \in [0, 1]$, and $f(x_i) = x_i^r$, with $r \in (0, 1]$. The optimal contest requires that no cap be imposed if the contest is sufficiently noisy—i.e., $r \leq v$—or, equivalently, if the contest is sufficiently even, i.e., $v \geq r$.*

By Remark 1, bid caps are suboptimal when (i) the contest is sufficiently noisy or (ii) players are sufficiently homogeneous, regardless of the designer's preference for tax revenue.

Multi-player Settings As stated above, the rationale that interprets Theorem 3 well extends to multi-player settings. The optimum continues to be subject to the trade-off between the cost and competition effects. For brevity, additional results are provided in Online Appendix B. The observations demonstrate how the optimum in a multi-player contest is shaped by the various environmental factors, and echo the findings presented above. However, it is noteworthy that the presence of multiple heterogeneous players triggers substantial nuances. First, player heterogeneity is inherently multidimensional and cannot readily be measured by a single parameter, as in a two-player contest. Second, bid caps continue to be a useful instrument for the designer to exploit players' heterogeneity; however, setting a bid cap gives rise to a hidden selection problem: The designer may deploy her instruments to selectively incentivize a certain subset of players.

The following result is obtained, which further illustrates the nature of a bid cap as a costly intervention.

Proposition 1 ***(Zero Marginal Tax Rate for the Minimum Bracket in Tullock Contests)*** *Suppose that $\lambda < 1$ and $f(x_i) = x_i^r$, with $r \in (0, 1]$. The optimal piecewise constant cap schedule sets $\tau_\kappa^\star = \cdots = \tau_n^\star = 0$, where κ is the number of active players in the optimum.*[16,17]

In a multi-player contest, the weakest players can be compelled to stay inactive, e.g., players i with $i \in \{\kappa + 1, \ldots, n\}$ in the context of Proposition 1. The optimally set flexible cap chosen from the set of piecewise constant tax functions $\overline{\mathcal{T}}$ requires a zero tax rate for the minimum bracket. The weakest active player—who is indexed by κ—does not pay a tax in the optimum. The trade-off between cost and competition effects emerges when setting the cap scheme: A sufficient spread between the highest and lowest marginal tax rates—i.e., between τ_1 and τ_κ—is required to even the race, while a tax burden erodes players' bidding incentives. Tax exemption for the underdog ensures a sufficiently large spread for an effective handicap without entailing excessive penalty, which mitigates the aforementioned cost effect.

4 Extension and Further Discussion

In this section, we extend our model to allow for convex impact functions, which sheds further light on the nature of bid caps and revives the linkage of our analysis to the conventional wisdom obtained in the literature.

[16]For the case of $\lambda = 1$, the optimum can be achieved by a piecewise constant cap scheme that satisfies $\tau_\kappa = \cdots = \tau_n = 0$.

[17]It is straightforward to verify that $\kappa = n$ if $r < 1$.

4.1 Tullock Contest with $r \in (1, 2]$

We now relax the assumption of concave impact function, while retaining a Tullock contest setting. It is well known that a pure-strategy equilibrium may not exist when r exceeds 1;[18] the properties of the resulting mixed-strategy equilibria largely remain elusive, especially for the case of $r > 2$.[19] We focus on the more tractable case of $r \in (1, 2]$. In a standard two-player contest, a pure-strategy equilibrium remains when r falls below a cutoff within the interval $(1, 2]$, while a semi-mixed equilibrium emerges when r exceeds it (Wang, 2010).[20]

In Appendix B, we establish that in parallel to Theorem 2, it is without loss of generality to search for the optimum within the class of two-part tax schedules that set strictly positive tax rates only for the bids in the higher bracket. We then characterize the equilibria under the prevailing bid caps, which lays a foundation for the analysis of optimal bid cap schemes. The following result can be obtained.

Proposition 2 ***(Equilibrium Property in Optimal Two-player Tullock Contests with $r \in (1, 2]$ and Bid Caps)*** *Suppose that $n = 2$ and $f(x_i) = x_i^r$, with $r \in (1, 2]$. The following statements hold whenever no cap is suboptimal:*

(i) For $\lambda = 0$, the optimum can take the form of either a rigid cap or a flexible cap: They generate the same amount of expected aggregate bid and induce the same bidding equilibrium.[21]

(ii) For $\lambda > 0$, the optimum requires a flexible cap, which strictly dominates all rigid caps.[22]

Proposition 2(i) states that whenever a binding cap is required, the optimum can equivalently be achieved by either a rigid cap or a flexible one when the designer only cares about the aggregate bid, i.e., $\lambda = 0$. However, a rigid cap remains suboptimal for $\lambda > 0$ because it does not generate tax revenue.

[18] For instance, Baye, Kovenock, and De Vries (1994) demonstrate that when $r > 2$, the local maximizer defined by the first-order conditions do not constitute a global maximum.

[19] With $r > 2$, the cumulative density function of a player's mixed strategy has infinite discontinuous points and has yet to be explicitly characterized in the literature. Baye, Kovenock, and De Vries (1994) employ a discretization approach to identify a symmetric mixed-strategy equilibrium in a two-player contest with $r \in (2, \infty)$. Alcalde and Dahm (2010) establish a mixed-strategy equilibrium in the case $r > 2$ and propose the notion of an all-pay-auction equilibrium. Ewerhart (2015) shows in a symmetric setting that the equilibria stand in contrast to the well-characterized pure-strategy equilibrium in a standard lottery contest and the mixed-strategy equilibrium in an all-pay auction.

[20] Feng and Lu (2017) and Ewerhart (2017) further verify equilibrium uniqueness in this case of $r \in (1, 2]$.

[21] For $\lambda = 0$, there could exist multiple equilibria under the optimal rigid cap. We can always construct a flexible cap that leads to a unique pure-strategy equilibrium to achieve the optimum.

[22] For $\lambda > 0$, whenever policy intervention is required, we can construct a flexible cap that results in a unique pure-strategy equilibrium to achieve the optimum.

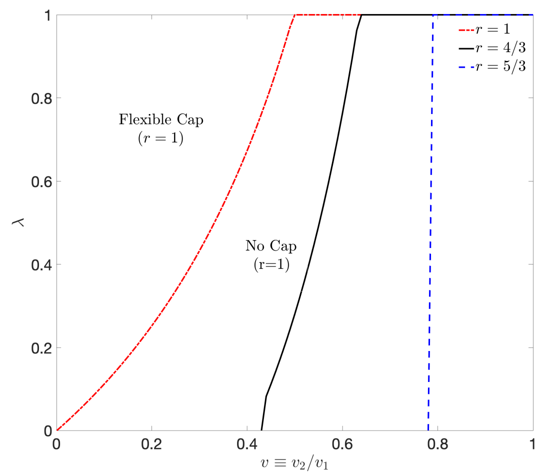

Figure 2: Optimal bid cap in two-player Tullock contests: $r \in \{1, 4/3, 5/3\}$.

Proposition 2(i) contrasts radically with Theorems 1 and 3. Recall that with concave impact functions, Theorem 1 states that for every rigid cap, there exists a flexible cap that generates a strictly larger amount of aggregate bid, which indicates the suboptimality of a rigid cap. By Proposition 2(i), an optimally set rigid cap and its flexible counterpart can induce the same equilibrium bidding outcome, although the equivalence dissolves for $\lambda > 0$ because a rigid cap does not generate tax revenue. As a result, a rigid cap can be a candidate for the optimum when the designer maximizes the aggregate bid only.

We conduct a numerical exercise that fully characterizes the optimum, which is illustrated in Figure 2. As in Figure 1, the horizontal axis measures $v \equiv v_2/v_1$ and the vertical axis traces the value of λ. The three positively sloped curves define the optimum for the three cases of $r = 1, 4/3$, and $5/3$, respectively, from left to right. For a given r, a binding cap is optimal when (v, λ) is located to the left of the curve, while no cap prevails for combinations of (v, λ) to the right. These observations are largely consistent with those obtained from Figure 1 and Remark 1. In particular, a larger r expands the set of parameterizations under which a binding cap prevails. For a given λ, a binding cap is less appealing when v increases, i.e., when the contest is more even.

In Appendix B, we provide sufficient conditions for a flexible cap or no cap to be optimal when the designer values tax revenue, i.e., $\lambda > 0$. When the designer cares only about the aggregate bid, however, we obtain unambiguous sufficient and necessary conditions under which either a binding cap or no cap emerges as the optimum. Define a cutoff value $r^* \in (1, 2]$ for r, which uniquely solves the equation $r = 1 + v^r$ for a given v; further, the equation can be solved by a unique $v^* \in (0,1)$ for a given r. Similarly, the equation

$$r + \frac{1}{1+v} - \frac{2r}{(1+v^r)} = 0$$

can be solved by a unique $r^{**} \in (1,2)$ for a given v or a unique $v^{**} \in (0,1)$ for a given r. The following can be obtained.

Proposition 3 ***(Optimal Aggregate-bid-maximizing Tax Schedule in Two-player Tullock Contests with $r \in (1,2]$)*** *Suppose that $n = 2$, $\lambda = 0$, and $f(x_i) = x_i^r$, with $r \in (1,2]$. The optimal contest requires no cap when $r \leq \min\{r^*, r^{**}\}$ or $v \geq \max\{v^*, v^{**}\}$; it requires a binding cap otherwise, in which case the optimum can be achieved by either a rigid cap or a flexible one.*

By Proposition 3, a binding cap, which can be either rigid or flexible, can maximize the aggregate bid. This observation further demonstrates the rationale laid out in Section 3.4. As a costly equalizing device, a binding bid cap is required to balance the playing field when the contest is sufficiently discriminatory—i.e., $r > \min\{r^*, r^{**}\}$—or the competition is sufficiently uneven, i.e., $v < \max\{v^*, v^{**}\}$.

4.2 Relation to the Literature: Further Discussion

Our main results sharply contrast with those obtained in two-player all-pay auctions, such as Che and Gale (1998); Kaplan and Wettstein (2006); and Sahuguet (2006). As previously stated, an all-pay auction can be viewed as the limiting form of a Tullock contest when the parameter r goes to infinity. The analysis in the case of $r \in (1,2]$ revives the linkage of our baseline analysis to the literature .

Note that objective function (3) can be rewritten as

$$\begin{aligned}\mathcal{F}(t) :=& \underbrace{\sum_{i=1}^n x_i^*(t)}_{\text{aggregate bid}} + \lambda \times \underbrace{\sum_{i=1}^n \int_0^{x_i^*(t)} t(s)ds}_{\text{tax revenue}} \\ =& (1-\lambda) \times \underbrace{\sum_{i=1}^n x_i^*(t)}_{\text{aggregate bid}} + \lambda \times \underbrace{\sum_{i=1}^n c\left(x_i^*(t)\right)}_{\text{aggregate bidding cost}}, \text{ with } 0 \leq \lambda \leq 1, \end{aligned} \tag{9}$$

which is a convex combination between the aggregate bid and the aggregate bidding cost.

Although neither Che and Gale (1998) nor Kaplan and Wettstein (2006) explicitly consider the design of a bid cap scheme, combining their results allows us to infer a complete ranking of flexible cap, rigid cap, and no cap in terms of maximizing $\mathcal{F}(t)$ in two-player all-pay auctions: An appropriately set rigid cap strictly outperforms no cap, while no cap is strictly preferred to any flexible cap, whenever they are enforceable. By Che and Gale (1998), an appropriately set bid cap elicits a higher aggregate bid and higher total expenditure than those under no cap; by Kaplan and Wettstein (2006), imposing a flexible cap causes the equilibrium aggregate bid—which varies players' bidding cost functions—to fall below that under no cap, while leaving equilibrium total bidding expenditure and winning probabilities constant, regardless of the effort cost function (binding flexible cap of any form or no cap), which refers to the well-known neutrality result.

Our Theorem 1, in contrast, establishes that a rigid cap must strictly underperform some flexible caps, regardless of the designer's preference, i.e., the size of λ. As a result, a rigid cap is always suboptimal in our noisy contest for all eligible objective functions. Furthermore, our Theorem 3 shows that when the designer aims to maximize only the aggregate bid, no cap outperforms all binding caps. A complete ranking immediately arises in our setting when the designer does not value tax revenue: No cap outperforms any flexible cap, while every rigid cap can be strictly dominated by a properly set flexible cap. These implications clearly depart from the conventional wisdom obtained in complete-information all-pay auction models.

The neutrality result of Kaplan and Wettstein (2006) obviously does not hold in our context. When the designer values tax revenue—i.e., $\lambda > 0$—either no cap or a flexible cap can be optimal (see Section 3.4). Players' equilibrium winning probabilities will also be affected by the imposition of a flexible cap. Given its progressive nature, a flexible cap subjects the favorite to a higher marginal effort cost and effectively imposes a handicap, which leads to a more even race than the competition under no cap.[23]

With r in an intermediate range—i.e., $r \in (1, 2]$—Proposition 2(i) establishes an equivalence between an optimally set rigid cap and its flexible counterpart, since they induce the same equilibrium outcome. Further, by Proposition 3, a rigid cap can be optimal in the case with $\lambda = 0$, although it does not strictly outperform the optimally set flexible cap. The setting of $r \in (1, 2]$ reconciles the two extreme cases of concave impact functions and all-pay auctions: Proposition 3 echoes Che and Gale (1998) and Kaplan and Wettstein (2006), although a rigid cap remains suboptimal in the case with $\lambda > 0$.

[23]We now briefly interpret the neutrality result of Kaplan and Wettstein (2006) and its limit. Note that a contest can be understood as an alternative but equivalent game in which each player chooses his bidding cost (or distribution of bidding costs in mixed strategy) instead of the bid itself. In an all-pay auction, the probability of winning with a given bidding cost is the same as the probability that the other bidder chooses a bid of lower cost, and a lower bid provided that the bidding cost function is monotone. As a result, manipulating a (common) bidding cost function—as a flexible cap does—does not vary players' distributions of bidding costs in equilibrium, which leads to the neutrality. Consider a two-player noisy contest; the probability of winning with a bid, however, is not the same as the probability that the other bidder chooses lower bids and also depends on the particular sizes of the bids. As a result, for a given pair of bids, if we vary players' bidding cost functions and change the bids to maintain the same bidding costs, players' winning probabilities would depart in response from those under the original bid profile, which dissolves the neutrality.

	$0 < r \leq 1$ (this paper)	$1 < r \leq 2$ (this paper)	$r = \infty$ (CG98, KW06)
Equilibrium Behavior	pure strategy	pure strategy or semi-mixed strategy under a two-part bid cap (pure strategy under optimal policy)	mixed strategy
Optimum ($\lambda = 0$)	no cap	no cap, rigid cap, or flexible cap	rigid cap
Optimum ($\lambda > 0$)	no cap, flexible cap	no cap, flexible cap	rigid cap

Table 1: Optimal bid cap in two-player Tullock contests.

Table 1 provides a summary of the results under different levels of r. CG98 and KW06 in the table stand for Che and Gale (1998) and Kaplan and Wettstein (2006), respectively.

5 Concluding Remarks

This paper explores the optimal bid cap scheme in multi-player generalized lottery contests. We model the bid cap as a progressive tax schedule that elevates players' bidding costs. In contrast to all-pay auctions, a bid cap, regardless of its form, enables an equalizing shift in the spirit of Che and Gale (2006) that levels the playing field and fuels competition, despite the discouragement caused by higher costs. We show that when the winner-selection mechanism of the contest involves substantial noise—i.e., when the impact function is concave—the discouragement caused by elevated costs always outweighs the competition effect, and a binding bid cap always reduces the aggregate bid. The observations in the case of concave impact functions notably contrast with results in the literature based on two-player all-pay auctions, which sheds light on the nature of bid caps and the complexity inherent in optimal contest design in multi-player noisy contests. We further extend the analysis to allow for convex impact functions. The results allow us to develop a rationale that reconciles our analysis with the literature based on all-pay auctions.

Our results also generate useful practical implications. Consider, for instance, the growing call to tax corporate special interest lobbying. The literature—e.g., Che and Gale (1998)—suggests that a rigid bid cap may paradoxically intensify the competition. Our results nevertheless demonstrate that a binding bid cap, regardless of its form, helps limit the aggregate bid. Further, our results show that a bid cap is more likely to be optimal when the winner-selection mechanism is more precise and the designer values tax revenue. Salary caps, along with luxury taxes, exemplify the flexible cap defined in our paper and are commonly adopted in professional sports leagues. Ben-Naim, Vazquez, and Redner (2007) empirically estimate and compare the degrees of predictability—i.e., the significance of luck in determining winners vis-à-vis skill or effort—in different sports. Our results could provide useful insights into the administration of salary cap schemes across different sports.

Caveats and room for future extensions remain. First, our results are obtained in a complete-information setting and should be interpreted with caution. For instance, the designer is required to know players' valuations to obtain the exact form of the optimal piecewise constant tax schedule. However, the majority of our results do not require the exact form of the optimal bid cap; rather, they yield qualitatively useful implications of broad relevance. Theorem 3, for instance, establishes the ranking between no cap and binding cap regardless of the specific form of the latter. That said, it is theoretically intriguing to extend the model to allow for more general information structures.

Second, our framework assumes risk-neutral players. The economics literature—e.g., Konrad and Schlesinger (1997); Cornes and Hartley (2003, 2012); and Treich (2010)—has broadly recognized the role played by risk aversion in shaping contenders' strategic behavior. This calls for efforts to reexamine the various classical issues of contest design under alternative risk attitudes. Fu, Wang, and Wu (2021) and Drugov and Ryvkin (2021), for instance, explore optimal prize allocation in contests with risk-averse players and demonstrate how such predictions may diverge from the conventional wisdom obtained under risk neutrality. Bid caps in contests also deserve to be investigated in a setting of richer risk preferences, which could be a fruitful avenue for future research.

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Appendix A: Proofs of Results in the Baseline Setting

Proof of Theorem 1

Proof. Consider a rigid cap and denote the corresponding equilibrium bid profile by $\boldsymbol{x}^* \equiv (x_1^*, \ldots, x_n^*)$. For the sake of exposition, we focus on the case in which players have different prize valuations—i.e., $v_1 > \cdots > v_n > 0$—and place different positive bids in equilibrium, i.e., $x_1^* > \cdots > x_n^* > 0$. The analysis is similar for cases in which (i) two or more players have the same prize valuation; (ii) some players remain inactive in equilibrium; and/or (iii) two or more players place the same bid.

Step 1 We first construct a series of bid profiles $\{\boldsymbol{x}^k\}_{k=1}^n := \{(x_1^k, \ldots, x_n^k)\}_{k=1}^n$, with $\boldsymbol{x}^1 = \boldsymbol{x}^*$. For $k \in \{2, \ldots, n\}$, the bid profile $\boldsymbol{x}^k \equiv (x_1^k, \ldots, x_n^k)$ can be derived from $\boldsymbol{x}^{k-1} \equiv (x_1^{k-1}, \ldots, x_n^{k-1})$ as follows:

$$x_j^k = \begin{cases} x_j^{k-1} - \epsilon_k^{k-1}, & j = k, \\ x_j^{k-1} + \epsilon_{k-1}^{k-1}, & j = k-1, \\ x_j^{k-1}, & j \in \mathcal{N} \setminus \{k-1, k\}, \end{cases} \tag{10}$$

where $\epsilon_k^{k-1} > 0$ and $\epsilon_{k-1}^{k-1} > 0$ are sufficiently small and satisfy

$$f(x_{k-1}^{k-1}) + f(x_k^{k-1}) = f(x_{k-1}^{k-1} + \epsilon_{k-1}^{k-1}) + f(x_k^{k-1} - \epsilon_k^{k-1}). \tag{11}$$

The above equation, together with the weak concavity of $f(\cdot)$ and Jensen's inequality, implies that $\epsilon_k^{k-1} \leq \epsilon_{k-1}^{k-1}$ and thus $\sum_{j=1}^n x_j^k \geq \sum_{j=1}^n x_j^{k-1}$.

Step 2 Next, we construct a tax schedule $t^k(\cdot)$, with $k \in \{1, \ldots, n\}$, to induce the bid profile $\boldsymbol{x}^k \equiv (x_1^k, \ldots, x_n^k)$. Let $t^1(\cdot)$ be the original rigid cap we consider. By construction, $\boldsymbol{x}^1$ is the equilibrium under $t^1(\cdot)$. Further, because $t^1(\cdot)$ is a rigid cap, we must have that

$$\left.\frac{\partial \pi_1}{\partial x_1}\right|_{x_1 = x_1^1} = \frac{f'\left(x_1^1\right) p_1\left(\boldsymbol{x}^1\right)\left[1 - p_1\left(\boldsymbol{x}^1\right)\right]}{f\left(x_1^1\right)} v_1 - 1 > 0, \tag{12}$$

and

$$\left.\frac{\partial \pi_2}{\partial x_2}\right|_{x_2 = x_2^1} = \frac{f'\left(x_2^1\right) p_2\left(\boldsymbol{x}^1\right)\left[1 - p_2\left(\boldsymbol{x}^1\right)\right]}{f\left(x_2^1\right)} v_2 - 1 = \frac{f'\left(x_2^1\right)\left[\sum_{j \neq 2} f(x_j^1)\right]}{\left[\sum_{j=1}^n f(x_j^1)\right]^2} v_2 - 1 = 0. \tag{13}$$

Combining (10), (11), (12), and (13), for sufficiently small ϵ_2^1 and ϵ_1^1, we can obtain that

$$\tau_1^2 := \frac{f'\left(x_1^2\right) p_1\left(\boldsymbol{x}^2\right)\left[1-p_1\left(\boldsymbol{x}^2\right)\right]}{f\left(x_1^2\right)} v_1 - 1 > 0, \tag{14}$$

and

$$\tau_2^2 := \frac{f'\left(x_2^2\right) p_2\left(\boldsymbol{x}^2\right)\left[1-p_2\left(\boldsymbol{x}^2\right)\right]}{f\left(x_2^2\right)} v_2 - 1 > 0. \tag{15}$$

By continuity, $\tau_1^2 > \tau_2^2 > 0$ when ϵ_2^1 and ϵ_1^1 are sufficiently small. Let $\boldsymbol{\tau}^2 := (\tau_1^2, \ldots, \tau_n^2)$, where τ_1^2 and τ_2^2 are defined in (14) and (15) above and $\tau_j^2 = 0$ for $j \in \{3, \ldots, n\}$. Next, we define the tax schedule $t^2(\cdot)$ based on the constructed $\boldsymbol{\tau}^2$ and $\boldsymbol{x}^2$ as follows:

$$t^2(x) = \begin{cases} \tau_n^2, & 0 \leq x \leq x_n^2, \\ \tau_i^2, & x_{i+1}^2 < x \leq x_i^2, i \in \{1, \ldots, n-1\}, \\ \tau_1^2, & x > x_1^2. \end{cases}$$

It is straightforward to verify that the bid schedule $t^2(\cdot)$ induces the bid profile $\boldsymbol{x}^2 \equiv (x_1^2, \ldots, x_n^2)$ in equilibrium and generates a positive amount of tax revenue. That is, $t^2(\cdot)$ is a flexible cap.

Similarly, for $k \in \{3, \ldots, n\}$, the first-order conditions for players $k-1$ and k under tax schedule $t^{k-1}(\cdot)$ imply that

$$v_k p_k(\boldsymbol{x}^{k-1})\left[1 - p_k(\boldsymbol{x}^{k-1})\right] \times \frac{f'(x_k^{k-1})}{f(x_k^{k-1})} = 1 + \tau_k^{k-1} = 1,$$

and

$$v_{k-1} p_{k-1}(\boldsymbol{x}^{k-1})\left[1 - p_{k-1}(\boldsymbol{x}^{k-1})\right] \times \frac{f'(x_{k-1}^{k-1})}{f(x_{k-1}^{k-1})} = 1 + \tau_{k-1}^{k-1} > 1.$$

Let $\boldsymbol{\tau}^k := (\tau_1^k, \ldots, \tau_n^k)$, where τ_j^k is given by

$$\tau_j^k := \begin{cases} v_j p_j(\boldsymbol{x}^k)\left[1 - p_j(\boldsymbol{x}^k)\right] \times \frac{f'(x_j^k)}{f(x_j^k)} - 1, & j \in \{k-1, k\}, \\ \tau_j^{k-1}, & j \in \mathcal{N} \setminus \{k-1, k\}. \end{cases}$$

Again, by continuity, $\tau_{k-1}^k > \tau_k^k > 0$ when ϵ_k^{k-1} and ϵ_{k-1}^{k-1} are sufficiently small. Define the tax schedule $t^k(\cdot)$ based on the constructed $\boldsymbol{\tau}^k \equiv (\tau_1^k, \ldots, \tau_n^k)$ and $\boldsymbol{x}^k \equiv (x_1^k, \ldots, x^k)$ as follows:

$$t^k(x) = \begin{cases} \tau_n^k, & 0 \le x \le x_n^k, \\ \tau_i^k, & x_{i+1}^k < x \le x_i^k, i \in \{1, \ldots, n-1\}, \\ \tau_1^k, & x > x_1^k. \end{cases}$$

Again, it can be verified that the bid schedule $t^k(\cdot)$ induces the bid profile $\boldsymbol{x}^k \equiv (x_1^k, \ldots, x_n^k)$ in equilibrium and generates a positive amount of tax revenue.

Step 3 From the above construction, we know that the tax schedule $t^n(\cdot)$ induces the bid profile $\boldsymbol{x}^n \equiv (x_1^n, \ldots, x_n^n)$ with $\sum_{j=1}^n x_j^n \ge \sum_{j=1}^n x_j^*$ and generates a positive amount of tax revenue. Note that $\tau_1^n > \cdots > \tau_n^n > 0$ by construction.

Define $\boldsymbol{x}^{n+1} \equiv (x_1^{n+1}, \ldots, x_n^{n+1})$, where x_j^{n+1} is given by

$$x_j^{n+1} = \begin{cases} x_j^n, & j \ne n, \\ x_j^n + \epsilon, & j = n. \end{cases}$$

Further, define τ_j^{n+1}, with $j \in \{1, \ldots, n\}$, as follows:

$$\tau_j^{n+1} := v_j p_j(\boldsymbol{x}^{n+1}) \left[1 - p_j(\boldsymbol{x}^{n+1})\right] \times \frac{f'(x_j^{n+1})}{f(x_j^{n+1})} - 1.$$

By continuity, we have that $\tau_1^{n+1} > \cdots > \tau_n^{n+1} > 0$ when ϵ is sufficiently small. Next, define $t^{n+1}(\cdot)$ as

$$t^{n+1}(x) = \begin{cases} \tau_n^{n+1}, & 0 \le x \le x_n^{n+1}, \\ \tau_i^{n+1}, & x_{i+1}^{n+1} < x \le x_i^{n+1}, i \in \{1, \ldots, n-1\}, \\ \tau_1^{n+1}, & x > x_1^{n+1}. \end{cases}$$

It can be verified that $\boldsymbol{x}^{n+1}$ is the unique equilibrium under the flexible cap $t^{n+1}(\cdot)$. Further, it is obvious that the aggregate bid under $t^{n+1}(\cdot)$ is strictly higher than that under the original rigid cap. This completes the proof. ■

Proof of Theorem 2 and Corollary 1

Proof. Consider an arbitrary tax schedule $t \in \mathcal{T}$ and denote the equilibrium bid profile by $\boldsymbol{x}^* \equiv (x_1^*, \ldots, x_n^*)$. For expositional convenience, we assume that $t(\cdot)$ is a continuous function. The proof can be easily adapted to the case in which $t(\cdot)$ is discontinuous.

It is evident that $x_1^* \ge \cdots \ge x_n^* \ge 0$. Consider the following piecewise constant tax schedule $\tilde{t}(\cdot)$:

$$\tilde{t}(x) = \begin{cases} t(x_n^*), & 0 \le x \le x_n^*, \\ t(x_i^*), & x_{i+1}^* < x \le x_i^*, i \in \{1, \dots, n-1\}, \\ t(x_1^*), & x > x_1^*. \end{cases}$$

It is straightforward to verify that $\boldsymbol{x}^* \equiv (x_1^*, \dots, x_n^*)$ constitutes the unique equilibrium under $\tilde{t}(\cdot)$. That is, the aggregate bid remains unchanged when the prevailing tax schedule is switched from $t(\cdot)$ to $\tilde{t}(\cdot)$. Further, $\tilde{t}(x) \ge t(x)$ for all $x \in [0, x_1^*]$ by our construction, implying that $\sum_{i=1}^n \int_0^{x_i^*} \tilde{t}(s)ds \ge \sum_{i=1}^n \int_0^{x_i^*} t(s)ds$. As a result, $\mathcal{F}(\tilde{t}) \ge \mathcal{F}(t)$. This completes the proof of Theorem 2. Corollary 1 can be proved by the same argument. ■

Proof of Theorem 3

Proof. For expositional convenience, let us consider a continuous tax schedule $t(\cdot)$ and denote the equilibrium by $\boldsymbol{x}^* \equiv (x_1^*, \dots, x_n^*)$. Let $\tau_i = t(x_i^*)$ for $i \in \mathcal{N}$ with slight abuse of notation. Clearly, we have $x_1^* \ge \cdots \ge x_n^*$ and thus $\tau_1 \ge \cdots \ge \tau_n$. If player i remains active in the contest, i.e., $x_i^* > 0$, then the following first-order condition holds:

$$\frac{v_i}{1+\tau_i} \times \big[1 - p_i(\boldsymbol{x}^*)\big] \times \frac{f'(x_i^*)}{\sum_{j \in \mathcal{N}} f(x_j^*)} = 1.$$

Similarly, if player i remains inactive in the equilibrium, i.e., $x_i^* = 0$, then we have that

$$\frac{v_i}{1+\tau_i} \times \big[1 - p_i(\boldsymbol{x}^*)\big] \times \frac{f'(x_i^*)}{\sum_{j \in \mathcal{N}} f(x_j^*)} \le 1.$$

The above conditions imply immediately that the equilibrium bid profile $\boldsymbol{x}^* \equiv (x_1^*, \dots, x_n^*)$ under the bid cap $t(\cdot)$ also constitutes the unique equilibrium in a contest without a cap in which the profile of players' prize valuations is $\big(v_1/(1+\tau_1), \dots, v_n/(1+\tau_n)\big)$. Denote by $X(v_1, \dots, v_n)$ the equilibrium aggregate bid in a contest with no cap. Given the profile of players' prize valuations $(v_1, \dots, v_n)$, it suffices to show that

$$X(v_1, \dots, v_n) \ge X\left(\frac{v_1}{1+\tau_1}, \dots, \frac{v_n}{1+\tau_n}\right),$$

where the inequality holds with equality if and only if $\tau_i = 0$ for all $i \in \mathcal{N}$.

It is useful to prove several intermediate results.

Lemma 1 *Let $g(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly increasing and convex function. Fix $k \in \{1, \dots, n\}$. Suppose that $\{y_i'\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ satisfy the following conditions:*

(i) $y_1' \ge y_2' \ge \cdots \ge y_n' \ge 0$ *and* $y_1 \ge \cdots \ge y_n \ge 0$.

(ii) $\sum_{i=1}^n y_i' \geq \sum_{i=1}^n y_i$.

(iii) $y_i' \geq y_i$ *for* $i \leq k$ *and* $\frac{y_i'}{\sum_{j=1}^n y_j'} \leq \frac{y_i}{\sum_{j=1}^n y_j}$ *for* $i \geq k+1$.

Then, we have that

$$\sum_{i=1}^n g(y_i') \geq \sum_{i=1}^n g(y_i).$$

Proof. The lemma follows directly from Karamata's inequality if we can show that

$$\sum_{i=1}^m y_i' \geq \sum_{i=1}^m y_i, \text{ for all } m \in \{1, 2, \ldots, n\}.$$

It is evident that the above inequality holds for $m \leq k$ because $y_i' \geq y_i$ for $i \leq k$, and it remains to prove the inequality for $m \geq k+1$. First, $\frac{y_i'}{\sum_{j=1}^n y_j'} \leq \frac{y_i}{\sum_{j=1}^n y_j}$ for $i \geq k+1$ implies that

$$\frac{\sum_{i=m+1}^n y_i'}{\sum_{i=1}^n y_i'} \leq \frac{\sum_{i=m+1}^n y_i}{\sum_{i=1}^n y_i},$$

which in turn implies that

$$\frac{\sum_{i=1}^m y_i'}{\sum_{i=1}^n y_i'} \geq \frac{\sum_{i=1}^m y_i}{\sum_{i=1}^n y_i}. \tag{16}$$

Moreover, note that

$$\sum_{i=1}^n y_i' \geq \sum_{i=1}^n y_i. \tag{17}$$

Combining (16) and (17), we can obtain that $\sum_{i=1}^m y_i' \geq \sum_{i=1}^m y_i$ for $m \geq k+1$. This completes the proof. ■

Lemma 2 *Suppose that* $w_1 \geq \cdots \geq w_n > 0$ *and let* $\boldsymbol{w} := (w_1, \ldots, w_n)$*. Fix* $k \in \{1, 2, \ldots, n\}$ *and* $\tau \geq 0$*, and define* $\boldsymbol{w}' := (w_1', \ldots, w_n') = \big(w_1(1+\tau), \ldots, w_k(1+\tau), w_{k+1}, \ldots, w_n\big)$*. Then*

$$X(\boldsymbol{w}') \geq X(\boldsymbol{w}).$$

Proof. With slight abuse of notation, let us denote the equilibrium bid profile and equilibrium winning probabilities in the absence of a bid cap under the profile of prize valuations $\boldsymbol{w} \equiv (w_1, \ldots, w_n)$ by $\boldsymbol{x}^* \equiv (x_1^*, \ldots, x_n^*)$ and $\boldsymbol{p}^* \equiv (p_1^*, \ldots, p_n^*)$, respectively. Further, let $y_i^* := f(x_i^*)$ and define $S^* := \sum_{i=1}^n y_i^*$.

An active player's first-order condition can be then be rewritten as

$$\frac{w_i(1-p_i^*)}{S^*} \times f'\left(f^{-1}(S^* p_i^*)\right) = 1, \text{ if } x_i^* > 0. \tag{18}$$

Note that the left-hand side of the above equation strictly increases with w_i and strictly decreases with S^* and p_i^*. Similarly, the following condition holds for an inactive player:

$$\frac{w_i}{S^*} \times f'(0) \leq 1, \text{ if } x_i^* = 0. \tag{19}$$

Next, define a function $\mathfrak{p}(w_i, S^*)$ based on (18) and (19) as follows:

$$\mathfrak{p}(w_i, S^*) = \begin{cases} 0, & \text{if } \frac{w_i}{S^*} f'(0) \leq 1, \\ \text{unique positive solution to } \frac{w_i(1-p_i)}{S^*} f'\left(f^{-1}(S^* p_i)\right) = 1, & \text{otherwise.} \end{cases}$$

It is straightforward to verify that $\mathfrak{p}(w_i, S^*)$ is decreasing in S^* and is increasing in w_i. Moreover, S^* is the unique solution to

$$\sum_{i \in \mathcal{N}} \mathfrak{p}(w_i, S^*) = 1,$$

and $p_i^* = \mathfrak{p}(w_i, S^*)$ for all $i \in \mathcal{N}$.

Note that for an inactive player, we can obtain $\mathfrak{p}(w_i(1+\tau), S^*(1+\tau)) = \mathfrak{p}(w_i, S^*) = 0$ from (19). Moreover, for an active player, the first-order condition (18) implies that

$$\begin{aligned} & \frac{w_i\left(1 - \mathfrak{p}(w_i, S^*)\right)}{S^*} \times f'\left(f^{-1}\left(S^* \mathfrak{p}(w_i, S^*)\right)\right) \\ =& 1 \\ =& \frac{w_i(1+\tau)\left[1 - \mathfrak{p}\left(w_i(1+\tau), S^*(1+\tau)\right)\right]}{S^*(1+\tau)} \times f'\left(f^{-1}\left(S^*(1+\tau)\mathfrak{p}\left(w_i(1+\tau), S^*(1+\tau)\right)\right)\right) \\ \leq& \frac{w_i\left[1 - \mathfrak{p}\left(w_i(1+\tau), S^*(1+\tau)\right)\right]}{S^*} \times f'\left(f^{-1}\left(S^* \mathfrak{p}\left(w_i(1+\tau), S^*(1+\tau)\right)\right)\right), \end{aligned}$$

where the second equality follows from the definition of $\mathfrak{p}(\cdot, \cdot)$ and the inequality follows from Assumption 1. The above inequality in turn implies that $\mathfrak{p}\left(w_i(1+\tau), S^*(1+\tau)\right) \leq \mathfrak{p}(w_i, S^*)$. Therefore, we have that

$$\sum_{i \in \mathcal{N}} \mathfrak{p}(w_i', S^*) \geq \sum_{i \in \mathcal{N}} \mathfrak{p}(w_i, S^*) = 1 \geq \sum_{i \in \mathcal{N}} \mathfrak{p}\left(w_i', S^*(1+\tau)\right), \tag{20}$$

where the first inequality follows from the fact that $\mathfrak{p}(w_i, S^*)$ is increasing in w_i, and the second inequality follows from $\mathfrak{p}(w_i(1+\tau), S^*(1+\tau)) \leq \mathfrak{p}(w_i, S^*)$ and that $\mathfrak{p}(w_i, S^*)$ is decreasing in S^*.

Denote the equilibrium bid profile and equilibrium winning probabilities in the absence of a bid cap under the profile of prize valuations $\boldsymbol{w}' \equiv (w_1', \ldots, w_n')$ by $\boldsymbol{x}^{*\prime} \equiv (x_1^{*\prime}, \ldots, x_n^{*\prime})$

and $\boldsymbol{p}^{*\prime} \equiv (p_1^{*\prime}, \ldots, p_n^{*\prime})$, respectively. Further, let $y_i^{*\prime} := f(x_i^{*\prime})$ and define $S^{*\prime} := \sum_{i=1}^n y_i^{*\prime}$. It follows from (20) that

$$S^* \leq S^{*\prime} \leq S^*(1+\tau), \tag{21}$$

where the first inequality holds with equality if and only if $\tau = 0$.

Note that for $i \geq k+1$, we have that

$$\frac{y_i^{*\prime}}{\sum_{j=1}^n y_j^{*\prime}} = p_i^{*\prime} = \mathfrak{p}(w_i', S^{*\prime}) = \mathfrak{p}(w_i, S^{*\prime}) \leq \mathfrak{p}(w_i, S^*) = p_i^* = \frac{y_i^*}{\sum_{j=1}^n y_j^*},$$

where the inequality follows from (21) and the fact that $\mathfrak{p}(w_i, S^*)$ is decreasing in S^*.

For $i \leq k$, we must have that $y_i^{*\prime} \geq y_i^*$. To see this, note that if $p_i^{*\prime} \geq p_i^*$, then $y_i^{*\prime} = S^{*\prime} p_i^{*\prime} \geq S^* p_i^* = y_i^*$, where the inequality follows from the postulated $p_i^{*\prime} \geq p_i^*$ and (21). Otherwise, suppose $p_i^{*\prime} < p_i^*$. Then we have that

$$\begin{aligned}
\frac{w_i(1-p_i^*)}{S^*} f'\left(f^{-1}(S^* p_i^*)\right) = 1 \geq & \frac{w_i'(1-p_i^{*\prime})}{S^{*\prime}} f'(f^{-1}\left(S^{*\prime} p_i^{*\prime}\right)) \\
\geq & \frac{w_i(1+\tau)(1-p_i^*)}{S^*(1+\tau)} f'\left(f^{-1}(S^{*\prime} p_i^{*\prime})\right) \\
= & \frac{w_i(1-p_i^*)}{S^*} f'\left(f^{-1}(S^{*\prime} p_i^{*\prime})\right),
\end{aligned}$$

where the first equality follows from $p_i^* > 0$ and (18); the first inequality follows from (18) and (19); the second inequality follows from (21). Therefore, we must have that $S^{*\prime} p_i^{*\prime} \geq S^* p_i^*$ by Assumption 1, which is equivalent to $y_i^{*\prime} \geq y_i^*$.

Applying Lemma 1 by taking $g = f^{-1}$ and considering $\{y_i^{*\prime}\}_{i=1}^n$ and $\{y_i^*\}_{i=1}^n$, we can obtain that

$$X(\boldsymbol{w}') = \sum_{i \in \mathcal{N}} x_i^{*\prime} = \sum_{i=1}^n f^{-1}(y_i^{*\prime}) \geq \sum_{i=1}^n f^{-1}(y_i^*) = \sum_{i \in \mathcal{N}} x_i^* = X(\boldsymbol{w}),$$

where the inequality holds with equality if and only if $\tau = 0$. This concludes the proof. ■

Now we can prove the theorem. For notational convenience, denote $\tau_{n+1} := 0$. For each $k \in \{1, 2, \ldots, n\}$, applying Lemma 2 by taking $\tau = (\tau_k - \tau_{k+1})/(1+\tau_{k+1})$ and $\boldsymbol{w} = \big(v_1/(1+\tau_k), \ldots, v_k/(1+\tau_k), v_{k+1}/(1+\tau_{k+1}), \ldots, v_n/(1+\tau_n)\big)$, we can obtain that

$$X\left(\frac{v_1}{1+\tau_{k+1}}, \ldots, \frac{v_{k+1}}{1+\tau_{k+1}}, \frac{v_{k+2}}{1+\tau_{k+2}}, \ldots, \frac{v_n}{1+\tau_n}\right) \geq X\left(\frac{v_1}{1+\tau_k}, \ldots, \frac{v_k}{1+\tau_k}, \frac{v_{k+1}}{1+\tau_{k+1}}, \ldots, \frac{v_n}{1+\tau_n}\right).$$

Combining the above n inequalities, we can obtain that

$$X(v_1, \ldots, v_n) \geq X\left(\frac{v_1}{1+\tau_1}, \ldots, \frac{v_n}{1+\tau_n}\right).$$

Clearly, the above inequality holds with equality if and only if $\tau_1 = \cdots = \tau_{n+1} = 0$, which implies that the optimum can only be achieved by imposing no cap on the contest. This completes the proof. ■

Proof of Proposition 1

Proof. It is useful to first prove the following lemma, which enables us to reformulate the designer's optimization problem for the case of $r < 1$.

Lemma 3 *Suppose that $\lambda \in [0,1]$ and $f(x_i) = x_i^r$, with $r \in (0,1)$. The optimal piecewise constant tax schedule $\langle \boldsymbol{\tau}^\star, \overline{\boldsymbol{x}}^\star \rangle \equiv \langle (\tau_1^\star, \ldots, \tau_n^\star), (\overline{x}_1^\star, \ldots, \overline{x}_n^\star) \rangle$ can be obtained via the following indirect approach. The designer first chooses the optimal equilibrium winning probability distribution, denoted by $\boldsymbol{p}^\star \equiv (p_1^\star, \ldots, p_n^\star)$, for the maximization problem*

$$\max_{\boldsymbol{p}^* \in \mathcal{P}} (1-\lambda) r v_n (p_n^*)^{1-\frac{1}{r}} (1-p_n^*) \sum_{i=1}^{n} (p_i^*)^{\frac{1}{r}} + \lambda \sum_{i=1}^{n} \frac{i r v_i p_i^* (1-p_i^*)}{(p_i^*)^{\frac{1}{r}}} \left[(p_i^*)^{\frac{1}{r}} - (p_{i+1}^*)^{\frac{1}{r}} \right], \tag{22}$$

where

$$\mathcal{P} := \left\{ \boldsymbol{p}^* \,\middle|\, \boldsymbol{p}^* \in \triangle^{n-1},\, p_1^* \geq \cdots \geq p_n^*,\, r v_1 (p_1^*)^{1-\frac{1}{r}} (1-p_1^*) \geq \cdots \geq r v_n (p_n^*)^{1-\frac{1}{r}} (1-p_n^*) \right\}. \tag{23}$$

The optimal piecewise constant tax rates $\boldsymbol{\tau}^\star \equiv (\tau_1^\star, \ldots, \tau_n^\star)$ and caps $\overline{\boldsymbol{x}}^\star \equiv (\overline{x}_1^\star, \ldots, \overline{x}_n^\star)$ can then be recovered by

$$\tau_i^\star = \frac{r v_i p_i^\star (1-p_i^\star)}{(S^\star p_i^\star)^{\frac{1}{r}}} - 1,\ i \in \mathcal{N},$$

and

$$\overline{x}_i^\star = x_i^\star = (S^\star p_i^\star)^{\frac{1}{r}},\ i \in \mathcal{N},$$

where $(S^\star)^{\frac{1}{r}} = r v_n p_n^\star (1-p_n^\star)/(p_n^\star)^{\frac{1}{r}}$.

Proof. The proof consists of the following five steps.

Step 1: Note that all players would remain active in the equilibrium for all possible finite piecewise constant tax rates when $r < 1$. Consider a potential piecewise constant tax schedule $\langle \boldsymbol{\tau}, \overline{\boldsymbol{x}} \rangle \equiv \langle (\tau_1, \ldots, \tau_n), (\overline{x}_1, \ldots, \overline{x}_n) \rangle$. By Corollary 1, the equilibrium bidding profile $\boldsymbol{x}^*$ coincides with the profile of taxation cutoffs $\overline{\boldsymbol{x}}$, which in turn implies

$$\frac{\partial \pi_i}{\partial x_i} = \frac{r p_i (1 - p_i)}{x_i} v_i - (1 + \tau_i) = 0,\ i \in \mathcal{N}. \tag{24}$$

Step 2: By the specified contest success function (1), the equilibrium bidding profile $\boldsymbol{x}^*$ can be written as a function of the equilibrium winning probability distribution $\boldsymbol{p}^* \equiv (p_1^*, \ldots, p_n^*)$ and the aggregate output $S^* := \sum_{j \in \mathcal{N}} (x_j^*)^r$:

$$x_i^* = (S^* p_i^*)^{\frac{1}{r}}, i \in \mathcal{N}. \tag{25}$$

Combining Equations (24) and (25), we can derive the set of marginal tax rates $\boldsymbol{\tau} \equiv (\tau_1, \ldots, \tau_n)$ as

$$\tau_i = \frac{r v_i p_i^* (1 - p_i^*)}{(S^* p_i^*)^{\frac{1}{r}}} - 1,\ i \in \mathcal{N}. \tag{26}$$

Step 3: The winning probability distribution and aggregate output in equilibrium, $(\boldsymbol{p}^*, S^*)$, must satisfy the following conditions:

$$p_1^* \geq \cdots \geq p_n^*, \text{ and } \frac{r v_1 p_1^* (1 - p_1^*)}{(S^* p_1^*)^{\frac{1}{r}}} \geq \cdots \geq \frac{r v_n p_n^* (1 - p_n^*)}{(S^* p_n^*)^{\frac{1}{r}}} \geq 1, \tag{27}$$

where the first inequality follows from the observation that a stronger player must place a higher bid in equilibrium, and the second inequality follows from the first-order condition Equation (7) and the assumption of progressive taxes $\tau_1 \geq \cdots \geq \tau_n \geq 0$.

Step 4: Let $x_{n+1}^* = 0$. The amount of tax that the designer is able to collect from player i can be expressed as

$$\sum_{i=1}^{n} \int_0^{x_i^*(t)} t(s) ds = \sum_{i=1}^{n} i \tau_i (x_i^* - x_{i+1}^*).$$

Define $p_{n+1}^* = 0$ for notational convenience. The above expression, together with Equations (25) and (26) , enables us to rewrite the objective function (3) and reformulate the designer's optimization problem as follows:

$$\max_{(\boldsymbol{p}^*, S^*) \in \mathcal{P}_S} (1 - \lambda)(S^*)^{\frac{1}{r}} \sum_{i=1}^{n} (p_i^*)^{\frac{1}{r}} + \lambda \sum_{i=1}^{n} \left\{ \frac{i r v_i p_i^* (1 - p_i^*)}{(p_i^*)^{\frac{1}{r}}} \left[(p_i^*)^{\frac{1}{r}} - (p_{i+1}^*)^{\frac{1}{r}} \right] \right\}, \tag{28}$$

where the set $\mathcal{P}_S$ is defined as

$$\mathcal{P}_S := \left\{ (\boldsymbol{p}^*, S^*) \,\middle|\, \boldsymbol{p}^* \in \triangle^{n-1},\, p_1^* \geq \cdots \geq p_n^*,\, \frac{rv_1 p_1^*(1-p_1^*)}{(S^* p_1^*)^{\frac{1}{r}}} \geq \cdots \geq \frac{rv_n p_n^*(1-p_n^*)}{(S^* p_n^*)^{\frac{1}{r}}} \geq 1 \right\}.$$

Step 5: Note that $(S^*)^{\frac{1}{r}} \leq rv_n p_n^*(1-p_n^*)/(p_n^*)^{\frac{1}{r}}$ by the constraints (27). Further, fixing a winning probability distribution $\boldsymbol{p}^*$, the reformulated objective (28) strictly increases with S^* for $\lambda < 1$. Therefore, the contest objective can be maximized only if S^* is set to satisfy $(S^*)^{\frac{1}{r}} = rv_n p_n^*(1-p_n^*)/(p_n^*)^{\frac{1}{r}}$. For $\lambda = 1$, the reformulated contest objective (28) is independent of S^*. It is evident that the maximum can again be achieved by setting $(S^*)^{\frac{1}{r}} = rv_n p_n^*(1-p_n^*)/(p_n^*)^{\frac{1}{r}}$.

From the above analysis, we can further reformulate the objective function (28) to eliminate S^*. Note that the choice set $\mathcal{P}$ defined in (23) is compact and the reformulated objective (22) is continuous in $\boldsymbol{p} \in \mathcal{P}$. As a result, a maximizer of the above optimization problem exists. This completes the proof. ■

Now we can prove Proposition 1. For $r < 1$, by the argument in Step 5, we have that $(S^\star)^{\frac{1}{r}} = rv_n p_n^\star(1-p_n^\star)/(p_n^\star)^{\frac{1}{r}}$, which implies that $\tau_n^\star = 0$.

For $r = 1$, suppose that κ players remain active in the optimal contest, i.e., $p_{\kappa+1} = \cdots = p_n^* = 0$. Similar to the analysis in Lemma 3, the designer's problem can be reformulated as follows:

$$\max_{(\boldsymbol{p}^*, S^*) \in \mathcal{P}_S} (1-\lambda) S^* \sum_{i=1}^{n} p_i^* + \lambda \sum_{i=1}^{n} \left\{ irv_i(1-p_i^*) \left[(p_i^*) - (p_{i+1}^*) \right] \right\},$$

where the set $\mathcal{P}_S$ is defined as

$$\mathcal{P}_S := \left\{ (\boldsymbol{p}^*, S^*) \,\middle|\, \begin{array}{l} \boldsymbol{p}^* \in \triangle^{n-1},\, p_1^* \geq \cdots \geq p_\kappa^* \geq p_{\kappa+1}^* = \cdots = p_n^* = 0, \\ \dfrac{rv_1(1-p_1^*)}{S^*} \geq \cdots \geq \dfrac{rv_\kappa(1-p_\kappa^*)}{S^*} \geq 1 \geq \dfrac{rv_{\kappa+1}}{S^*} \end{array} \right\}.$$

It is evident that the optimum requires that $\frac{rv_\kappa(1-p_\kappa^*)}{S^*} = 1$, which implies that $\tau_\kappa^\star = 0$. This concludes the proof. ■

Appendix B: Analysis and Additional Results of Tullock Contests with $r \in (1,2]$

B.1 Preliminary Analysis

We consider a case in which the impact function is moderately convex—with $r \in (1,2]$—which alludes to a more discriminatory winner-selection mechanism than in our baseline setting. The literature has yet to characterize the equilibrium under bid caps. We first obtain the following lemma, which, in parallel to Theorem 2, greatly simplifies our search for the optimum without fully characterizing the equilibria for all eligible bid caps $t(\cdot) \in \mathcal{T}$.

Lemma 4 ***(Optimality of Two-part Tax Schedules)*** *Suppose that $n = 2$ and $r \in (1,2]$. The optimum can be achieved by a bid cap scheme $t(\cdot)$ that takes the form of*

$$t(x) = \begin{cases} 0, & x \le \overline{x}, \\ \tau, & x > \overline{x}. \end{cases} \tag{29}$$

Proof. A mixed strategy for player $i \in \{1,2\}$ is a probability measure μ_i on the interval $[0, v_i]$. For any measurable set $\mathfrak{X}_i \subseteq [0, v_i]$, $\mu_i(\mathfrak{X}_i)$ specifies the probability that the bid realization chosen by player i is contained in $\mathfrak{X}_i$. Pure strategies can be considered as degenerate probability measures. The support of player i's mixed strategy is denoted by $\text{supp}(\mu_i)$ and players' equilibrium bidding strategies under a tax schedule $t(\cdot)$ are denoted by (μ_1^*, μ_2^*). We use μ_{-i} and x_{-i} to respectively denote the strategy and bid of player i's opponent.

The proof proceeds in five steps. We first show in Lemma 6 that one player $i_H \in \{1,2\}$ would employ a pure strategy in equilibrium and the bid, which we denote by $X_{i_H}^*$, is higher than all of his opponent's possible bids, i.e., $X_{i_H}^* \geq \max_{x_{-i_H} \in \text{supp}(\mu_{-i_H}^*)} \{x_{-i_H}\} =: X_{-i_H}^*$. Second, we show in Lemma 7 that player i_H is the stronger player, i.e., $i_H = 1$. Third, we construct a tax schedule $\check{t}(\cdot)$ in Lemma 8 that induces a pure-strategy equilibrium $(x_1^*, x_2^*) = (X_1^*, X_2^*)$ and generates more aggregate bid and tax revenue than the equilibrium (μ_1^*, μ_2^*) under the tax schedule $t(\cdot)$. Fourth, we construct a two-part tax schedule $\check{t}(\cdot)$ to replicate the equilibrium bid profile and tax revenue under $\check{t}(\cdot)$. Last, we construct a two-part tax schedule with zero marginal tax rate for the minimum tax bracket to further improve the contest performance.

Step 1 We show that at least one player uses a pure strategy in equilibrium. Recall that bidding $x_i > v_i$ is strictly dominated by bidding $x_i = 0$ and thus $\operatorname{supp}(\mu_i^*) \subseteq [0, v_i]$. Let

$$i_H := \arg\max_{i\in\{1,2\}} \max_{x_i\in\operatorname{supp}(\mu_i^*)} \{x_i\}.$$

In the case of $\max_{x_1\in\operatorname{supp}(\mu_1^*)}\{x_1\} = \max_{x_2\in\operatorname{supp}(\mu_2^*)}\{x_2\}$, we set $i_H = 1$. Next, define $X_i^* := \max_{x_i\in\operatorname{supp}(\mu_i^*)}\{x_i\}$.

Because $X_{i_H}^* \in \operatorname{supp}(\mu_{i_H}^*)$, we can obtain the following first-order condition:

$$1 + \lim_{x\nearrow X_{i_H}^*} t(x) \le \mathbb{E}_{\mu_{-i_H}^*}\left[\frac{rv_{i_H}(X_{i_H}^*)^{r-1}(x_{-i_H})^r}{\left[(X_{i_H}^*)^r + (x_{-i_H})^r\right]^2}\right] \le 1 + \lim_{x\searrow X_{i_H}^*} t(x),$$

where we use the notation $\mathbb{E}_{\mu_{-i_H}^*}[\cdot]$ to denote the expectation under $\mu_{-i_H}^*$. Let

$$\tau_{i_H} := \mathbb{E}_{\mu_{-i_H}^*}\left[\frac{rv_{i_H}(X_{i_H}^*)^{r-1}(x_{-i_H})^r}{\left[(X_{i_H}^*)^r + (x_{-i_H})^r\right]^2}\right] - 1, \tag{30}$$

and consider the following auxiliary function

$$\psi(x) := \mathbb{E}_{\mu_{-i_H}^*}\left[\frac{v_{i_H}(x)^r}{(x)^r + (x_{-i_H})^r}\right] - (1+\tau_{i_H})x. \tag{31}$$

In words, $\psi(x)$ is player i_H's expected payoff given his opponent's strategy $\mu_{-i_H}^*$ and that the marginal tax rate is constant and is equal to τ_{i_H}. By construction, we have that $\psi'(X_{i_H}^*) = 0$.

Lemma 5 *$\psi(X_{i_H}^*) > \psi(x)$ for all $x \in (0, X_{i_H}^*)$ and $\psi(X_{i_H}^*) \ge \psi(0)$.*

Proof. We first show that $\psi(X_{i_H}^*) \ge \psi(0)$. In fact, we have that

$$\begin{aligned}
\psi(0) = 0 = X_{i_H}^*\psi'(X_{i_H}^*)& \\
&= \mathbb{E}_{\mu_{-i_H}^*}\left[\frac{rv_{i_H}(X_{i_H}^*)^r(x_{-i_H})^r}{\left[(X_{i_H}^*)^r + (x_{-i_H})^r\right]^2}\right] - (1+\tau_{i_H})X_{i_H}^* \\
&\le rv_{i_H} \times \mathbb{E}_{\mu_{-i_H}^*}\left[\frac{(X_{i_H}^*)^r}{(X_{i_H}^*)^r + (x_{-i_H})^r}\right] \times \mathbb{E}_{\mu_{-i_H}^*}\left[1 - \frac{(X_{i_H}^*)^r}{(X_{i_H}^*)^r + (x_{-i_H})^r}\right] - (1+\tau_{i_H})X_{i_H}^* \\
&\le \frac{rv_{i_H}}{2} \times \mathbb{E}_{\mu_{-i_H}^*}\left[\frac{(X_{i_H}^*)^r}{(X_{i_H}^*)^r + (x_{-i_H})^r}\right] - (1+\tau_{i_H})X_{i_H}^* \\
&\le v_{i_H} \times \mathbb{E}_{\mu_{-i_H}^*}\left[\frac{(X_{i_H}^*)^r}{(X_{i_H}^*)^r + (x_{-i_H})^r}\right] - (1+\tau_{i_H})X_{i_H}^*
\end{aligned}$$

$$= \psi(X^*_{i_H}),$$

where the second equality follows from $\psi'(X^*_{i_H}) = 0$; the first inequality follows from the strict concavity of the function $z(1-z)$; the second inequality follows from the fact that $X^*_{i_H} \geq x_{-i_H}$ for all $x_{-i_H} \in \text{supp}\,(\mu^*_{-i_H})$; and the third inequality follows from $r \leq 2$. Moreover, it is straightforward to see that $\psi(0) = \psi(X^*_{i_H})$, i.e., all of the above equalities hold with equality, if and only if (i) player $-i_H$ uses a pure strategy; (ii) $X^*_{i_H} = \max_{x_{-i_H} \in \text{supp}\,(\mu^*_{-i_H})}\{x_{-i_H}\} \equiv X^*_{-i_H}$; and (iii) $r = 2$.

To prove the lemma, it suffices to show that $\psi(X^*_{i_H}) > \psi(x)$ for all $x \in (0, X^*_{i_H})$ such that $\psi'(x) = 0$. It follows from (30) that

$$v_{i_H} \times \mathbb{E}_{\mu^*_{-i_H}}\left[\frac{r(X^*_{i_H})^r(x_{-i_H})^r}{\left[(X^*_{i_H})^r + (x_{-i_H})^r\right]^2}\right] = (1+\tau_{i_H})X^*_{i_H}.$$

The above equation, together with (31), implies that

$$\begin{aligned}
\psi(X^*_{i_H}) &= v_{i_H} \times \mathbb{E}_{\mu^*_{-i_H}}\left[\frac{(X^*_{i_H})^r}{(X^*_{i_H})^r + (x_{-i_H})^r}\right] - (1+\tau_{i_H})X^*_{i_H} \\
&= v_{i_H} \times \mathbb{E}_{\mu^*_{-i_H}}\left[\frac{(X^*_{i_H})^r}{(X^*_{i_H})^r + (x_{-i_H})^r}\right] - \mathbb{E}_{\mu^*_{-i_H}}\left[\frac{rv_{i_H}(X^*_{i_H})^r(x_{-i_H})^r}{\left[(X^*_{i_H})^r + (x_{-i_H})^r\right]^2}\right] \\
&= v_{i_H} \times \mathbb{E}_{\mu^*_{-i_H}}\left[p\left(X^*_{i_H}; x_{-i_H}\right) \times \left\{1 - r\left[1 - p\left(X^*_{i_H}; x_{-i_H}\right)\right]\right\}\right], \qquad (32)
\end{aligned}$$

where $p(\cdot;\cdot)$ is defined as

$$p\left(x; x_{-i_H}\right) := \frac{(x)^r}{(x)^r + (x_{-i_H})^r}. \qquad (33)$$

Analogously, it follows from $\psi'(x) = 0$ that

$$\psi(x) = v_{i_H} \times \mathbb{E}_{\mu^*_{-i_H}}\left[p\left(x; x_{-i_H}\right) \times \left\{1 - r\left[1 - p\left(x; x_{-i_H}\right)\right]\right\}\right]. \qquad (34)$$

It is evident that $p(X^*_{i_H}; x_{-i_H}) \geq \frac{1}{2}$ for $x_{-i_H} \in \text{supp}\,(\mu^*_{-i_H})$ and $p(X^*_{i_H}; x_{-i_H}) > p(x; x_{-i_H})$ for $x < X^*_{i_H}$. Together with (32) and (34), we can obtain that

$$\psi(X^*_{i_H}) = v_{i_H} \times \mathbb{E}_{\mu^*_{-i_H}}\left[p\left(X^*_{i_H}; x_{-i_H}\right) \times \left\{1 - r\left[1 - p\left(X^*_{i_H}; x_{-i_H}\right)\right]\right\}\right]$$

$$
\begin{aligned}
&> v_{i_H} \times \mathbb{E}_{\mu^*_{-i_H}} \left[p(x; x_{-i_H}) \times \left\{ 1 - r\left[1 - p(x; x_{-i_H})\right] \right\} \right] \\
&= \psi(x), \text{ for all } x \in \left(0, X^*_{i_H}\right),
\end{aligned}
$$

where the inequality follows from the fact that $p[1-r(1-p)] > p'[1-r(1-p')]$ for $r \in (1,2]$ if $p > p'$ and $p \geq \frac{1}{2}$. This concludes the proof. ■

Lemma 6 *Player i_H would use a pure strategy in equilibrium, i.e., supp $(\mu^*_{i_H}) = \{X^*_{i_H}\}$. Moreover, $X^*_{i_H} \geq X^*_{-i_H}$.*

Proof. Recall that $X^*_{i_H}$ is player i_H's maximum bid in equilibrium, and thus it suffices to show that player i_H will not place a bid that is strictly below $X^*_{i_H}$. Denote player i's expected payoff of placing a bid x_i under $t(\cdot)$ given his opponent's strategy μ_{-i} by $\Pi_i(x_i; \mu_{-i})$. Then we can obtain that

$$
\begin{aligned}
&\Pi_{i_H}\left(X^*_{i_H}; \mu^*_{-i_H}\right) - \Pi_{i_H}\left(x; \mu^*_{-i_H}\right) \\
=& v_{i_H} \times \mathbb{E}_{\mu^*_{-i_H}} \left[\frac{(X^*_{i_H})^r}{(X^*_{i_H})^r + (x_{-i_H})^r} \right] - v_{i_H} \times \mathbb{E}_{\mu^*_{-i_H}} \left[\frac{(x)^r}{(x)^r + (x_{-i_H})^r} \right] - \int_x^{X^*_{i_H}} \left[1 + t(s)\right] ds \\
\geq& v_{i_H} \times \mathbb{E}_{\mu^*_{-i_H}} \left[\frac{(X^*_{i_H})^r}{(X^*_{i_H})^r + (x_{-i_H})^r} \right] - v_{i_H} \times \mathbb{E}_{\mu^*_{-i_H}} \left[\frac{(x)^r}{(x)^r + (x_{-i_H})^r} \right] - (X^*_{i_H} - x)(1 + \tau_{i_H}) \\
=& \psi(X^*_{i_H}) - \psi(x) \geq 0,
\end{aligned}
$$

where the first inequality follows from the fact that $t(s) \leq \lim_{x \nearrow X^*_{i_H}} t(x) \leq \tau_{i_H}$ for all $s \in (x, X^*_{i_H})$ and the second inequality follows from Lemma 5.

From the above analysis, we know that player i_H can only randomize between 0 and $X^*_{i_H}$ in the equilibrium. If all of the above equalities hold with equality, then $\psi(X^*_{i_H}) = \psi(0)$, which implies that player $-i_H$ must use a pure strategy and $X^*_{i_H} = X^*_{-i_H}$ from the proof of Lemma 5. In that case, we can simply relabel player $-i_H$ as i_H. This concludes the proof. ■

Step 2 We show that the stronger player always uses a pure strategy in equilibrium.

Lemma 7 *Player 1 is the one that uses a pure strategy, i.e., $i_H = 1$.*

Proof. Recall by definition that $X^*_{i_H} \geq \max_{x_{-i_H} \in \text{supp}\,(\mu^*_{-i_H})}\{x_{-i_H}\} \equiv X^*_{-i_H}$. We consider the following two cases:

Case I: $X^*_{i_H} > X^*_{-i_H}$. It follows immediately from (33) that

$$p\left(X^*_{i_H}; x_{-i_H}\right) \geq p\left(X^*_{i_H}; X^*_{-i_H}\right) > \frac{1}{2}, \text{ for all } x_{-i_H} \in \text{supp}\,(\mu^*_{-i_H}). \tag{35}$$

Moreover, we can obtain that

$$\begin{aligned}
v_{i_H} =& \frac{(1+\tau_{i_H})X^*_{i_H}}{r} \times \left\{\mathbb{E}_{\mu^*_{-i_H}}\left[\frac{(X^*_{i_H})^r (x_{-i_H})^r}{\left[(X^*_{i_H})^r + (x_{-i_H})^r\right]^2}\right]\right\}^{-1} \\
=& \frac{(1+\tau_{i_H})X^*_{i_H}}{r} \times \left\{\mathbb{E}_{\mu^*_{-i_H}}\left[p\left(X^*_{i_H}; x_{-i_H}\right)\left[1 - p\left(X^*_{i_H}; x_{-i_H}\right)\right]\right]\right\}^{-1} \\
\geq& \frac{(1+\tau_{i_H})X^*_{i_H}}{r} \times \left\{p\left(X^*_{i_H}; X^*_{-i_H}\right)\left[1 - p\left(X^*_{i_H}; X^*_{-i_H}\right)\right]\right\}^{-1} \\
>& \frac{\left[1 + \lim_{x \searrow X^*_{-i_H}} t(x)\right] X^*_{-i_H}}{r} \times \left\{p\left(X^*_{i_H}; X^*_{-i_H}\right)\left[1 - p\left(X^*_{i_H}; X^*_{-i_H}\right)\right]\right\}^{-1} \\
\geq& v_{-i_H},
\end{aligned} \tag{36}$$

where the first equality follows from the rearrangement of the first-order condition (30); the first inequality follows from (35) and the fact that $p(1-p)$ decreases with p if $p > 1/2$; the second inequality follows from the fact that $X^*_{i_H} > X^*_{-i_H}$; and the last inequality follows from the first-order condition of player $-i_H$. Therefore, we must have that $v_{i_H} > v_{-i_H}$, which implies that $i_H = 1$.

Case II: $X^*_{i_H} = X^*_{-i_H}$. From the proof of Lemmata 5 and 6, we know that player $-i_H$ either chooses a pure strategy or randomizes between 0 and $X^*_{-i_H}$. If player $-i_H$ chooses a pure strategy, then both players choose pure strategies and thus player 1 uses a pure strategy. Otherwise, it must be the case that player $-i_H$ is indifferent between bidding 0—which generates an expected payoff of 0 regardless of the tax schedule—and bidding $X^*_{-i_H}$. As a result, we can obtain that

$$\begin{aligned}
0 &= v_{-i_H} \times p\left(X^*_{-i_H}; X^*_{i_H}\right) - \int_0^{X^*_{-i_H}} \left[1 + t(s)\right] ds \\
&\geq v_{-i_H} \times p\left(X^*_{-i_H}; X^*_{i_H}\right) - \left[1 + \lim_{x \nearrow X^*_{-i_H}} t(x)\right] X^*_{-i_H},
\end{aligned}$$

which in turn implies that

$$v_{-i_H} \leq \left[p\left(X^*_{-i_H}; X^*_{i_H}\right)\right]^{-1} \times \left[1 + \lim_{x \nearrow X^*_{-i_H}} t(x)\right] X^*_{-i_H} = 2\left[1 + \lim_{x \nearrow X^*_{-i_H}} t(x)\right] X^*_{-i_H}, \tag{37}$$

where the last equality follows from the postulated $X^*_{i_H} = X^*_{-i_H}$.

Therefore, we have that

$$\begin{aligned} v_{i_H} &= \frac{(1+\tau_{i_H})X^*_{i_H}}{r} \times \left\{\mathbb{E}_{\mu^*_{-i_H}}\left[p\left(X^*_{i_H}; x_{-i_H}\right)\left[1 - p\left(X^*_{i_H}; x_{-i_H}\right)\right]\right]\right\}^{-1} \\ &\geq \frac{4(1+\tau_{i_H})X^*_{i_H}}{r} \\ &\geq 2\left[1 + \lim_{x \nearrow X^*_{-i_H}} t(x)\right] X^*_{-i_H} \geq v_{-i_H}, \end{aligned}$$

where the first equality follows from (36); the second inequality follows from $r \leq 2$, $\lim_{x \nearrow X^*_{-i_H}} t(x) \leq \tau_{i_H}$, and the postulated $X^*_{i_H} = X^*_{-i_H}$; and the third inequality follows from (37). Therefore, we have that $i_H = 1$. This completes the proof. ■

Step 3 From Steps 1 and 2, fixing a tax schedule $t(\cdot)$, we know that player 1 would employ a pure strategy X^*_1 and $X^*_1 \geq X^*_2$ in an equilibrium (μ^*_1, μ^*_2). Consider the following tax schedule

$$\check{t}(x) := \begin{cases} t(x), & x < X^*_1, \\ \frac{rv_1}{X^*_1} \times p\left(X^*_1; X^*_2\right)\left[1 - p\left(X^*_1; X^*_2\right)\right] - 1, & x \geq X^*_1. \end{cases} \tag{38}$$

The following result can then be obtained.

Lemma 8 *Consider the tax schedule $\check{t}(x)$ specified in* (38). *There exists a pure-strategy equilibrium in which player* 1 *bids X^*_1 and player* 2 *bids X^*_2.*

Proof. Clearly, bidding X^*_2 is optimal to player 2 under the constructed tax schedule $\check{t}(\cdot)$ given that player 1 bids X^*_1 with certainty because $\check{t}(x) = t(x)$ for $x < X^*_1$, and it thus remains to examine player 1's incentive.

By the same argument used in Lemma 6, we can show that $\pi_1(x_1, X^*_2) \leq \pi_1(X^*_1, X^*_2)$ for all $x_1 \in [0, X^*_1)$. Moreover, fixing $x_2 = X^*_2$, the first derivative of player 1's expected payoff with respect to his bid x_1 is given by

$$\lim_{x_1 \nearrow X^*_1} \frac{\partial \pi_1(x_1, X^*_2)}{\partial x_1} = \frac{rv_1}{X^*_1} \times p\left(X^*_1; X^*_2\right)\left[1 - p\left(X^*_1; X^*_2\right)\right] - \left[1 + \lim_{x_1 \nearrow X^*_1} t(x_1)\right]$$

$$\geq \frac{rv_1}{X_1^*} \times \mathbb{E}_{\mu_2^*}\Big[p\left(X_1^*; x_2\right)\left[1 - p\left(X_1^*; x_2\right)\right]\Big] - 1 - \tau_1 = 0,$$

where the first equality follows from (36); the inequality follows from the fact that $p\left(X_1^*; x_2\right)$ is strictly decreasing in x_2 and is greater than $1/2$ for all $x_2 \in \operatorname{supp}\left(\mu_2^*\right)$; and the last equality follows from (31). Therefore, player 1 has no strict incentive to place a bid that is strictly less than X_1^* and it remains to show that bidding $x_1 > X_1^*$ is suboptimal to player 1.

Note that player 1's expected payoff is twice differentiable for $x_1 > X_1^*$. Carrying out the algebra, we have that

$$\begin{aligned}
\frac{\partial^2 \pi_1(x_1, X_2^*)}{\partial x_1^2} &= \frac{rv_1(x_1)^{r-2}(X_2^*)^r}{\big[(x_1)^r + (X_2^*)^r\big]^3} \times \left[(r-1)(X_2^*)^r - (r+1)(x_1)^r\right] \\
&< \frac{rv_1(x_1)^{r-2}(X_2^*)^r}{\big[(x_1)^r + (X_2^*)^r\big]^3} \times (r-1) \times \left[(X_2^*)^r - (x_1)^r\right] < 0, \text{ for all } x_1 > X_1^*.
\end{aligned}$$

Note that $\lim_{x_1 \searrow X_1^*} \frac{\partial \pi_1(x_1, X_2^*)}{\partial x_1} = 0$ by our construction of the tax schedule $\check{t}(\cdot)$. Together with the above condition, we know that that $\pi_1(x_1, X_2^*)$ is strictly decreasing in x_1 for $x_1 \in (X_1^*, +\infty)$. Therefore, bidding X_1^* is optimal to player 1 under the tax schedule $\check{t}(\cdot)$ given that player 2 bids X_2^*. This concludes the proof. ■

Step 4 Next, we construct a two-part tax schedule $\breve{t}(\cdot)$ to induce the bid profile (X_1^*, X_2^*). Specifically, let $\breve{\tau}_2 := \big[\int_0^{X_2^*} \check{t}(s)ds\big]/X_2^*$ and $\breve{\tau}_1 := \check{t}(X_1^*)$ and consider the following two-part tax schedule:

$$\breve{t}(x) := \begin{cases} \breve{\tau}_2, & x \leq X_2^*, \\ \breve{\tau}_1, & x > X_2^*. \end{cases}$$

It is straightforward to verify that

$$\int_0^x \Big[1 + \breve{t}(s)\Big]\, ds \geq \int_0^x \left[1 + \check{t}(s)\right] ds, \text{ for all } x \in \left[0, X_1^*\right],$$

where the inequality holds with equality at $x = X_2^*$. Therefore, given that player 1's bid is X_1^*, player 2's expected payoff under $\breve{t}(\cdot)$ is weakly less than that under $\check{t}(\cdot)$; and his expected payoff at (X_1^*, X_2^*) under $\breve{t}(\cdot)$ coincides with that under $\check{t}(\cdot)$. Therefore, player 2 has no strict incentive to deviate from X_2^* when player 1 chooses $x_1 = X_1^*$.

For player 1, note that the first-order condition (38) is satisfied at $x_1 = X_1^*$ under $\breve{t}(\cdot)$. By the same argument used in the proof of Lemma 6, we can show that player 1 has no strict incentive to choose a bid below X_1^*. Moreover, player 1's equilibrium behavior is governed by the first-order condition and his marginal cost remains unchanged when we switch from $\check{t}(\cdot)$ to $\breve{t}(\cdot)$, implying that player 1 has no strict incentive to place a bid that is greater than

X_1^*. Therefore, (X_1^*, X_2^*) also constitutes an equilibrium under the constructed two-part tax schedule $\breve{t}(\cdot)$.

Step 5 Last, we construct a two-part tax schedule $\tilde{t}(\cdot)$ with zero marginal tax rate for the minimum bracket that outperforms $\breve{t}(\cdot)$. Specifically, let

$$\tilde{t}(x) = \begin{cases} 0, & x \leq (1+\breve{\tau}_2)X_2^*, \\ (\breve{\tau}_1 - \breve{\tau}_2)/(1+\breve{\tau}_2), & x > (1+\breve{\tau}_2)X_2^*. \end{cases}$$

It can be verified that the bid profile $\big((1+\breve{\tau}_2)X_1^*, (1+\breve{\tau}_2)X_2^*\big)$ constitutes an equilibrium under $\tilde{t}(\cdot)$, which generates a larger aggregate bid amount than that under $\breve{t}(\cdot)$. Further, the equilibrium aggregate bidding cost under $\tilde{t}(\cdot)$ is $(1+\breve{\tau}_2)(X_1^*+X_2^*)+(\breve{\tau}_1-\breve{\tau}_2)(X_1^*-X_2^*)$, which is the same as that under $\breve{t}(\cdot)$. Note that the designer's objective function can be rewritten as a convex combination between the aggregate bid and the aggregate bidding cost by (9). Therefore, the tax schedule $\tilde{t}(\cdot)$ generates a higher payoff to the designer than $\breve{t}(\cdot)$ does. This completes the proof. ■

By Lemma 4, a candidate optimal bid cap is fully defined by a tuple $(\tau, \overline{x})$. We are now ready to characterize the equilibria under an arbitrary $(\tau, \overline{x})$.

Lemma 9 ***(Equilibrium in Two-player Contests with Bid Caps)*** *Consider a two-player Tullock contest with $r \in (1, 2]$. Fix an arbitrary two-part tax schedule $t(\cdot)$ in the form of (29), as defined in Lemma 4. Whenever an equilibrium exists, it must take one of the following forms.*

(i) A pure-strategy equilibrium with a unique bid profile (x_1^*, x_2^*).

(ii) A semi-mixed equilibrium in which player 1 chooses x_1^* and player 2 randomizes between 0 and x_2^*. In case multiple equilibria exist, they share the same support and only differ in the probabilities of player 2's remaining inactive.

(iii) A semi-mixed equilibrium in which player 1 chooses x_1^* and player 2 randomizes between 0, x_{2L}^*, and x_{2H}^*, with $0 < x_{2L}^* < x_{2H}^*$. In case multiple equilibria exist, they share the same support and only differ in the probabilities of player 2's choosing each bid level.

Proof. Consider an arbitrary equilibrium (μ_1^*, μ_2^*). By Lemmata 6 and 7, player 1 always employs a pure strategy in equilibrium. A closer look at player 2's expected payoff enables us to conclude that player 2 can randomize between zero and at most two positive bids. More formally, the following lemma can be established.

Lemma 10 *Suppose that* $r \in (1, 2]$ *and consider a two-part tax schedule in the form of* (29)*. Fixing player* 1*'s bid* $x_1 > 0$*, player* 2*'s expected payoff*

$$\pi_2(x_2) = \frac{v_2(x_2)^r}{(x_1)^r + (x_2)^r} - \int_0^{x_2} \left[1 + t(s)\right] ds$$

has at most two positive local maxima.

Proof. Note that $\pi_2(\cdot)$ is differentiable on $(0, \overline{x})$ and $(\overline{x}, +\infty)$. Carrying out the algebra, we have that

$$\pi_2'(x_2) = \begin{cases} \frac{rv_2(x_1)^r(x_2)^{r-1}}{\left[(x_1)^r+(x_2)^r\right]^2} - 1, & x_2 < \overline{x}, \\ \frac{rv_2(x_1)^r(x_2)^{r-1}}{\left[(x_1)^r+(x_2)^r\right]^2} - 1 - \tau, & x_2 > \overline{x}. \end{cases}$$

It can be verified that

$$h(x_2) := \frac{rv_2(x_1)^r(x_2)^{r-1}}{\left[(x_1)^r + (x_2)^r\right]^2}$$

increases with x_2 on $\big(0, [(r-1)/(r+1)]^{1/r}x_1\big)$ and decreases with x_2 on $\big([(r-1)/(r+1)]^{1/r}x_1, +\infty\big)$, which implies that there exist at most three solutions to $\pi_2'(x_2) = 0$. Therefore, $\pi_2(x_2)$ has at most 2 positive local maxima. This concludes the proof. ■

The next lemma, which is essentially the same as Lemma A.1 in Ewerhart (2017), demonstrates that all equilibria must share the same support. The proof is a straightforward adaptation of an argument detailed in Klumpp and Polborn (2006), and is presented below with the notations in our paper.

Lemma 11 *Consider a two-player contest with an arbitrary tax schedule* $t(\cdot)$*. Suppose that* (μ_1^*, μ_2^*) *and* (μ_1^{**}, μ_2^{**}) *are (mixed-strategy) equilibria of the contest game. Then* (μ_1^*, μ_2^{**}) *and* (μ_1^{**}, μ_2^*) *are equilibria as well.*

Proof. Recall that $p_i(x_1, x_2)$ is player i's winning probability given the bid profile (x_1, x_2). For notational convenience, fixing a mixed-strategy (μ_1, μ_2), denote player i's ex ante winning probability by $\mathbb{E}\left[p_i(x_1, x_2)|\mu_1, \mu_2\right]$ and his expected cost by $\mathbb{E}\left[c(x_i)|\mu_i\right]$.

Because (μ_1^*, μ_2^*) and (μ_1^{**}, μ_2^{**}) are equilibria, we have that

$$v_1 \times \mathbb{E}\left[p_1(x_1, x_2)|\mu_1^*, \mu_2^*\right] - \mathbb{E}\left[c(x_1)|\mu_1^*\right] \geq v_1 \times \mathbb{E}\left[p_1(x_1, x_2)|\mu_1^{**}, \mu_2^*\right] - \mathbb{E}\left[c(x_1)|\mu_1^{**}\right], \tag{39}$$

$$v_1 \times \mathbb{E}\left[p_1(x_1, x_2)|\mu_1^{**}, \mu_2^{**}\right] - \mathbb{E}\left[c(x_1)|\mu_1^{**}\right] \geq v_1 \times \mathbb{E}\left[p_1(x_1, x_2)|\mu_1^*, \mu_2^{**}\right] - \mathbb{E}\left[c(x_1)|\mu_1^*\right], \tag{40}$$

$$v_2 \times \mathbb{E}\left[p_2(x_1, x_2)|\mu_1^*, \mu_2^*\right] - \mathbb{E}\left[c(x_2)|\mu_2^*\right] \geq v_2 \times \mathbb{E}\left[p_2(x_1, x_2)|\mu_1^*, \mu_2^{**}\right] - \mathbb{E}\left[c(x_2)|\mu_2^{**}\right], \tag{41}$$

$$v_2 \times \mathbb{E}\left[p_2(x_1,x_2)|\mu_1^{**},\mu_2^{**}\right] - \mathbb{E}\left[c(x_2)|\mu_2^{**}\right] \geq v_2 \times \mathbb{E}\left[p_2(x_1,x_2)|\mu_1^{**},\mu_2^{*}\right] - \mathbb{E}\left[c(x_2)|\mu_2^{*}\right]. \tag{42}$$

Summing up (39) and (40), we can obtain that

$$\mathbb{E}\left[p_1(x_1,x_2)|\mu_1^{*},\mu_2^{*}\right] + \mathbb{E}\left[p_1(x_1,x_2)|\mu_1^{**},\mu_2^{**}\right] \geq \mathbb{E}\left[p_1(x_1,x_2)|\mu_1^{**},\mu_2^{*}\right] + \mathbb{E}\left[p_1(x_1,x_2)|\mu_1^{*},\mu_2^{**}\right]. \tag{43}$$

Similarly, summing up (41) and (42), we can obtain that

$$\mathbb{E}\left[p_2(x_1,x_2)|\mu_1^{*},\mu_2^{*}\right] + \mathbb{E}\left[p_2(x_1,x_2)|\mu_1^{**},\mu_2^{**}\right] \geq \mathbb{E}\left[p_2(x_1,x_2)|\mu_1^{*},\mu_2^{**}\right] + \mathbb{E}\left[p_2(x_1,x_2)|\mu_1^{**},\mu_2^{*}\right]. \tag{44}$$

Note that

$$\mathbb{E}\left[p_1(x_1,x_2)|\mu_1^{*},\mu_2^{*}\right] + \mathbb{E}\left[p_1(x_1,x_2)|\mu_1^{**},\mu_2^{**}\right] + \mathbb{E}\left[p_2(x_1,x_2)|\mu_1^{*},\mu_2^{*}\right] + \mathbb{E}\left[p_2(x_1,x_2)|\mu_1^{**},\mu_2^{**}\right] = 2,$$
$$\mathbb{E}\left[p_1(x_1,x_2)|\mu_1^{**},\mu_2^{*}\right] + \mathbb{E}\left[p_1(x_1,x_2)|\mu_1^{*},\mu_2^{**}\right] + \mathbb{E}\left[p_2(x_1,x_2)|\mu_1^{*},\mu_2^{**}\right] + \mathbb{E}\left[p_2(x_1,x_2)|\mu_1^{**},\mu_2^{*}\right] = 2.$$

Therefore, the inequalities in (43) and (44) must hold with equality, which in turn implies that the inequalities in (39)-(42) must hold with equality as well. Therefore, (μ_1^*, μ_2^{**}) and (μ_1^{**}, μ_2^*) are equilibria of the contest game. ■

By Lemma 11, we can conclude that (i) the equilibrium is unique if it is in pure strategy; (ii) all equilibria share the same support if there exist multiple semi-mixed equilibria. This completes the proof. ■

Lemma 9 shows that one of the three cases would emerge in the equilibrium when a two-part bid cap $(\tau, \overline{x})$ is imposed. As previously mentioned, the equilibrium in a two-player contest without bid caps is unique for $r \in (1, 2]$. Bid caps alter equilibrium plays substantially and could cause multiple equilibria. In what follows, we assume that the equilibrium most favorable to the designer is selected whenever multiple equilibria exist. It is noteworthy that equilibrium selection is not essential for contest design because the optimum can always be achieved by a flexible tax schedule that leads to a unique equilibrium, as pointed out in Footnotes 21 and 22 in the main text.

B.2 Proof of Proposition 2

Proof. By Lemma 4, the optimum can be achieved via a two-part bid cap in the form of

$$t^\star(x) = \begin{cases} 0, & x \leq \overline{x}^\star, \\ \tau^\star, & x > \overline{x}^\star. \end{cases}$$

Note that Step 3 in the proof of Lemma 4 suggests that the selected equilibrium in the optimum must be in pure strategy, which we denote by $(x_1^\star, x_2^\star)$. Moreover, Steps 4 and 5 in the proof of Lemma 4 imply that $\overline{x}^\star = x_2^\star$.

It is useful to first prove an intermediate result.

Lemma 12 *Suppose that $n = 2$, $r \in (1, 2]$, and no cap is suboptimal. Then $x_1^\star > x_2^\star$.*

Proof. Note that $x_1^\star \geq x_2^\star$ by Lemmata 6 and 7. Suppose to the contrary that $x_1^\star = x_2^\star$. The first-order condition for player 2 implies that

$$rv_2 p_1(x_1^\star, x_2^\star) p_2(x_1^\star, x_2^\star) \geq x_2^\star. \tag{45}$$

We consider the following two cases:

Case I The inequality in (45) is strict. Then the designer can further increase the contest performance by increasing the threshold $\overline{x}^\star$. Specifically, consider the following tax schedule:

$$t_\epsilon^\star(x) = \begin{cases} 0, & x \leq \overline{x}^\star + \epsilon, \\ \tau^\star, & x > \overline{x}^\star + \epsilon. \end{cases} \tag{46}$$

It is can be verified that the bid profile $(x_1^\star + \epsilon, x_2^\star + \epsilon)$ constitutes an equilibrium under $t_\epsilon^\star(\cdot)$ for a sufficiently small $\epsilon > 0$, which yields a higher aggregate bid than that under $t^\star(\cdot)$. Note that the tax revenue under $t_\epsilon^\star(\cdot)$ is zero, which is the same as that under $t^\star(\cdot)$. Therefore, the contest performance is higher under $t_\epsilon^\star(\cdot)$, which contradicts the optimality of $t^\star(\cdot)$.

Case II The inequality in (45) holds with equality. In this case, we must have $v_1 > v_2$. Otherwise, suppose that $v_1 = v_2$. Then we have that

$$rv_1 p_1(x_1^\star, x_2^\star) p_2(x_1^\star, x_2^\star) = rv_2 p_1(x_1^\star, x_2^\star) p_2(x_1^\star, x_2^\star) = x_2^\star = x_1^\star,$$

and thus the optimal tax schedule $t^\star(\cdot)$ is equivalent to having no cap. A contradiction.

From the above analysis, we have that

$$rv_1 p_1(x_1^\star, x_2^\star) p_2(x_1^\star, x_2^\star) > x_1^\star,$$

and

$$rv_2 p_1(x_1^\star, x_2^\star) p_2(x_1^\star, x_2^\star) = x_2^\star.$$

Similar to the analysis in Case I, we consider the tax schedule $t_\epsilon^\star(\cdot)$ specified in (46) and denote the equilibrium by $\big(x_1^*(\epsilon), x_2^*(\epsilon)\big)$. It is straightforward to verify that $x_1^*(\epsilon) = \overline{x}^\star + \epsilon$

and $x_2^*(\epsilon)$ is governed by the following first-order condition:

$$rv_2 p_1\left(x_1^*(\epsilon), x_2^*(\epsilon)\right)\left[1 - p_1\left(x_1^*(\epsilon), x_2^*(\epsilon)\right)\right] = x_2^*(\epsilon).$$

Note that $p_1\left(x_1^*(0), x_2^*(0)\right) = \frac{1}{2}$. Therefore, we have that

$$\left.\frac{dx_2^*(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = 0.$$

Moreover, it follows from $x_1^*(\epsilon) = \overline{x}^\star + \epsilon$ that

$$\left.\frac{dx_1^*(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = 1.$$

Therefore, we have that

$$\left.\frac{dx_1^*(\epsilon)}{d\epsilon}\right|_{\epsilon=0} + \left.\frac{dx_2^*(\epsilon)}{d\epsilon}\right|_{\epsilon=0} = 1 > 0.$$

The above condition implies that the aggregate bid increases when we switch from $t^\star(x)$ to $t_\epsilon^\star(x)$ while the tax revenue remains at zero, which is a contradiction to the optimality of $t^\star(x)$. This completes the proof. ■

Now we can prove Proposition 2. Consider the following two cases:

Case I: $\boldsymbol{\lambda = 0}$. In this case, the designer is concerned only with the aggregate bid. Consider the following rigid cap:

$$t^{\star\star}(x) = \begin{cases} 0, & x \le x_1^\star, \\ \infty, & x > x_1^\star. \end{cases}$$

It can be verified that the equilibrium bid profile $(x_1^\star, x_2^\star)$ under $t^\star(\cdot)$ is also an equilibrium under $t^{\star\star}(\cdot)$. Next, note that $x_1^\star > x_2^\star$ by Lemma 12. Following Step 4 in the proof of Lemma 4, we can construct a flexible cap to induce the bid profile $(x_1^\star, x_2^\star)$. Therefore, the optimum can take the form of either a rigid cap or a flexible cap.

Case II: $\boldsymbol{\lambda > 0}$. Because $x_1^\star > x_2^\star$ by Lemma 12, we can always adopt the construction in Step 4 in the proof of Lemma 4 to induce the bid profile $(x_1^\star, x_2^\star)$ and generate a positive amount of tax revenue. Therefore, a rigid cap that generates zero tax revenue is suboptimal. ■

B.3 Optimal Tax Schedule in Tullock Contests with $r \in (1, 2]$

We are ready to spell out the following sufficient conditions for either a flexible cap or no cap to be optimal.

Proposition 4 ***(Optimal Tax Schedule in Two-player Tullock Contests with $r \in (1, 2]$)*** *Suppose that $n = 2$, $\lambda \in [0, 1]$, and $r \in (1, 2]$. The following statements hold.*

(i) If $r > 1 + v^r$ or

$$\frac{r(1 - v^r)}{1 + v^r} + \frac{(1 - v)\lambda - 1}{1 + v} > 0,$$

the optimum requires a binding cap.

(ii) If $r \leq 1 + v^r$ and

$$(2r + 1)v^{r+1} - r + 1 - rv > \lambda \left[1 + rv + v^{r+1} - (r + 2)v^r\right],$$

then no cap is optimal.

Proof. We first prove part (i) of the proposition. It is well known that the equilibrium without policy intervention is unique and is in mixed strategy when $r > 1 + v^r$ (Feng and Lu, 2017; Ewerhart, 2017). Recall that the selected equilibrium in the optimal contest is in pure strategy by Proposition 2. Therefore, a binding cap is required in the optimum.

In what follows, we focus on the case of $r \leq 1 + v^r$. First, note that player 1's bid x_1^* in the selected pure-strategy equilibrium is pinned down by the first-order condition according to the construction in Step 3 in the proof of Lemma 4. Second, by the same argument used in the proof of Theorem 2, player 2's equilibrium bid x_2^* is also governed by the first-order condition in the optimal contest. Therefore, the argument in Lemma 3 applies if we ignore players' participation constraints—i.e., $p_i(x_1^*, x_2^*)v_i - x_i^* \geq 0$, with $i \in \{1, 2\}$—and the designer's optimization problem can be reformulated as

$$\max_{p_1^* \in [1/2, p_1^\dagger]} \mathcal{F}(p_1^*; \lambda),$$

where

$$\mathcal{F}(p_1^*; \lambda) = r\Bigg\{(1 - \lambda)vp_1^*(1 - p_1^*)^{1-\frac{1}{r}} \left[(p_1^*)^{\frac{1}{r}} + (1 - p_1^*)^{\frac{1}{r}}\right] + \lambda \left[2vp_1^*(1 - p_1^*) + (1 - p_1^*) \left[p_1^* - (p_1^*)^{1-\frac{1}{r}}(1 - p_1^*)^{\frac{1}{r}}\right]\right] \Bigg\}, \tag{47}$$

and $p_1^\dagger \equiv 1/(1+v^r)$ is player 1's equilibrium winning probability in a contest under no cap. It can verified that both players' participation constraints are indeed satisfied for all $p_1^* \in [1/2, p_1^\dagger]$.

A sufficient condition for a flexible cap to be optimal is $\left.\frac{\partial \mathcal{F}(p_1^*;\lambda)}{\partial p_1^*}\right|_{p_1^*=p_1^\dagger} < 0$, which is equivalent to $\mathcal{G}(v^{-r}) < 0$, where $\mathcal{G}(\cdot)$ is defined as

$$\mathcal{G}(\eta) := r\left\{(1-\lambda)v\left[\left(1+\frac{1}{r}\right)\eta^{\frac{1}{r}} + \left(\frac{1}{r}-1\right)\eta^{1+\frac{1}{r}} + 1 - \eta\right]\right.$$
$$\left. + \lambda\left[2v\left(1-\eta\right) + \left(1-\eta+\left(\frac{1}{r}-1\right)\left(\frac{1}{\eta}\right)^{\frac{1}{r}} + \left(\frac{1}{r}+1\right)\left(\frac{1}{\eta}\right)^{\frac{1}{r}-1}\right)\right]\right\}.$$

It is straightforward to verify that $\mathcal{G}(v^{-r}) < 0$ is equivalent to

$$\frac{r(1-v^r)}{1+v^r} + \frac{(1-v)\lambda - 1}{1+v} > 0.$$

Next, we prove part (ii). As previously discussed, a flexible cap is optimal if $r > 1 + v^r$. Therefore, the condition $r \leq 1 + v^r$ is required for having no cap to be optimal. A sufficient condition for having no cap to be optimal is

$$\mathcal{G}(\eta) > 0, \text{ for all } \eta \in [1, v^{-r}].$$

Carrying out the algebra, $\mathcal{G}(\eta)$ can be bounded from below by

$$\begin{aligned}
\mathcal{G}(\eta) =& (1-\lambda)v\left[\left(1+\frac{1}{r}\right)\eta^{\frac{1}{r}} + \left(\frac{1}{r}-1\right)\eta^{\frac{1}{r}+1} + 1 - \eta\right] \\
& + \lambda\left[2v(1-\eta) + 1 - \eta + \left(\frac{1}{r}-1\right)\eta^{-\frac{1}{r}} + \left(\frac{1}{r}+1\right)\eta^{1-\frac{1}{r}}\right] \\
\geq& (1-\lambda)v\left[\left(1+\frac{1}{r}\right) + \left(\frac{1}{r}-1\right)v^{-r-1} + 1 - v^{-r}\right] \\
& + \lambda\left[2v(1-v^{-r}) + 1 - v^{-r} + \left(\frac{1}{r}-1\right) + \left(\frac{1}{r}+1\right)\right] \\
=& \frac{1}{rv^r} \times \left\{(2r+1)v^{r+1} - r + 1 - rv + \lambda\left[-v^{r+1} + (r+2)v^r - rv - 1\right]\right\}.
\end{aligned}$$

Therefore, if

$$(2r+1)v^{r+1} - r + 1 - rv > \lambda\left[1 + rv + v^{r+1} - (r+2)v^r\right],$$

then it is optimal to have no cap. This concludes the proof. ■

B.4 Proof of Proposition 3

Proof. Recall that if $v < (r-1)^{\frac{1}{r}}$, or equivalently, if $r > r^*$ or $v < v^*$, then the equilibrium under no cap is semi-mixed, which is suboptimal by Proposition 2. In what follows, we assume that $v \geq (r-1)^{\frac{1}{r}}$.

Substituting $\lambda = 0$ into the contest objective (47) yields that

$$\mathcal{F}_0(p_1^*) := \mathcal{F}(p_1^*; 0) = rv_2 p_1^*(1-p_1^*) \times \left[1 + \left(\frac{p_1^*}{1-p_1^*}\right)^{\frac{1}{r}}\right].$$

Carrying out the algebra, we can obtain that

$$\begin{aligned}
\mathcal{F}_0'(p_1^*) =& \frac{\mathcal{F}_0(p_1^*)}{r} \times \left[\frac{r}{p_1^*} - \frac{r-1}{1-p_1^*} + \frac{(p_1^*)^{\frac{1}{r}-1} - (1-p_1^*)^{\frac{1}{r}-1}}{(p_1^*)^{\frac{1}{r}} + (1-p_1^*)^{\frac{1}{r}}}\right] \\
=& \frac{\mathcal{F}_0(p_1^*)}{rp_1^*} \times \left[r - (r-1)\eta + \frac{\eta^{\frac{1}{r}} - \eta}{\eta^{\frac{1}{r}} + 1}\right] \\
=& \frac{\mathcal{F}_0(p_1^*)\eta^{\frac{1}{r}+1}}{rp_1^*(\eta^{\frac{1}{r}} + 1)} \times \left[r(\eta^{-1} - 1)(1 + \eta^{-\frac{1}{r}}) + (1 + \eta^{-1})\right],
\end{aligned}$$

where $\eta := p_1^*/(1-p_1^*) \in [1, v^{-r}]$. It can be verified that $\mathcal{F}_0'(p_1^*) > 0$ is equivalent to

$$\phi(\eta) := r(\eta^{-1} - 1)(1 + \eta^{-\frac{1}{r}}) + (1 + \eta^{-1}) > 0.$$

If $\phi(v^{-r}) < 0$, or equivalently, if $v < v^{**}$ or $r > r^{**}$, then $\mathcal{F}_0'(p_1^\dagger) < 0$ and thus having no cap is suboptimal. It remains to show that if $\phi(v^{-r}) \geq 0$, then $\phi(\eta) > 0$ for $\eta \in [1, v^{-r})$, which implies that $\mathcal{F}_0(p_1^*)$ strictly increases with p_1^* for $p_1^* \in [1/2, p_1^\dagger]$ and hence it is optimal to have no cap.

Note that $v \geq (r-1)^{\frac{1}{r}}$ implies that $v^{-r} \leq 1/(r-1)$ and thus $[1, v^{-r}] \subseteq [1, 1/(r-1)]$. Next, we show that $\phi'(\eta) < 0$ for all $\eta \in [1, 1/(r-1)]$. Carrying out the algebra, we have that

$$\phi'(\eta) = -r\eta^{-2}(1 + \eta^{-\frac{1}{r}}) - \eta^{-\frac{1}{r}-1}(\eta^{-1} - 1) - \eta^{-2}$$

$$
\begin{aligned}
&= \eta^{-\frac{1}{r}-1} \times \left[1-(r+1)\eta^{\frac{1}{r}-1}(1+\eta^{-\frac{1}{r}}) \right] \\
&\le \eta^{-\frac{1}{r}-1} \times \left\{ 1-(r+1)(r-1)^{\frac{r-1}{r}} \left[1+(r-1)^{\frac{1}{r}} \right] \right\} \\
&\le \eta^{-\frac{1}{r}-1} \times \left[1-2(r-1)^{r-1} \right] \\
&\le \eta^{-\frac{1}{r}-1} \times \left[1-2e^{-\frac{1}{e}} \right] < 0,
\end{aligned}
$$

where the first inequality follows from $\eta \le v^{-r} \le 1/(r-1)$; the second inequality follows from $r > 1$, $(r-1)^{\frac{r-1}{r}} \ge (r-1)^{r-1}$, and $1+(r-1)^{\frac{1}{r}} \ge 1$, and the third inequality follows from the fact that $(r-1)^{r-1} \ge e^{-\frac{1}{e}}$ for $r \in (1,2]$.

From the above analysis, we know that $\phi(\eta)$ is strictly decreasing in η for $\eta \in [1, v^{-r}]$. Therefore, $\phi(\eta) > \phi(v^{-r}) \ge 0$ for $\eta \in [1, v^{-r})$. This completes the proof. ■

Bid Caps in Noisy Contests

ONLINE APPENDIX

(Not Intended for Publication)

Qiang Fu* Zenan Wu† Yuxuan Zhu‡

In this online appendix, we collect the analyses and discussions omitted from the main text.[1] Online Appendix A provides sufficient condition under which a flexible cap or no cap can be optimal in a two-player Tullock contest setting. Online Appendix B characterizes the optimal cap schemes in a multi-player contest with two player types. Online Appendix C collects the proofs of propositions.

A Optimal Cap Schemes in Two-Player Contests

Proposition A1 ***(Flexible Cap vs. No Cap in Two-player Tullock Contests)*** *Suppose that $n = 2$, $\lambda \in [0, 1]$, and $r \in (0, 1]$. The following statements hold.*

(i) If

$$\frac{r(1-v^r)}{1+v^r} + \frac{(1-v)\lambda - 1}{1+v} > 0, \tag{A1}$$

then the optimal contest imposes a flexible cap.

(ii) If

$$v\left[(2+r)v^r - r\right] > \lambda(1-v^r)(r-v), \tag{A2}$$

then the optimal contest imposes no cap.

Remark 1 follows immediately from Proposition A1.

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[1]This note is not self-contained; it is the online appendix of the paper "Bid Caps in Noisy Contests."

B Optimal Cap Schemes in Multi-player Contests with Two Player Types

The two-player example in Section 3.4 and Figure 1 provide an intuitive account of the fundamental trade-off between the cost and competition effects in asymmetric contests, as well as how the optimum depends on players' type differential and the noisiness of the winner-selection mechanism. However, a multi-player contest differs substantially from its bilateral counterpart. In a two-player contest, player heterogeneity can be captured by a single parameter, $v \equiv v_2/v_1$. In contrast, heterogeneity is inherently multidimensional with three or more players, which cannot readily be defined or measured without imposing a specific structure on the profile of prize valuations $(v_1, \ldots, v_n)$. This nuance prevents handy comparative statics.

We consider a simple Tullock contest setting with a two-type distribution—i.e., stronger and weaker—to demonstrate the complications. There are $n_s \geq 1$ stronger players and $n_w \geq 1$ weaker players, with $n_s + n_w = n \geq 3$. The former type values the prize at v_s, while the latter values it at v_w, with $v_s \geq v_w > 0$. Despite the vast simplification, it is difficult to provide a simple account of the heterogeneity between players, as in the previous section: This depends on prize valuations across types—i.e., the ratio between v_s and v_w—and also the composition of types within the pool, i.e., (n_s, n_w). We analyze two simple cases, which demonstrate that a variation in either dimension may change the optimum fundamentally.

Case I: $n_s = 1$. We first assume one stronger player vs. $n - 1$ weaker opponents. The following result can be obtained.

Proposition A2 ***(Optimal Contest with One Strong Player)*** *Suppose that $n_s = 1$, $n_w \geq 2$, and $\lambda + r > 1$. There exist two cutoffs $\hat{v}_h(\lambda, r) \in (0, 1)$ and $\hat{v}_l(\lambda, r) \in (0, 1)$ such that a flexible cap is optimal if $v_w/v_s < \hat{v}_l(\lambda, r)$ and no cap is optimal if $v_w/v_s > \hat{v}_h(\lambda, r)$.*

The prediction is largely in line with that of Proposition A1 in a two-player setting. When v_w/v_s is sufficiently small, a flexible cap plays a more significant equalizing role. Conversely, the optimum requires no cap when v_w/v_s is sufficiently large: The direct discount on bidding incentives outweighs the limited equalizing role of a bid cap; as a result, the contest needs no intervention.

Case II: $n_s \geq 2$. The prediction drastically differs in the case of two or more stronger players, and the optimum with respect to the ratio v_w/v_s can be nonmonotone.

Proposition A3 ***(Optimal Contest with Two or More Strong Players)*** *Suppose that $n_s \geq 2$ and $n_w \geq 1$. Fixing $\lambda < 1$ and $r < 1$, there exists a lower threshold $\underline{v}(\lambda, r) \in (0, 1)$*

and an upper threshold $\bar{v}(\lambda, r) \in (0,1)$, with $\bar{v}(\lambda, r) \geq \underline{v}(\lambda, r)$, such that no cap is optimal if $v_w/v_s < \underline{v}(\lambda, r)$ or $v_w/v_s > \bar{v}(\lambda, r)$.

Although a sufficiently large ratio of v_w/v_s—i.e., $v_w/v_s > \bar{v}(\lambda, r)$—implies no policy intervention, as in Propositions A1 and A2, no cap also emerges as the optimum when v_w/v_s is sufficiently small, i.e., $v_w/v_s < \underline{v}(\lambda, r)$, which overturns the predictions of Propositions A1 and A2. Proposition A3 suggests that a flexible cap can be optimal only if v_w/v_s is in an intermediate range. This result reveals the complexity involved in a multi-player setting.

The competition effect loses its appeal when multiple stronger players are present. Suppose that $(n_s, n_w) = (2, 1)$. In this case, a stronger player has to outperform his equally competent peer to secure the prize, which may help discipline him from shirking regardless of the prevailing cap scheme. Meanwhile, a cap that handicaps the stronger may not effectively revive the weaker's momentum, as a win is difficult regardless when outnumbered by more competent opponents. A smaller v_w/v_s turns out to elevate the cost of a flexible cap: To level the playing field and incentivize the single underdog, a sufficiently high marginal tax rate is required to offset the initial asymmetry, which may cause excessive incentive loss from the two stronger players. In this scenario, contest design involves a hidden selection problem: The designer may simply "abandon" the weaker, while sustaining the competition between the stronger. This effect would not come into play in a bilateral contest.

C Proofs

Proof of Proposition A1

Proof. Clearly, with $n = 2$, both players are active in equilibrium and the set $\mathcal{P}$ defined in (23) can be simplified as

$$\mathcal{P} = \left\{ (p_1^*, p_2^*) : p_1^* + p_2^* = 1, \frac{1}{2} \leq p_1^* \leq \frac{1}{1+v^r} \right\}.$$

For notational convenience, define $p_1^\dagger := 1/(1+v^r)$. Substituting $p_2^* = 1 - p_1^*$ into the contest objective (22), the maximization problem degenerates to a single-variable optimization problem as follows:

$$\max_{p_1^* \in [1/2, p_1^\dagger]} \mathcal{F}(p_1^*),$$

where

$$\mathcal{F}(p_1^*) = r \Bigg\{ (1-\lambda) v p_1^* (1-p_1^*)^{1-\frac{1}{r}} \left[(p_1^*)^{\frac{1}{r}} + (1-p_1^*)^{\frac{1}{r}} \right]$$

$$+\lambda\left[2vp_1^*(1-p_1^*)+(1-p_1^*)\left[p_1^*-(p_1^*)^{1-\frac{1}{r}}(1-p_1^*)^{\frac{1}{r}}\right]\right]\Bigg\}.$$

Carrying out the algebra, we can obtain that

$$\mathcal{F}'(p_1^*)=(1-p_1^*)\mathcal{G}(\eta),$$

where $\eta := p_1^*/(1-p_1^*) \in [1, v^{-r}]$ and

$$\mathcal{G}(\eta) := r\Bigg\{(1-\lambda)v\left[\left(1+\frac{1}{r}\right)\eta^{\frac{1}{r}}+\left(\frac{1}{r}-1\right)\eta^{1+\frac{1}{r}}+1-\eta\right]$$

$$+\lambda\left[2v\,(1-\eta)+\left(1-\eta+\left(\frac{1}{r}-1\right)\left(\frac{1}{\eta}\right)^{\frac{1}{r}}+\left(\frac{1}{r}+1\right)\left(\frac{1}{\eta}\right)^{\frac{1}{r}-1}\right)\right]\Bigg\}.$$

It can be verified that $p_1^* = p_1^\dagger = 1/(1+v^r)$, or equivalently, $\eta = v^{-r}$, in a two-player contest without a cap. Therefore, a sufficient condition for a flexible cap to be optimal is $\mathcal{F}'(p_1^\dagger) < 0$, or equivalently, $\mathcal{G}(v^{-r}) < 0$. Carrying out the algebra, we can obtain that

$$\begin{aligned}\mathcal{G}(v^{-r}) =& v^{-r}\times\Bigg\{(1-\lambda)\left[(r+1)v^r+1-r+rv^{r+1}-rv\right]\\ &+\lambda\times\left[(r+1)v^{r+1}+rv^r+(1-r)v-r\right]\Bigg\}\\ =& v^{-r}\times\left[\lambda(v^r+1)(v-1)+r(v+1)(v^r-1)+(v^r+1)\right]\\ =& -(1+v^{-r})(v+1)\times\left[\frac{r(1-v^r)}{1+v^r}+\frac{(1-v)\lambda-1}{1+v}\right].\end{aligned}$$

It is evident that $\mathcal{G}(v^{-r}) < 0$ if

$$\frac{r(1-v^r)}{1+v^r}+\frac{(1-v)\lambda-1}{1+v}>0,$$

which corresponds to (A1) in Proposition A1(i).

Next, note that $\mathcal{G}(\eta)$ can be bounded from below by

$$\mathcal{G}(\eta) =(1-\lambda)v\left[\left(1+\frac{1}{r}\right)\eta^{\frac{1}{r}}+\left(\frac{1}{r}-1\right)\eta^{\frac{1}{r}+1}+1-\eta\right]$$

$$
\begin{aligned}
&+\lambda\left[2v(1-\eta)+1-\eta+\left(\frac{1}{r}-1\right)\eta^{-\frac{1}{r}}+\left(\frac{1}{r}+1\right)\eta^{1-\frac{1}{r}}\right]\\
\geq&(1-\lambda)v\left[\left(1+\frac{1}{r}\right)+\left(\frac{1}{r}-1\right)+1-v^{-r}\right]\\
&+\lambda\left[2v(1-v^{-r})+1-v^{-r}+\left(\frac{1}{r}-1\right)v+\left(\frac{1}{r}+1\right)v^{1-r}\right]\\
=&\frac{v^{-r}}{r}\left\{v\left[(2+r)v^r-r\right]+\lambda(v^r-1)(r-v)\right\},
\end{aligned}
$$

where the inequality follows from $\eta \in [1, v^{-r}]$. Clearly, $\mathcal{G}(\eta) > 0$ for all $\eta \in [1, v^{-r}]$, or equivalently, $\mathcal{F}'(p_1^*) > 0$ for all $p_1^* \in [\frac{1}{2}, p_1^\dagger]$, if

$$
v\left[(2+r)v^r-r\right] > \lambda(1-v^r)(r-v),
$$

which implies that $\mathcal{F}(p_1^*)$ is uniquely maximized at $p_1^* = p_1^\dagger$ on $[\frac{1}{2}, p_1^\dagger]$ and it is optimal to have no cap. Note that the above inequality corresponds to (A2) in Proposition A1(ii). This completes the proof. ■

Proof of Proposition A2

Proof. Note that players of the same type must win with equal probabilities in equilibrium. Therefore, the winning probability distribution $\boldsymbol{p}^* \equiv (p_1^*, \ldots, p_n^*)$ is fully characterized by (p_s^*, p_w^*), where p_s^* and p_w^* respectively represent the stronger players' and the weaker players' equilibrium winning probabilities. With slight abuse of notation, the set $\mathcal{P}$ defined in (23) can then be simplified as

$$
\mathcal{P} = \left\{(p_s^*, p_w^*) : n_s p_s^* + n_w p_w^* = 1,\ 1/n \geq p_w^* \geq p_w^\dagger\right\},
$$

where $p_w^\dagger$ is the equilibrium winning probability of each weaker player under no cap. Normalizing v_s to 1 without loss of generality and substituting $p_s^* = (1 - n_w p_w^*)/n_s$ into the contest objective (22), the designer's optimization problem boils down to

$$
\max_{p_w^* \in [p_w^\dagger, 1/n]} \mathcal{F}(p_w^*),
$$

where $\mathcal{F}(\cdot)$ is given by

$$
\mathcal{F}(p_w^*) := (1-\lambda)v_w(p_w^*)^{1-\frac{1}{r}}(1-p_w^*)\left[n_s\left(\frac{1-n_w p_w^*}{n_s}\right)^{\frac{1}{r}}+n_w(p_w^*)^{\frac{1}{r}}\right]
$$

$$+\lambda\left\{n_s\left(\frac{1-n_w p_w^*}{n_s}\right)^{1-\frac{1}{r}}\left[1-\left(\frac{1-n_w p_w^*}{n_s}\right)\right]\left[\left(\frac{1-n_w p_w^*}{n_s}\right)^{\frac{1}{r}}-(p_w^*)^{\frac{1}{r}}\right]+nv_w p_w^*(1-p_w^*)\right\}. \tag{A3}$$

Carrying out the algebra, we can obtain that

$$\begin{aligned}\mathcal{F}'(p_w^*) =&(1-\lambda)v_w\times\left\{\left(1-\frac{1}{r}\right)(p_w^*)^{-\frac{1}{r}}(1-p_w^*)\left[n_s(p_s^*)^{\frac{1}{r}}+n_w(p_w^*)^{\frac{1}{r}}\right]\right.\\ &\left.-(p_w^*)^{1-\frac{1}{r}}\left[n_s(p_s^*)^{\frac{1}{r}}+n_w(p_w^*)^{\frac{1}{r}}\right]+(p_w^*)^{1-\frac{1}{r}}(1-p_w^*)n_w\frac{1}{r}\left[-(p_s^*)^{\frac{1}{r}-1}+(p_w^*)^{\frac{1}{r}-1}\right]\right\}\\ &+\lambda\times\left\{\left(\frac{1}{r}-1\right)n_w(p_s^*)^{-\frac{1}{r}}(1-p_s^*)\left[(p_s^*)^{\frac{1}{r}}-(p_w^*)^{\frac{1}{r}}\right]+n_w(p_s^*)^{1-\frac{1}{r}}\left[(p_s^*)^{\frac{1}{r}}-(p_w^*)^{\frac{1}{r}}\right]\right.\\ &\left.-n_s(p_s^*)^{1-\frac{1}{r}}(1-p_s^*)\frac{1}{r}\left[\frac{n_w}{n_s}(p_s^*)^{\frac{1}{r}-1}+(p_w^*)^{\frac{1}{r}-1}\right]+nv_w(1-2p_w^*)\right\}.\end{aligned} \tag{A4}$$

Recall that $p_w^\dagger$ is the equilibrium winning probability of each weaker player under no cap. Therefore, for a flexible cap to be optimal, it suffices to show that $\mathcal{F}'(p_w^\dagger)>0$ when v_w is sufficiently small.

Denote the equilibrium winning probability of each strong player by $p_s^\dagger$. We first take a closer look at the equilibrium winning probability $(p_s^\dagger, p_w^\dagger)$ under no cap. From the first-order conditions for each type of players, we have that

$$(p_s^\dagger)^{1-\frac{1}{r}}(1-p_s^\dagger)=v_w(p_w^\dagger)^{1-\frac{1}{r}}(1-p_w^\dagger). \tag{A5}$$

Note that $n_s=1$ by assumption. Therefore, we have that $p_s^\dagger=1-n_w p_w^\dagger$. Substituting the expression of p_s into the above condition, for a sufficiently small v_w, we can obtain that

$$p_w^\dagger=\left(\frac{v_w}{n_w}\right)^r\left[1+o(1)\right].$$

Carrying out the algebra, for a sufficiently small v_w, we have that

$$\begin{aligned}\mathcal{F}'(p_w^\dagger) =&(1-\lambda)\times\left\{v_w\left(1-\frac{1}{r}\right)\left(\frac{v_w}{n_w}\right)^{-1}\left[1+o(1)\right]+o(1)\right\}\\ &+\lambda\times\left\{n_w\left[1+o(1)\right]+o(1)\right\}\\ =&\frac{n_w}{r}(\lambda+r-1)+o(1)>0,\end{aligned}$$

where the strict inequality follows from the condition $\lambda + r > 1$ assumed in Proposition A2. In other words, there exists a threshold $\hat{v}_l(\lambda, r) > 0$ such that imposing a flexible cap is optimal to the designer for all $v_w/v_s < \hat{v}_l(\lambda, r)$.

Next, we show that having no cap is optimal if v_w is sufficiently large. It is evident that $p_s^\dagger = 1/n + o(1)$ and $p_w^\dagger = 1/n + o(1)$ in this case. Therefore, $\mathcal{F}'(p_w^*)$ in (A4) can be bounded from above by

$$
\begin{aligned}
\mathcal{F}'(p_w^*) =&(1-\lambda) \times n \times \left[\left(1-\frac{1}{r}\right)\left(1-\frac{1}{n}\right) - n \times \frac{1}{n} + o(1)\right] \\
&+ \lambda \times \left[-n \times \frac{1}{r}\left(1-\frac{1}{n}\right) + n \times \left(1-\frac{2}{n}\right) + o(1)\right] < 0, \text{ for all } p_w^* \in [p_w^\dagger, 1/n].
\end{aligned}
$$

Therefore, there exists a threshold $\hat{v}_h(\lambda, r) > 0$ such that having no cap is optimal for all $v_w/v_s > \hat{v}_h(\lambda, r)$. This concludes the proof. ■

Proof of Proposition A3

Proof. Similar to the proof of Proposition A2, we normalize v_s to 1 without loss of generality.

We first consider the case in which v_w is sufficiently small. It is evident that $p_w^\dagger = o(1)$ and $p_s^\dagger = 1/n_s + o(1)$. It follows from the first-order conditions (A5) that

$$
p_w^\dagger = \frac{1}{n_s}\left(\frac{v_w n_s}{n_s - 1}\right)^{\frac{r}{1-r}}\left[1 + o(1)\right].
$$

By the above equation and (A3), when v_w is sufficiently small, we can obtain that

$$
\begin{aligned}
\mathcal{F}(p_w^\dagger) =&(1-\lambda) v_w \left\{\frac{1}{n_s}\left(\frac{v_w n_s}{n_s - 1}\right)^{\frac{r}{1-r}}\left[1 + o(1)\right]\right\}^{1-\frac{1}{r}} n_s^{1-\frac{1}{r}}\left[1 + o(1)\right] \\
&+ \lambda \times \left\{n_s (p_s^\dagger)^{-1}(1 - p_s^\dagger)\left[1 + o(1)\right] + o(1)\right\} \\
=&(1-\lambda)\left(1 - \frac{1}{n_s}\right) + \lambda\left(1 - \frac{1}{n_s}\right) + o(1) = 1 - \frac{1}{n_s} + o(1).
\end{aligned}
$$

For $p_w^* > v_w^{\frac{2r}{2-r}}$, we have that

$$
\begin{aligned}
\mathcal{F}(p_w^*) =&(1-\lambda) v_w (p_w^*)^{1-\frac{1}{r}}(1 - p_w^*)\left[n_s (p_s^*)^{\frac{1}{r}} + n_w (p_w^*)^{\frac{1}{r}}\right] \\
&+ \lambda \left\{n_s (p_s^*)^{1-\frac{1}{r}}(1 - p_s^*)\left[(p_s^*)^{\frac{1}{r}} - (p_w^*)^{\frac{1}{r}}\right] + n v_w p_w^* (1 - p_w^*)\right\} \\
\leq&(1-\lambda) v_w (p_w^*)^{1-\frac{1}{r}}(n_s p_s^* + n_w p_w^*) + \lambda\left[n_s (p_s^*)^{1-\frac{1}{r}}(p_s^*)^{\frac{1}{r}}(1 - p_s^*) + n v_w p_w^*\right]
\end{aligned}
$$

$$
\begin{aligned}
=&(1-\lambda)v_w(p_w^*)^{1-\frac{1}{r}} + \lambda\left[n_s p_s^*(1-p_s^*) + n v_w p_w^*\right] \\
\leq&(1-\lambda)v_w^{\frac{1}{2-r}} + \lambda\left(1-\frac{1}{n_s} + n v_w^{\frac{2+r}{2-r}}\right) \\
=&\lambda\left(1-\frac{1}{n_s}\right) + o(1) < \mathcal{F}(p_w^\dagger),
\end{aligned}
$$

where the last inequality follows from $\lambda < 1$.

For $p_w^* \leq v_w^{\frac{2r}{2-r}}$, it follows from (A4) that

$$
\begin{aligned}
\mathcal{F}'(p_w^*) =&(1-\lambda)\times\left\{\left(1-\frac{1}{r}\right)v_w(p_w^*)^{-\frac{1}{r}} n_s^{1-\frac{1}{r}}\left[1+o(1)\right]\right\} + \lambda\times O(1) \\
\leq&(1-\lambda)\left(1-\frac{1}{r}\right)n_s^{1-\frac{1}{r}} v_w^{-\frac{r}{2-r}}\left[1+o(1)\right] < 0.
\end{aligned}
$$

To summarize, $\mathcal{F}(p_w^*)$ is strictly decreasing in p_w^* for $p_w^* \in [p_w^\dagger, v_w^{\frac{2r}{2-r}}]$ and $\mathcal{F}(p_w^*) < \mathcal{F}(p_w^\dagger)$ for all $p_w^* \in (v_w^{\frac{2r}{2-r}}, 1/n]$ if v_w is sufficiently small, which in turn implies that there exists a threshold $\underline{v}(\lambda, r) > 0$ such that having no cap is optimal for all $v_w/v_s < \underline{v}(\lambda, r)$.

Next, we consider the case where v_w is sufficiently large. In this case, we have that $p_w^\dagger = 1/n + o(1)$ and $p_s^\dagger = 1/n + o(1)$. Therefore, for all $p_w^* \in [p_w^\dagger, 1/n]$, we have that

$$
\begin{aligned}
\mathcal{F}'(p_w^*) =&(1-\lambda)\times n\times\left[\left(1-\frac{1}{r}\right)\left(1-\frac{1}{n}\right) - n\times\frac{1}{n} + o(1)\right] \\
&+\lambda\times\left[-n\times\frac{1}{r}\left(1-\frac{1}{n}\right) + n\times\left(1-\frac{2}{n}\right) + o(1)\right] < 0,
\end{aligned}
$$

and thus $\mathcal{F}(p_w^*)$ is strictly decreasing in p_w^*, which implies the optimality of imposing no cap on the contest. Therefore, there exists $\bar{v}(\lambda, r)$ such that having no cap is optimal for all $v_w/v_s > \bar{v}(\lambda, r)$. This concludes the proof. ■