Competitive Personalized Pricing with Multidimensional Characteristics*

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Abstract

Consumers differ in both their brand-dependent preferences (loyalty) and the intensity of their brand-independent preferences (choosiness). Firms produce horizontally differentiated products and, depending on data availability or competition policy, tailor their prices based on their learning about consumer characteristics. With a fully covered market, either fully personalized pricing—i.e., price discrimination based on both loyalty and choosiness—or loyalty-based pricing—i.e., price discrimination based only on loyalty—maximizes consumer welfare. The latter pricing regime is more likely to be optimal when the market involves more firms. In contrast, partially personalized pricing based on choosiness always maximizes industry profit.

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1 Introduction

The widespread use of commercial surveillance technology, data analytics, and AI-enabled algorithmic tools has provided firms with unprecedented flexibility in drawing precise inferences about consumer preferences and customizing their prices. Traditional second- and third-degree price discrimination—based on quantity or broad market segmentation—has evolved into more granular pricing schedules tailored to individual consumer characteristics, which brings first-degree price discrimination much closer to reality (Dubé and Misra, 2023; Spann et al., 2025).

Meanwhile, commercial surveillance and personalized pricing are often perceived as unfair, opaque, or even intrusive, which has sparked controversy and provoked backlash. Various regulatory responses have been triggered to limit firms' ability to process personal data and ensure consumers' right to transparent pricing, such as the European Union's General Data Protection Regulation and the California Privacy Rights Act. The U.S. Federal Trade Commission has also launched investigations into firms that use artificial intelligence and data analytics to help clients implement personalized pricing, accusing them of "putting people's privacy at risk" and "exploiting vast troves of personal information to charge people higher prices."[1]

The controversy calls for systematic research on the distributional effects of consumer information disclosure and price personalization—specifically, how firms' access to consumer information and their ability to tailor prices influence economic efficiency, consumer welfare, and firm profitability (Acquisti, Taylor, and Wagman, 2016; Baye and Sappington, 2020; Ali, Lewis, and Vasserman, 2023; Anderson, Baik, and Larson, 2023; Rhodes and Zhou, 2024), which lie at the core of regulatory debates regarding consumer privacy protection and competition policy (OECD, 2018; Rott, Strycharz, and Alleweldt, 2022). This paper seeks to address these concerns.

The scope of our paper goes beyond the conventional binary debate over whether access to consumer data should be allowed for pricing purposes or whether personalized pricing harms consumers. We take a step further and examine the implications of different pricing regimes that rely on inferences about different dimensions of consumer preferences. This enables us to contribute to the broader question of which types of consumer data should be subject to stricter trade regulations (Tucker, 2024).

[1] Source: `https://www.reuters.com/world/us/us-ftc-looking-into-targeted-pricing-based-personal-data-2024-07-23/`.

Consumer characteristics are arguably multidimensional, and different types of consumer information could be more relevant for inferences about different dimensions of consumer characteristics. For instance, a consumer may prefer one brand over another due to taste, whether aesthetic or functional. Meanwhile, consumers may also vary in their willingness to pay for products that better match their preferences. A consumer's socioeconomic status may strongly indicate how much they would pay for a closer match, but reveal little about how they rank similar brands. Conversely, browsing history or search queries may be predictive of a consumer's brand-specific preferences while providing less insight into the premium they are willing to pay for a preferred brand.[2] Due to regulatory constraints on data collection and pricing practices, firms may be unable to fully infer consumer characteristics, and prices may be tailored along only certain dimensions rather than being fully personalized. We show that such partially personalized pricing can benefit or harm consumers relative to fully personalized and/or uniform pricing, depending on the nature of the available data and the inferences it supports. The literature typically assumes unidimensional consumer characteristics and compares two extremes: fully personalized versus uniform pricing. We instead explore a setting with multidimensional consumer characteristics, which opens a new avenue for understanding how consumer information along different dimensions shapes pricing practices and market outcomes.

We construct a general discrete choice model à la Perloff and Salop (1985). Each consumer purchases one unit of a differentiated good from one firm, and consumers can be heterogeneous in two uncorrelated dimensions. One's gross valuation for the product offered by a firm $i \in \{1, 2\}$ is given by $v_i = v + tx_i$. The expression comprises (i) a "horizontal" element x_i, which describes the consumer's brand-specific preferences—i.e., how well each product i matches her taste—and (ii) a "vertical" element t, which measures her marginal valuation for consuming a product with a closer match—i.e., the intensity of her brand-specific preferences—and indicates how a consumer weighs price against a better fit.

Borrowing the terminology of Armstrong (2006), we call $\boldsymbol{x} = (x_1, x_2)$ one's *brand loyalty* and t her *choosiness*. Four pricing regimes are possible. Under *uniform pricing*—in which price discrimination is banned or consumer data on neither $\boldsymbol{x}$ nor

[2] Shiller (2020) analyzes personalized pricing for Netflix's movie rentals by mail. He distinguishes between demographic and web-browsing data, and finds that price discrimination based on different data sources has different profit implications for firms.

t is available—firms set a single price for all consumers. Under *fully personalized pricing*, firms perfectly observe both $\boldsymbol{x}$ and t for each consumer and tailor their prices accordingly. Under *loyalty-based pricing* (pricing based on $\boldsymbol{x}$ but not t) or *choosiness-based pricing* (pricing based on t but not $\boldsymbol{x}$), firms use partial preference information. We characterize the equilibrium under each regime and rank the resulting consumer welfare and industry profit.

Under full market coverage, fully personalized pricing sharpens market competition and benefits consumers compared with uniform pricing: Each consumer is rendered contestable under the former, which intensifies competition. This reaffirms the wisdom of Thisse and Vives (1988) and Rhodes and Zhou (2024) established with unidimensional consumer characteristics. Notably, we show that partially personalized pricing may either outperform fully personalized pricing in terms of consumer welfare or underperform relative to uniform pricing, depending on which dimension of consumer characteristics—loyalty or choosiness—firms can infer and customize their prices accordingly. Our key observations are as follows.

First, if firms learn loyalty $\boldsymbol{x}$ but not choosiness t, consumer welfare can exceed that under fully personalized pricing. Firms' inability to observe choosiness triggers a subtle marginal-inframarginal trade-off. A firm would refrain from setting a high price even if its product is most preferred by the consumers because some of them may not be sufficiently choosy to justify a high willingness to pay for the better fit. This uncertainty forces the price down, since the firm must avoid alienating less choosy consumers. Put differently, less choosy consumers generate a positive externality that benefits choosy consumers when t is unknown. The comparison between loyalty-based pricing and fully personalized pricing ultimately depends on the distribution of consumer types. We provide plausible conditions under which either maximizes overall consumer welfare.

Second, if firms learn t but not $\boldsymbol{x}$—i.e., under choosiness-based pricing—the expected consumer welfare falls below that of uniform pricing. Information about t softens competition between firms for choosy consumers while intensifying it for less choosy ones: A large t implies a greater advantage for the preferred firm, which renders undercutting less effective for poaching loyal consumers from competitors. Firms can thus extract more surplus from choosy consumers—who are more willing to pay for a better-matched product and generate greater value to firms—while losing out on less choosy consumers, who are less valuable to firms regardless. Information about

t hurts consumers but benefits firms.

Third, equilibrium industry profit can be unambiguously ranked: Choosiness-based pricing always maximizes industry profit, while loyalty-based pricing minimizes it. Furthermore, industry profit always gains from firms' learning about choosiness. This affirms the intuition we outlined above: The inability to observe choosiness entails the marginal-inframarginal trade-off and prevents firms from extracting surplus.

Our paper presents the first comprehensive study of personalized pricing with multidimensional consumer characteristics, which broadens the scope of the literature on competitive personalized pricing: The implications of consumer characteristics for distribution outcomes are not simply binary. Learning and inferring consumers' brand preferences (loyalty) versus their non-brand preferences (choosiness) yields qualitatively different effects. The former creates asymmetry among firms and renders each individual consumer more contestable; in contrast, the latter dismisses the marginal-inframarginal trade-off and enables more effective surplus extraction. These findings call for a more thorough examination of data and privacy regulations—as well as competition policies—in light of richer consumer preferences. A proper policy analysis should move beyond the binary question of whether price personalization should be allowed, and understand how different types of consumer information enable firms to learn about consumer preferences, shape their pricing strategies, and distribute surplus across different parties.

Section 2 sets up the model. Section 3 analyzes the duopoly case in detail and highlights these mechanisms with fully fledged analytical results. Section 4 extends the model to an oligopoly setting and shows that our key insights remain valid with more firms in the market. Notably, loyalty-based pricing becomes even more likely to emerge as the consumer-optimal regime. Our baseline model assumes small production costs, which ensures full market coverage. Section 5 allows for larger production costs, which cause some consumers to opt out.

Link to the Literature A vast amount of scholarly effort has been devoted to the study of competitive price discrimination. One strand of the literature models personalized pricing in the form of imperfect price discrimination, wherein firms set different prices for different consumer segments (e.g., Shaffer and Zhang, 1995; Fudenberg and Tirole, 2000; Chen, Narasimhan, and Zhang, 2001; Iyer, Soberman, and Villas-Boas, 2005; Esteves and Resende, 2016). The rapid development of information technology has sparked interest in more granular pricing strategies (see, e.g., Acquisti, Taylor, and Wagman, 2016). The seminal study of Thisse and Vives (1988) compares uniform pricing with fully personalized pricing in a spatial duopoly model, in which each consumer's location is either perfectly revealed to firms or entirely unknown. A similar approach to modeling price personalization in duopoly is adopted by Chen and Iyer (2002); Shaffer and Zhang (2002); and Chen, Choe, and Matsushima (2020).

Most studies on personalized pricing focus on duopolistic competition. In a general discrete choice oligopoly model, Rhodes and Zhou (2024) show that personalized pricing benefits consumers and harms firms under full market coverage—which generalizes the insight of Thisse and Vives (1988)—while the comparison can be overturned otherwise. Our paper is more closely related to Rhodes and Zhou (2024) in terms of modeling approach. However, we aim to explore the ramifications of partially personalized pricing when consumers are characterized along multiple dimensions.

Anderson, Baik, and Larson (2023) also adopt a general discrete choice model. They let firms first set a listing price in the first stage and then send personalized offers in the second stage, while assuming that targeting each individual consumer is costly. Both Rhodes and Zhou (2024) and our paper assume costless targeting and a single-stage structure for pricing.

The literature predominantly assumes that consumer heterogeneity can be adequately captured through variations along a single dimension of consumer characteristics. Each consumer's type is defined solely by her location in the spatial competition model of Thisse and Vives (1988). Anderson, Baik, and Larson (2023) and Rhodes and Zhou (2024) assume that consumers differ only in $\boldsymbol{v} = (v_1, \ldots, v_n)$, their respective gross valuations for different firms' products. As a result, these studies typically focus on a binary comparison between uniform pricing and personalized pricing. A notable exception is Armstrong (2006). In a Hotelling duopolistic setting, Armstrong (2006) considers choosiness-based pricing and compares it with uniform pricing and fully personalized pricing. We incorporate Armstrong's conceptual notions of choosiness versus loyalty and analyze all possible pricing regimes. Our study thereby highlights the qualitatively contrasting roles played by information on different dimensions of consumer characteristics.

Ali, Lewis, and Vasserman (2023) allow a consumer to voluntarily disclose information about her preferences. This may lead to partial information disclosure and coarse inference by firms, which could improve consumer welfare relative to both fully personalized pricing and uniform pricing. We also show that partial access to consumer information can benefit consumers. Our setting and results differ subtly from theirs. They assume that a consumer's type is defined by her location and focus on the precision of the information she voluntarily discloses, which may lead firms to make coarse inferences. In contrast, we characterize a consumer's type along two dimensions, allowing for one or both to be unknown to firms.

2 Model and Preliminaries

Each of $n \geq 2$ firms, indexed by $i \in \mathcal{N} \equiv \{1, \ldots, n\}$, offers a horizontally differentiated product at a constant marginal cost $c \geq 0$. The market involves a unit mass of consumers. Each consumer wishes to buy one unit of the product, and her gross valuation for the products supplied by these firms is given by $\boldsymbol{v} \equiv (v_1, \ldots, v_n)$.

A consumer's gross valuation for the product supplied by a firm i is determined by $v_i = v + tx_i$, where v is the base utility she derives from consumption of the product and tx_i measures the additional utility she gains from the product supplied by firm $i \in \mathcal{N}$. The vector $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$ captures the consumer's firm- or brand-specific preferences, which measures each product i's match to her taste. The parameter t, which is common to all firms for a given consumer, indicates the intensity of her brand-specific preferences and measures the marginal valuation for consuming a better-matched product. This also reflects the relative importance she assigns to taste vis-à-vis price in her purchasing decision. Alternatively, the parameter t can be interpreted as an indicator for income, since a consumer with a larger t tends to be less price sensitive.[3] Following the literature (Armstrong, 2006), we call the former brand-specific preferences *loyalty* and the latter *choosiness*.

Each consumer's loyalty $\boldsymbol{x} \equiv (x_1, \cdots, x_n)$ is distributed on $[\underline{x}, \overline{x}]$, with $0 \leq \underline{x} < \overline{x} < \infty$, according to a joint cumulative density function (CDF) $\tilde{G}(\boldsymbol{x})$ with a probability density function (PDF) $\tilde{g}(\boldsymbol{x})$. The CDF $\tilde{G}(\boldsymbol{x})$ is *exchangeable*, such that the joint CDF is independent of any permutation of $(x_1, \ldots, x_n)$. The exchangeability, as in Rhodes and Zhou (2024), assumes away systematic quality differences across firms. The marginal CDF and PDF of x_i are denoted by $G(\cdot)$ and $g(\cdot)$, respectively. Further, consumers' choosiness t is distributed on $[\underline{t}, \overline{t}]$, with $0 \leq \underline{t} < \overline{t} < \infty$, according to a

[3] To see this, note that a consumer's utility of buying from firm i is proportional to $v/t + x_i - p_i/t$: A larger t implies a lower price sensitivity.

differentiable CDF $F(\cdot)$ with a PDF $f(\cdot)$. For simplicity, we assume that t and $\boldsymbol{x}$ are independent: Firms cannot draw inferences about one dimension of a consumer's preferences from data regarding the other.[4]

We assume that the base utility v is commonly known, but firms may not observe $\boldsymbol{x}$ and/or t.[5] The availability of consumer information determines firms' ability to tailor their prices. We consider four pricing regimes. Under *uniform pricing*, each firm, lacking access to consumer data, sets a single price $p_i^{\mathcal{U}}$ for all. Under *loyalty-based pricing* (resp., *choosiness-based pricing*), each firm sets (partially) personalized pricing $p_i^{\mathcal{L}}(\boldsymbol{x})$ (resp., $p_i^{\mathcal{C}}(t)$) based on $\boldsymbol{x}$ (resp., t). Under *fully personalized pricing*, a firm offers each consumer a price fully customized according to both $\boldsymbol{x}$ and t.[6]

A consumer $(\boldsymbol{x}, t)$ purchases the product provided by firm i if

$$v + tx_i - p_i \geq \max_{j \neq i}\{v + tx_j - p_j\}. \tag{1}$$

In case a consumer is indifferent between multiple products, she chooses the one with the highest gross valuation. We assume throughout the paper, except in Section 5, that v is sufficiently large—more precisely, $v \geq c + 2\overline{x}\overline{t}$—such that the market is fully covered.

We adopt Nash equilibrium as the solution concept for all our analyses of pricing competition and focus on pure-strategy equilibrium.

2.1 Pricing Equilibrium

We now proceed to equilibrium analysis of each pricing regime. Under uniform pricing, a symmetric pure-strategy equilibrium requires $p_j = p^{\mathcal{U}}$ for each $j \neq i$ and

[4] Miklós-Thal, Goldfarb, Haviv, and Tucker (2024) examine users' data-sharing decisions in a setting in which different dimensions of consumer data are correlated, which allows firms to infer one aspect of a user's type based on data from another. In contrast, we assume that different dimensions of consumer preferences are independently distributed. They focus on users' data-sharing choices, while we investigate the implications of personalized pricing for market and distribution outcomes under trade regulation rules.

[5] Allowing for heterogeneous base utility, v, has no effect in the presence of market competition under the premise of full market coverage. See Section 3.1 in Armstrong (2006) for more discussion.

[6] We take a "third-party" rather than a "first-party" approach, in the sense that a third-party data provider collects and provides information to firms. In practice, firms can learn about consumers' preferences based on their purchase history, from which they may be able to infer (partial) information about both dimensions. We leave this extension for future research.

(1) reduces to

$$p_i - p^{\mathcal{U}} \leq t\left(x_i - \max_{j\neq i} x_j\right).$$

Define $\hat{x}_i := x_i - \max_{j\neq i} x_j$ and denote its CDF and PDF by $\widehat{G}(\cdot)$ and $\hat{g}(\cdot)$, respectively. Further, define $z := t\hat{x}_i$ and denote the CDF and PDF of z by $H(\cdot)$ and $h(\cdot)$. We impose the following assumptions throughout the paper, except in Section 5:

Assumption 1 *$1 - \widehat{G}(\hat{x}_i)$ is log-concave in $\hat{x}_i$.*

Assumption 2 *$f(t)/t$ is log-concave in t.*

Assumption 3 *$1 - H(z)$ is log-concave in z.*

Assumptions 1, 2, and 3 ensure the existence of pure-strategy equilibria in the pricing game under $\mathcal{C}$, $\mathcal{L}$, and $\mathcal{U}$, respectively. Two remarks are in order. First, in principle, $H(\cdot)$ is known for given $\widehat{G}(\cdot)$ and $f(\cdot)$. For example, it can be verified that Assumption 3 follows from Assumption 2 if $\hat{x}_i$ is uniformly distributed (which satisfies Assumption 1). However, to the best of our knowledge, the literature provides no general conditions under which the survival function of two independent random variables' product—i.e., $1 - H(z)$ in our context—is log-concave. As a result, we directly impose Assumption 3 in our analysis. Second and relatedly, cautious readers may have noted that $f(t)/t$ is required to be log-concave in Assumption 2, which is not satisfied when t follows a uniform distribution. Relaxing this restriction—e.g., assuming $f(t)$ or $1 - F(t)$ to be log-concave—would render Assumption 3 less likely to be satisfied. To see this, suppose that both $\hat{x}_i$ and t follow uniform distributions. It can be verified that Assumption 3 would be violated.

Lemma 1 ***(Equilibrium Characterization under $\mathcal{U}$, $\mathcal{C}$, and $\mathcal{F}$)*** *The following statements hold:*

(i) There exists a unique symmetric equilibrium under uniform pricing $\mathcal{U}$, in which each firm sets a price $p^{\mathcal{U}} = c + \frac{1}{nh(0)} = c + \frac{1}{n\hat{g}(0)\mathbb{E}[\frac{1}{t}]}$.

(ii) Fix a realized choosiness level t. There exists a unique symmetric equilibrium under choosiness-based pricing $\mathcal{C}$, in which each firm sets a price $p^{\mathcal{C}}(t) = c + \frac{t}{n\hat{g}(0)}$.

(iii) *Consider fully personalized pricing $\mathcal{F}$. Without loss of generality, fix a realized profile of consumer brand preferences $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$ with $x_1 > \cdots > x_n$. The pricing game yields a unique equilibrium outcome: The most preferred firm—i.e., firm 1—charges an equilibrium price $p_1^{\mathcal{F}}(\boldsymbol{x}, t) = c + t(x_1 - x_2)$ and monopolizes the market; the second most preferred firm—i.e., firm 2—charges an equilibrium price $p_2^{\mathcal{F}}(\boldsymbol{x}, t) = c$.*[7]

The equilibrium characterization for uniform pricing $\mathcal{U}$ and fully personalized pricing $\mathcal{F}$ can be obtained by adapting Lemmas 1 and 2 in Rhodes and Zhou (2024), respectively: Firms are uninformed under the former and perfectly informed in the latter. Similarly, choosiness-based pricing $\mathcal{C}$ is equivalent to uniform pricing in Rhodes and Zhou (2024) with a fixed t.

Firms remain symmetric under choosiness-based pricing $\mathcal{C}$. In the symmetric equilibrium, each firm ultimately matters to a consumer only if she ranks its product in the first or second place. Recall that $\hat{x}_i := x_i - \max_{j \neq i} x_j$ with a PDF $\hat{g}(\cdot)$. The equilibrium price is thus determined by $\hat{g}(0)$—i.e., the density of consumers "neutral" to the two most preferred firms. Loyalty-based pricing $\mathcal{L}$, in contrast, renders firms asymmetric once $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$ is realized. Fixing $\boldsymbol{x}$, denote by $k(\boldsymbol{x})$ the number of firms with positive demand in the equilibrium. The following result ensues.

Lemma 2 ***(Equilibrium Characterization under $\mathcal{L}$)*** *Consider loyalty-based pricing $\mathcal{L}$. Without loss of generality, fix a realized profile of consumer brand preferences $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$ with $x_1 > \cdots > x_n$. The pricing game yields a unique equilibrium outcome: If $f(\underline{t}) \geq 1/\underline{t}$, then $k(\boldsymbol{x}) = 1$ and the market is monopolized by consumers' most preferred firm 1; otherwise, $k(\boldsymbol{x}) \geq 2$ and the market is segmented by a set of consumers' most preferred firms, i.e., $\{1, \ldots, k(\boldsymbol{x})\}$.*[8] *In particular, if $\underline{t} = 0$, then $k(\boldsymbol{x}) = n$.*

For a given realization of $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$, the interim equilibrium outcome for the case of $f(\underline{t}) < 1/\underline{t}$ can be intuitively described as follows. There exist a set of cutoffs $\big(\alpha_1(\boldsymbol{x}), \ldots, \alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\big)$, with $\alpha_0(\boldsymbol{x}) := \bar{t} > \alpha_1(\boldsymbol{x}) > \cdots > \alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x}) > \alpha_{k(\boldsymbol{x})}(\boldsymbol{x}) := \underline{t}$. A consumer purchases the product from her i-th most preferred firm—i.e., firm i in

[7] All other firms charge $p_i^{\mathcal{F}}(\boldsymbol{x}, t) \geq c$ for $i \in \{3, \ldots, n\}$.

[8] The interim equilibrium characterization established in Lemma 2 resembles that in a vertical differentiation model (e.g., Shaked and Sutton, 1982, 1983): x_i can alternatively be interpreted as firm i's product quality, and t as consumers' marginal valuation of quality.

this context—if and only if her choosiness level t falls in the interval $\big[\alpha_i(\boldsymbol{x}), \alpha_{i-1}(\boldsymbol{x})\big)$, which represents the i-th choosiest consumer segment in the equilibrium. Details of the equilibrium are provided in the proof of the lemma in the Appendix.

Without knowing t, a firm would be subject to the usual marginal-inframarginal trade-off: A competitive price allows the firm to sell to less choosy consumers—i.e., those with low t—but prevents the firm from extracting surplus from their choosier counterparts, who are more willing to pay for better-matched products. By Lemma 2, the equilibrium outcome sensitively depends on the amount of the least choosy consumers in the population—i.e., the value of $f(\underline{t})$. Imagine that $f(\underline{t}) \geq 1/\underline{t}$, which implies significant presence of such consumers. It is too costly to lose these consumers, which forces firm 1 to lower its price and induces more intense price competition. In the Appendix, we show that firms 2 to n each charge their marginal cost c in the most intuitive equilibrium, while the preferred firm 1 responds by charging a markup $\underline{t}(x_1 - x_2)$ to remain competitive, even for the least choosy consumers. The resultant equilibrium is efficient, since all consumers buy their favorite product.

However, retaining these consumers is costly to firm 1 because of this marginal-inframarginal trade-off. When the density of the least choosy consumers is small—i.e., when $f(\underline{t}) < 1/\underline{t}$—firm 1 would instead raise its price to secure a premium from choosy consumers, while leaving a positive residual demand to others. This softens the price competition: As Lemma 2 and the Appendix show, at least two firms charge a price above the marginal cost c. The segmented market causes inefficiency, because some consumers are lured by lower prices and end up consuming less preferred products.

2.2 Preliminaries of Welfare and Profit Ranking

Next, we compare consumer welfare and industry profit across pricing regimes. Denote, respectively, by W^j, V^j, and Π^j the expected equilibrium total surplus, consumer welfare, and industry profit under pricing regime $j \in \{\mathcal{U}, \mathcal{C}, \mathcal{L}, \mathcal{F}\}$. The following ensues from Lemmas 1 and 2.

Lemma 3 ***(Potential Market Inefficiency with Full Market Coverage)*** *The following statements hold:* $W^{\mathcal{U}} = W^{\mathcal{C}} = W^{\mathcal{F}} = W^{\mathcal{L}}$ *if* $f(\underline{t}) \geq 1/\underline{t}$, *and* $W^{\mathcal{U}} = W^{\mathcal{C}} = W^{\mathcal{F}} > W^{\mathcal{L}}$ *if* $f(\underline{t}) < 1/\underline{t}$.

With full market coverage, the equilibrium outcome is efficient as long as every consumer buys from their preferred firm. All of these pricing regimes generate efficient outcomes except for the case of loyalty-based pricing $\mathcal{L}$ when $f(\underline{t}) < 1/\underline{t}$: As Lemma 2 shows, less choosy consumers—i.e., those with $t < \alpha_1(\boldsymbol{x})$—end up with a less preferred product, which causes efficiency loss.

Lemma 4 ***(Armstrong, 2006; Rhodes and Zhou, 2024)*** *The following statements hold:* $V^{\mathcal{F}} > V^{\mathcal{U}} > V^{\mathcal{C}}$ *and* $\Pi^{\mathcal{C}} > \Pi^{\mathcal{U}} > \Pi^{\mathcal{F}}$.

This setting is equivalent to that of Rhodes and Zhou (2024) when comparing uniform pricing $\mathcal{U}$ with fully personalized pricing $\mathcal{F}$: Firms are completely uninformed in the former case and perfectly informed in the latter, which suggests $V^{\mathcal{F}} > V^{\mathcal{U}}$. This affirms the conventional wisdom in the literature: Competitive personalized pricing renders every consumer contestable, which enables firms to poach other firms' loyal consumers and thus intensifies market competition (see, e.g., Thisse and Vives, 1988; Shaffer and Zhang, 2002; Chen and Iyer, 2002; Montes, Sand-Zantman, and Valletti, 2019; Chen, Choe, and Matsushima, 2020). Given $W^{\mathcal{F}} = W^{\mathcal{U}}$, we can conclude $\Pi^{\mathcal{F}} < \Pi^{\mathcal{U}}$.

However, allowing firms to acquire information about choosiness t harms consumers and renders them even worse off than under uniform pricing $\mathcal{U}$.[9] Without knowing t, a firm prices by the average, which yields $p^{\mathcal{U}} = c + 1/\left[n\hat{g}(0)\mathbb{E}[1/t]\right]$. Firms remain symmetric upon knowing t and continue to compete for "neutral" consumers, but they customize their prices for every t, with $p^{\mathcal{C}}(t) = c + t/\left[n\hat{g}(0)\right]$. Choosy consumers—i.e., those with $t > 1/\mathbb{E}[1/t]$—are charged a higher price under choosiness-based pricing $\mathcal{C}$, while the less choosy—i.e., those with $t < 1/\mathbb{E}[1/t]$—pay less. The conclusion is mathematically straightforward by Cauchy-Schwartz inequality; its economic rationale is also intuitive. Revealing t softens the price competition for choosy consumers: A larger t amplifies the advantage of the more preferred product and renders undercutting less effective when poaching others' loyal customers. In contrast, a smaller t limits the degree of perceived product differentiation and encourages competition, since a lower price is more likely to lure a consumer away from her most preferred product. Compared with uniform pricing $\mathcal{U}$, firms gain by knowing t from the more valuable consumers—i.e., those with larger t and higher willingness to pay—while losing from the less valuable. The information thus benefits firms in general, but harms consumers as a whole.

[9]This observation was first noted by Armstrong (2006) in a standard Hotelling duopoly model, but extending it to a general discrete choice model is straightforward.

Lemma 4 paves the way for our search for the consumer-optimal pricing regime. It suffices to compare the case of fully personalized pricing $\mathcal{F}$ with that of loyalty-based pricing $\mathcal{L}$. In what follows, we first present a case of duopoly with $n = 2$, then proceed to a case of oligopoly with $n \geq 3$.

3 Case of Duopoly

With $n = 2$, the pricing equilibrium in Lemma 2 can be simplified as follows.

Corollary 1 *Consider loyalty-based pricing $\mathcal{L}$ and fix $\boldsymbol{x} \equiv (x_1, x_2)$, with $x_1 > x_2$. If $f(\underline{t}) \geq 1/\underline{t}$, the equilibrium prices satisfy*

$$p_1^{\mathcal{L}}(\boldsymbol{x}) = c + \underline{t}(x_1 - x_2) \text{ and } p_2^{\mathcal{L}}(\boldsymbol{x}) = c. \tag{2}$$

If $f(\underline{t}) < 1/\underline{t}$, the equilibrium prices satisfy

$$p_1^{\mathcal{L}}(\boldsymbol{x}) = p_2^{\mathcal{L}}(\boldsymbol{x}) + (x_1 - x_2)\alpha^* \text{ and } p_2^{\mathcal{L}}(\boldsymbol{x}) = c + (x_1 - x_2)\frac{F(\alpha^*)}{f(\alpha^*)}, \tag{3}$$

where $\alpha^* \in (\underline{t}, \bar{t}\,)$ uniquely solves

$$\alpha^* f(\alpha^*) = 1 - 2F(\alpha^*). \tag{4}$$

Specifically, when $f(\underline{t}) < 1/\underline{t}$, firm 1 sells to consumers with a choosiness level above the cutoff α^* and leaves firm 2 to sell to the rest. By Corollary 1, $x_1 - x_2$ is a sufficient statistic for consumers' brand preferences $\boldsymbol{x} \equiv (x_1, x_2)$ under duopoly. Comparing $V^{\mathcal{L}}$ with $V^{\mathcal{F}}$, together with Lemma 4, yields the following main result.

Proposition 1 ***(Consumer Welfare Comparison under Duopoly)*** *Fix $n = 2$. Either fully personalized pricing $\mathcal{F}$ or loyalty-based pricing $\mathcal{L}$ maximizes consumer welfare. Comparing $V^{\mathcal{L}}$ with $V^{\mathcal{F}}$ yields the following:*

(i) If $f(\underline{t}) \geq 1/\underline{t}$ or

$$\int_{\alpha^*}^{\bar{t}} [1 - F(t)]dt > \frac{F(\alpha^*)}{f(\alpha^*)}, \tag{5}$$

then $V^{\mathcal{L}} > V^{\mathcal{F}}$, with loyalty-based pricing $\mathcal{L}$ maximizing consumer welfare.

(ii) If $f(\underline{t}) < 1/\underline{t}$ and

$$\int_{\alpha^*}^{\bar{t}} [1 - F(t)]dt < \frac{F(\alpha^*)}{f(\alpha^*)}, \tag{6}$$

then $V^{\mathcal{F}} > V^{\mathcal{L}}$, with fully personalized pricing $\mathcal{F}$ maximizing consumer welfare.

Proposition 1 provides necessary and sufficient conditions for the consumer welfare comparison between fully personalized pricing $\mathcal{F}$ and loyalty-based pricing $\mathcal{L}$ under duopoly. As noted above, the literature on competitive personalized pricing suggests that firms' access to consumer information intensifies market competition and benefits consumers (see, e.g., Thisse and Vives, 1988; Rhodes and Zhou, 2024). These studies typically assume unidimensional consumer characteristics. Proposition 1 shows that with multidimensional characteristics, consumer welfare can be maximized by *partial* access to consumer data and limited inferences regarding consumer attributes.

We now elaborate on the underlying logic. Fixing $(\boldsymbol{x}, t)$, recall by Lemma 1 that in the equilibrium under fully personalized pricing $\mathcal{F}$, firms charge

$$p_1^{\mathcal{F}}(\boldsymbol{x}, t) = c + t(x_1 - x_2) \text{ and } p_2^{\mathcal{F}}(\boldsymbol{x}, t) = c,$$

with firm 1 monopolizing the market.

To compare this with loyalty-based pricing $\mathcal{L}$, we begin with the case of $f(\underline{t}) < 1/\underline{t}$, which implies limited presence of the least choosy consumers—i.e., those with $t = \underline{t}$. By (3), under loyalty-based pricing, firms charge

$$p_1^{\mathcal{L}}(\boldsymbol{x}) = c + (x_1 - x_2)\left[\alpha^* + \frac{F(\alpha^*)}{f(\alpha^*)}\right] \text{ and } p_2^{\mathcal{L}}(\boldsymbol{x}) = c + (x_1 - x_2)\frac{F(\alpha^*)}{f(\alpha^*)}.$$

By Lemma 2 and Corollary 1, the market is segmented in the equilibrium, and we illustrate the market outcome in Figure 1(a). Without knowing t, a firm has to set a flat price for all, which triggers the marginal-inframarginal trade-off: To capture a larger market share, firm 1 has to forgo the surplus it could otherwise extract from choosier consumers. As noted above, the firm surrenders the segment of less choosy consumers—i.e., those with $t \in [\underline{t}, \alpha^*)$—to firm 2 and sells exclusively to those with higher gross valuations, i.e., those with $t \in [\alpha^*, \bar{t}]$.

Less choosy consumers are surely worse off under loyalty-based pricing $\mathcal{L}$. Under fully personalized pricing $\mathcal{F}$, these consumers would be indifferent between purchasing from firm 1 for a price $p_1^{\mathcal{F}}(\boldsymbol{x}, t) = c + t(x_1 - x_2)$ and firm 2 for a price $p_2^{\mathcal{F}}(\boldsymbol{x}, t) = c$.

Now, under loyalty-based pricing $\mathcal{L}$, they have to purchase from the less preferred firm 2, while paying more than c.

Choosy consumers—i.e., those with $t \in [\alpha^*, \bar{t}]$—purchase from their preferred firm 1 under loyalty-based pricing $\mathcal{L}$. The flat price they pay—$p_1^{\mathcal{L}}(\boldsymbol{x}) = c + (x_1 - x_2)[\alpha^* + F(\alpha^*)/f(\alpha^*)]$—can be either higher or lower than $p_1^{\mathcal{F}}(\boldsymbol{x}, t) = c + t(x_1 - x_2)$, as illustrated in Figure 1(a). Moderately choosy consumers—i.e., $t \in [\alpha^*, t^*)$—pay more, i.e., $p_1^{\mathcal{L}}(\boldsymbol{x}) > p_1^{\mathcal{F}}(\boldsymbol{x}, t)$; while very choosy consumers—i.e., $t \in (t^*, \bar{t}]$—pay less, i.e., $p_1^{\mathcal{L}}(\boldsymbol{x}) < p_1^{\mathcal{F}}(\boldsymbol{x}, t)$, which reflects the marginal-inframarginal trade-off.

Very choosy consumers benefit from the presence of their moderately choosy counterpart: Firm 1 must keep its flat price sufficiently low to retain those with $t \in [\alpha^*, t^*)$. In contrast, moderately choosy consumers are caught in the middle: They end up paying more, because firm 1 is unwilling to forgo too much of the surplus it can extract from the very choosy. At the same time, their valuations for the preferred product are high enough such that they are unwilling to settle for their second choice.

In summary, consumers with $t \in (t^*, \bar{t}]$ benefit from loyalty-based pricing $\mathcal{L}$ compared with fully personalized pricing $\mathcal{F}$, while all others are worse off.[10]

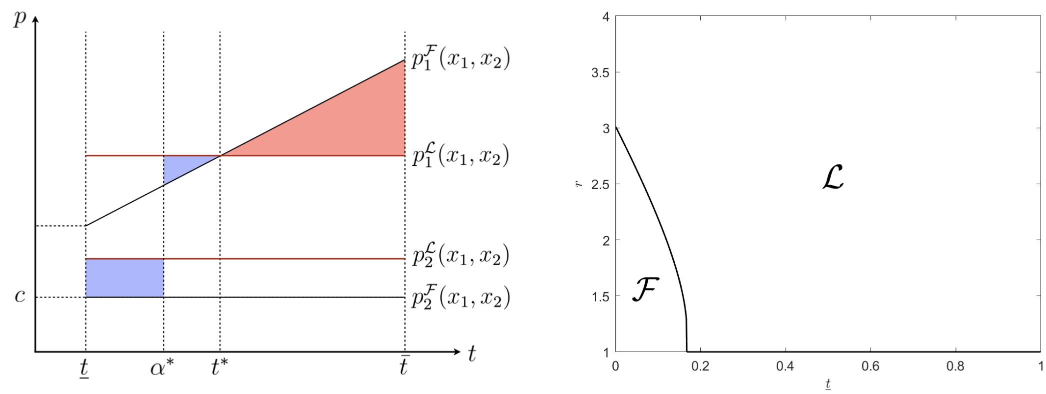
Summing up the relative gains and losses leads to condition (5). Since only very choosy consumers prefer loyalty-based pricing $\mathcal{L}$, $V^{\mathcal{L}}$ is more likely to exceed $V^{\mathcal{F}}$ when they make up a larger share of the population, such that more individuals benefit while fewer suffer. This can be reflected by an upward shift in the probability mass for t above the cutoff α^* toward the upper end of its support. In that case, equilibrium prices remain unchanged, while $F(t)$ decreases for all $t \in [\alpha^*, \bar{t})$ and renders the condition $\int_{\alpha^*}^{\bar{t}} [1 - F(t)]dt > F(\alpha^*)/f(\alpha^*)$ more likely to hold. We construct the following example to illustrate this logic.

Example 1 *We set $\bar{t} = 1$ and parameterize the CDF $F(\cdot)$ as follows:*

$$F(t; \underline{t}, r) = \left(\frac{t - \underline{t}}{1 - \underline{t}} \right)^r, \text{ with } \underline{t} \in (0,1) \text{ and } r \geq 1.$$

It is straightforward to verify that $f(\underline{t}) = 0 < 1/\underline{t}$ for all $\underline{t} \in (0,1)$ and $r \geq 1$. As $\underline{t}$ or r increases, the probability densities would be concentrated more on large values of

[10] Recall that a larger t indicates a consumer with higher income and lower price sensitivity. These observations point to a (less desirable) distributional outcome of loyalty-based pricing: Although it can maximize aggregate consumer welfare, the gains relative to fully personalized pricing accrue to high-income consumers, whereas the losses are borne by their low-income counterparts.



(a) Equilibrium pricing schedules

(b) Consumer welfare

Figure 1: Loyalty-based pricing vs. fully personalized pricing: $f(\underline{t}) < 1/\underline{t}$.

t, which implies that condition (5) *is more likely to be satisfied. Figure 1(b) depicts the pricing regime that maximizes consumers' welfare. The horizontal axis represents $\underline{t}$ and the vertical axis r, with the former ranging from 0 to 1 and the latter from 0 to 4. The solid curve is defined by the condition $\int_{\alpha^*}^{\bar{t}}[1-F(t)]dt = F(\alpha^*)/f(\alpha^*)$; the region to its right collects all $(\underline{t}, r)$ under which loyalty-based pricing outperforms fully personalized pricing in terms of consumer welfare. That is, loyalty-based pricing $\mathcal{L}$ tends to prevail for large $\underline{t}$ and r, which affirms the intuition laid out above.*

Suppose instead that $f(\underline{t}) \geq 1/\underline{t}$, which implies a substantial presence of the least choosy consumers. In this scenario, forgoing these consumers becomes excessively costly for firm 1 and prompts it to lower its price so that even the least choosy consumer would prefer its product. As a result, consumers pay $p_1^{\mathcal{L}}(\boldsymbol{x}) = c+\underline{t}(x_1-x_2)$, which is lower than the price under fully personalized pricing $\mathcal{F}$—i.e., $p_1^{\mathcal{F}}(\boldsymbol{x}, t) = c+t(x_1-x_2)$—for all $t \in [\underline{t}, \bar{t}\,]$. Without revealing t, the marginal-inframarginal trade-off benefits *every* consumer: The significant presence of the least choosy consumers generates a positive externality to the rest of the population.

Recall that the cutoff type α^* in the case $f(\underline{t}) < 1/\underline{t}$ is determined by $\alpha^* f(\alpha^*) = 1-2F(\alpha^*)$. If $f(\underline{t})$ rises to $1/\underline{t}$—i.e., when the condition $f(\underline{t}) \geq 1/\underline{t}$ is met—then $F(\alpha^*)$ must equal zero, and both α^* and t^* depicted in Figure 1(a) degenerate to $\underline{t}$. In other words, every consumer is sufficiently choosy (i.e., $t \geq t^* = \underline{t}$) and pays less under loyalty-based pricing $\mathcal{L}$ than fully personalized pricing $\mathcal{F}$, which ensures $V^{\mathcal{L}} > V^{\mathcal{F}}$.

Although ranking consumer welfare across pricing regimes causes complications, firms' profits can be ranked unambiguously.

Proposition 2 ***(Industry Profit Comparison under Duopoly)*** *Fix* $n = 2$. *The equilibrium profits under the four pricing regimes can be ranked as follows:* $\Pi^{\mathcal{C}} > \Pi^{\mathcal{U}} > \Pi^{\mathcal{F}} > \Pi^{\mathcal{L}}$. *That is, choosiness-based pricing maximizes industry profit, while loyalty-based pricing minimizes it.*

The first two inequalities follow directly from Lemmas 3 and 4. However, the comparison between $\Pi^{\mathcal{F}}$ and $\Pi^{\mathcal{L}}$ is less explicit. Under the condition $f(\underline{t}) < 1/\underline{t}$, loyalty-based pricing $\mathcal{L}$ generates lower total surplus than fully personalized pricing $\mathcal{F}$ due to inefficient allocation—i.e., $W^{\mathcal{L}} < W^{\mathcal{F}}$; it also generates less consumer welfare—i.e., $V^{\mathcal{L}} < V^{\mathcal{F}}$—provided condition (6) holds.

We fix $\boldsymbol{x} \equiv (x_1, x_2)$ and compare industry profit under loyalty-based pricing $\mathcal{L}$ with that under fully personalized pricing $\mathcal{F}$. As discussed above, under loyalty-based pricing $\mathcal{L}$, firm 1 earns more from moderately choosy consumers—i.e., those with $t \in [\alpha^*, t^*)$—but loses on very choosy consumers—i.e., those with $t \in [t^*, \bar{t}\,]$—relative to fully personalized pricing $\mathcal{F}$. Moreover, firm 1 gives up less choosy consumers—i.e., those with $t \in [\underline{t}, \alpha^*)$—while firm 2 is now able to secure positive profit from this consumer segment.

However, the industry's relative gain under loyalty-based pricing $\mathcal{L}$ does not compensate for its corresponding loss. Firm 1 surrenders surplus from its most valuable consumers; its gain only comes from those with $t \in [\alpha^*, t^*)$—a segment whose value is constrained by their moderate willingness to pay and the competitive pressure from firm 2. Meanwhile, firm 2 captures less choosy consumers but its profitability is also limited: These consumers now end up with their less preferred product, which further suppresses their willingness to pay. Proposition 2 verifies that these losses exceed the industry's gain in profit under loyalty-based pricing $\mathcal{L}$, which leads to $\Pi^{\mathcal{L}} < \Pi^{\mathcal{F}}$.

It is worth noting that since $\Pi^{\mathcal{C}} > \Pi^{\mathcal{U}}$ and $\Pi^{\mathcal{F}} > \Pi^{\mathcal{L}}$, the industry always benefits from knowing consumers' choosiness t, regardless of the availability of information about $\boldsymbol{x}$. Intuitively, knowing t alleviates each firm's marginal-inframarginal trade-off, which enables more flexible pricing strategies and more effective surplus extraction.

4 Case of Oligopoly

We now extend our analysis to the case with $n \geq 3$. As before, we focus on the comparison between loyalty-based pricing $\mathcal{L}$ and fully personalized pricing $\mathcal{F}$, since consumer welfare is maximized under either regime. We first outline the key preliminaries, which set the stage for our main predictions.

4.1 Preliminaries: Interim Consumer Welfare

Fix a realized profile of consumer brand preferences $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$, with $x_1 > \cdots > x_n$. With slight abuse of notation, let $V^{\mathcal{L}}(\boldsymbol{x})$ and $V^{\mathcal{F}}(\boldsymbol{x})$ denote the *interim* consumer welfare in the equilibrium under loyalty-based pricing $\mathcal{L}$ and fully personalized pricing $\mathcal{F}$, respectively. More formally, define

$$V^{\mathcal{L}}(\boldsymbol{x}) := \int_{\underline{t}}^{\bar{t}} \max\left\{v + tx_1 - p_1^{\mathcal{L}}(\boldsymbol{x}), \ldots, v + tx_n - p_n^{\mathcal{L}}(\boldsymbol{x})\right\} dF(t), \tag{7}$$

$$V^{\mathcal{F}}(\boldsymbol{x}) := \int_{\underline{t}}^{\bar{t}} (v + tx_2 - c)dF(t). \tag{8}$$

We begin with the following thought experiment. Suppose hypothetically that a least preferred firm $n+1$, with $x_{n+1} < x_n$, is introduced to the market. This addition leads to the following.

Lemma 5 ***(Interim Consumer Welfare and Number of Firms)*** *Fix $n \geq 2$ and $\boldsymbol{x}_n \equiv (x_1, \ldots, x_n)$, with $x_1 > \cdots > x_n$. For each realized profile of consumer brand preferences $\boldsymbol{x}_{n+1} := (x_1, \ldots, x_n, x_{n+1})$ with $x_{n+1} < x_n$, the following holds:*

$$V^{\mathcal{L}}(\boldsymbol{x}_{n+1}) \geq V^{\mathcal{L}}(\boldsymbol{x}_n) \text{ and } V^{\mathcal{F}}(\boldsymbol{x}_{n+1}) = V^{\mathcal{F}}(\boldsymbol{x}_n).$$

With a realized $\boldsymbol{x}_n$, when the hypothetical least preferred firm is introduced to the market, loyalty-based pricing $\mathcal{L}$ is more likely to prevail over fully personalized pricing $\mathcal{F}$ in terms of interim equilibrium consumer welfare. Under fully personalized pricing $\mathcal{F}$, consumer welfare remains unchanged, whereas it weakly improves under loyalty-based pricing $\mathcal{L}$.

Recall from Lemma 1(iii) that firms engage in asymmetric Bertrand competition under fully personalized pricing $\mathcal{F}$; so the interim equilibrium boils down to the head-to-head competition between the two most preferred firms, which implies that the additional firm does not affect the market outcome. In contrast, as shown in Lemma 2, under loyalty-based pricing $\mathcal{L}$, the market can be split by *three or more firms* when $f(\underline{t}) < 1/\underline{t}$. The additional firm could reshape the equilibrium, intensify market competition, and ultimately benefit consumers at the cost of firms.

We continue our thought experiment to explore how this least preferred firm improves consumer welfare under loyalty-based pricing $\mathcal{L}$. Define

$$\widetilde{V}^{\mathcal{L}}(\boldsymbol{x}_{n+1}) := \int_{\underline{t}}^{\bar{t}} \max\left\{v + tx_1 - p_1^{\mathcal{L}}(\boldsymbol{x}_{n+1}), \ldots, v + tx_n - p_n^{\mathcal{L}}(\boldsymbol{x}_{n+1})\right\} dF(t). \tag{9}$$

This represents the hypothetical interim consumer welfare when consumers are required to purchase from the first n firms at prices determined by the equilibrium that involves $n+1$ firms. The impact on consumer welfare can be decomposed as follows:

$$V^{\mathcal{L}}(\boldsymbol{x}_{n+1}) - V^{\mathcal{L}}(\boldsymbol{x}_n) = \underbrace{\left[\widetilde{V}^{\mathcal{L}}(\boldsymbol{x}_{n+1}) - V^{\mathcal{L}}(\boldsymbol{x}_n)\right]}_{\text{price effect}} + \underbrace{\left[V^{\mathcal{L}}(\boldsymbol{x}_{n+1}) - \widetilde{V}^{\mathcal{L}}(\boldsymbol{x}_{n+1})\right]}_{\text{choice effect}}.$$

Two effects come into play when the least preferred firm enters the market under loyalty-based pricing $\mathcal{L}$. The first is a *price effect*: Consumers choose from the same set of products $(x_1, \ldots, x_n)$ at the new equilibrium prices $(p_1^{\mathcal{L}}(\boldsymbol{x}_{n+1}), \ldots, p_n^{\mathcal{L}}(\boldsymbol{x}_{n+1}))$, which are formed after the addition of firm $n+1$. The second is a *choice effect*: With equilibrium prices fixed at $(p_1^{\mathcal{L}}(\boldsymbol{x}_{n+1}), \ldots, p_{n+1}^{\mathcal{L}}(\boldsymbol{x}_{n+1}))$, consumers now have the option of purchasing from firm $n+1$.

Lemma 6 ***(Price Effect and Choice Effect)*** *Both the price effect and the choice effect are nonnegative, i.e.,*

$$\widetilde{V}^{\mathcal{L}}(\boldsymbol{x}_{n+1}) - V^{\mathcal{L}}(\boldsymbol{x}_n) \geq 0 \text{ and } V^{\mathcal{L}}(\boldsymbol{x}_{n+1}) - \widetilde{V}^{\mathcal{L}}(\boldsymbol{x}_{n+1}) \geq 0.$$

Moreover, the price effect is weakly increasing in x_{n+1}, while the choice effect is nonmonotone.

By Lemma 6, consumers benefit from both channels in the interim equilibrium. The price effect is more pronounced when the newly added firm better matches consumers' taste—i.e., when its brand-specific value approaches x_n. However, the choice effect—i.e., the value of an additional option—is nonmonotone with respect to its brand-specific value of the new firm: When x_{n+1} is much smaller than x_n, the new option is largely irrelevant for consumers. Conversely, when x_{n+1} is very close to x_n, the additional choice provides little benefit, since purchasing from firm $n+1$ is almost the same as purchasing from firm n.

4.2 Main Results

We now compare ex ante expected consumer welfare under loyalty-based pricing $\mathcal{L}$ and fully personalized pricing $\mathcal{F}$. The analysis extends from the case of duopoly and examines how the comparison evolves as more firms are included.

For this purpose, we introduce two indices that are critical for our analysis. Denote the j-th order statistics of $\boldsymbol{x}$ by $x^{(j)}$, with $j \in \{1, \dots, n\}$, so that $x^{(1)} > \cdots > x^{(n)}$. Fixing a constant $\kappa \in (0,1)$ and a joint distribution of consumers' brand loyalty $\tilde{g}(\cdot)$, we define

$$\mathcal{C}_1(\tilde{g}, \kappa) := \frac{\mathbb{E}_{\boldsymbol{x}\sim\tilde{g}(\cdot)}\left[\mathbb{1}\left(\frac{x^{(2)}-x^{(3)}}{x^{(1)}-x^{(2)}} \leq \kappa\right) \times \left(x^{(1)} - x^{(2)}\right)\right]}{\mathbb{E}_{\boldsymbol{x}\sim\tilde{g}(\cdot)}\left[\mathbb{1}\left(\frac{x^{(2)}-x^{(3)}}{x^{(1)}-x^{(2)}} \geq \kappa\right) \times \left(x^{(1)} - x^{(2)}\right)\right]}.$$

The index $\mathcal{C}_1(\tilde{g}, \kappa)$ is an intuitive measure of the impact of the third most preferred firm under loyalty-based pricing $\mathcal{L}$. Ceteris paribus, a higher value of the index implies a greater likelihood of the event $[x^{(2)} - x^{(3)}]/[x^{(1)} - x^{(2)}] \leq \kappa$, given the distribution $\tilde{g}(\cdot)$. A smaller numerator $x^{(2)} - x^{(3)}$ indicates that the third most preferred firm is closer to the second most preferred firm in terms of consumers' loyalty; this implies a stronger price effect, as shown in Lemma 6. Similarly, a large denominator $x^{(1)} - x^{(2)}$ implies a more lopsided and softer competition between the two most preferred firms, which implies that an additional firm can play a more significant role in intensifying competition. In short, a larger $\mathcal{C}_1(\tilde{g}, \kappa)$ indicates greater likelihood of an impactful third firm under loyalty-based pricing $\mathcal{L}$ in favor of consumers through the price effect, given a constant κ and distribution $\tilde{g}(\cdot)$.

Next, fixing a distribution of consumer choosiness $f(\cdot)$, we define

$$\mathcal{C}_2(f, \kappa) := \frac{\widetilde{V}^{\mathcal{L}}(\kappa + 1, \kappa, 0) - V^{\mathcal{F}}(\kappa + 1, \kappa, 0)}{V^{\mathcal{F}}(\kappa + 1, \kappa) - V^{\mathcal{L}}(\kappa + 1, \kappa)}.$$

We now present the main result; the index $\mathcal{C}_2(f, \kappa)$ will be interpreted later.

Proposition 3 ***(Consumer Welfare Comparison under Oligopoly)*** *Either* *fully personalized pricing $\mathcal{F}$ or loyalty-based pricing $\mathcal{L}$ maximizes consumer welfare. The following statements hold regarding the comparison between $V^{\mathcal{L}}$ and $V^{\mathcal{F}}$:*

(i) Suppose $V^{\mathcal{L}} > V^{\mathcal{F}}$ for the case when $n = 2$; that is, $f(\underline{t}) \geq 1/\underline{t}$ or condition (5) holds. Then $V^{\mathcal{L}} > V^{\mathcal{F}}$ for all $n \geq 3$.

(ii) Suppose $V^{\mathcal{L}} < V^{\mathcal{F}}$ for the case when $n = 2$; that is, $f(\underline{t}) < 1/\underline{t}$ and condition (6) holds. If there exists $\kappa \in (0, 1)$ such that

$$\mathcal{C}_1(\tilde{g}, \kappa) \times \mathcal{C}_2(f, \kappa) > 1, \tag{10}$$

then $V^{\mathcal{L}} > V^{\mathcal{F}}$ for all $n \geq 3$.

Proposition 3 establishes the conditions under which loyalty-based pricing $\mathcal{L}$ or fully personalized pricing $\mathcal{F}$ maximizes consumer welfare in an oligopoly. Comparing Proposition 3 with Proposition 1, we can conclude that increasing the number of firms in the market strengthens the case for loyalty-based pricing $\mathcal{L}$.[11] By Proposition 3(i), if loyalty-based pricing $\mathcal{L}$ maximizes consumer welfare under duopoly, this ranking remains unchanged in an oligopoly. Moreover, Proposition 3(ii) further states that if loyalty-based pricing underperforms under duopoly, the comparison can be reversed in an oligopoly, provided condition (10) is satisfied. An example that illustrates this possibility is provided in Section 4.3.

As in the duopoly case, equilibrium industry profit can again be unambiguously ranked across the four pricing regimes.

Proposition 4 ***(Industry Profit Comparison under Oligopoly)*** *Fix $n \geq 3$. The equilibrium profits under the four pricing regimes can be ranked as follows: $\Pi^{\mathcal{C}} > \Pi^{\mathcal{U}} > \Pi^{\mathcal{F}} > \Pi^{\mathcal{L}}$. That is, choosiness-based pricing $\mathcal{C}$ maximizes industry profit, and loyalty-based pricing $\mathcal{L}$ minimizes it.*

4.3 Further Discussion

We now interpret our results. First, the comparison is relatively straightforward when $f(\underline{t}) \geq 1/\underline{t}$ or when condition (5) holds. In this case, we have $V^{\mathcal{L}} > V^{\mathcal{F}}$ for

[11] Interestingly, adopting an oligopoly framework, Zhou (2017) highlights the importance of market structure within a context of product bundling. Specifically, Zhou finds that compared with separate sales, bundling tends to raise market prices, thus benefiting firms while harming consumers as the number of firms increases.

duopoly. Lemma 5 further implies that this would persist for $n \geq 3$, since additional firms are more likely to affect the market outcome under loyalty-based pricing $\mathcal{L}$, which leads to greater improvement in $V^{\mathcal{L}}$ compared with $V^{\mathcal{F}}$: Under fully personalized pricing $\mathcal{F}$, additional firms affect the interim equilibrium only when they rank among a consumer's two most preferred firms.

A more nuanced discussion is required for the case in which $f(\underline{t}) < 1/\underline{t}$ and condition (6) holds. The comparison under duopoly can be reversed—i.e., loyalty-based pricing $\mathcal{L}$ can surpass fully personalized pricing $\mathcal{F}$—whenever condition (10) holds. This occurs when there exists some constant $\kappa \in (0,1)$ that generates sufficiently large $\mathcal{C}_1(\tilde{g}, \kappa)$ and $\mathcal{C}_2(f, \kappa)$.

To understand the index $\mathcal{C}_2(f, \kappa)$, imagine the following: The market initially consists of two firms with brand-specific values $x_1 = \kappa + 1$ and $x_2 = \kappa$; then a third firm with $x_3 = 0$ is introduced. This index measures the impact of the third most preferred firm on consumer welfare under loyalty-based pricing $\mathcal{L}$ in this hypothetical scenario, in which $[x^{(2)} - x^{(3)}]/[x^{(1)} - x^{(2)}] = \kappa$. If $f(\underline{t}) < 1/\underline{t}$ and condition (6) holds, the denominator of the index, $V^{\mathcal{F}}(\kappa+1, \kappa) - V^{\mathcal{L}}(\kappa+1, \kappa)$, is positive for $\boldsymbol{x} = (\kappa+1, \kappa)$ under duopoly. Meanwhile, the value of the numerator, $\widetilde{V}^{\mathcal{L}}(\kappa+1, \kappa, 0) - V^{\mathcal{F}}(\kappa+1, \kappa, 0)$, is bounded from above by $V^{\mathcal{L}}(\kappa+1, \kappa, 0) - V^{\mathcal{F}}(\kappa+1, \kappa, 0)$ because the addition of the third firm also causes a nonnegative choice effect, i.e., $V^{\mathcal{L}}(\kappa+1, \kappa, 0) - \widetilde{V}^{\mathcal{L}}(\kappa+1, \kappa, 0)$: Intuitively, a larger $\mathcal{C}_2(f, \kappa)$ suggests a more significant improvement in consumer welfare under loyalty-based pricing $\mathcal{L}$ relative to fully personalized pricing $\mathcal{F}$ through the price effect in this hypothetical scenario.

The index $\mathcal{C}_2(f, \kappa)$ thus provides a lower bound for the impact of additional firms under loyalty-based pricing $\mathcal{L}$ on consumer welfare. First, it isolates the improvement in consumer welfare due to the price effect while disregarding the choice effect. Second, it assumes a third firm with $x_3 = 0$ in a duopolistic market with $(x_1, x_2) = (\kappa+1, \kappa)$. By Lemma 6, the price effect is stronger when the third most preferred firm is a closer substitute for the second. Thus, if $\mathcal{C}_2(f, \kappa)$ is sufficiently large—which indicates a strong price effect when $[x^{(2)} - x^{(3)}]/[x^{(1)} - x^{(2)}] = \kappa$—we can conclude that the price effect is even more pronounced if a new added firm has a brand-specific value greater than zero, which leads to the event $[x^{(2)} - x^{(3)}]/[x^{(1)} - x^{(2)}] < \kappa$.

Therefore, a large $\mathcal{C}_1(\tilde{g}, \kappa) \times \mathcal{C}_2(f, \kappa)$ indicates that for a given κ, having additional firms is sufficiently likely to significantly improve consumer welfare under loyalty-based pricing $\mathcal{L}$ through the price effect alone. This leads to Proposition 3(ii): Whenever certain $\kappa \in (0,1)$ exists that satisfies condition (10)—i.e., $\mathcal{C}_1(\tilde{g},\kappa)\times\mathcal{C}_2(f,\kappa) > 1$—additional firms would cause $V^{\mathcal{L}}$ to surpass $V^{\mathcal{F}}$.

We now impose additional structure on the distribution of brand loyalty $\boldsymbol{x} \equiv (x_1,\ldots,x_n)$ to elucidate the role played by condition (10).

Corollary 2 *Suppose that x_i's are independent and identically distributed with a weakly decreasing PDF $g(\cdot)$. It can be shown that $\mathcal{C}_1(\tilde{g},1/2) \geq 5/4$ for $n \geq 3$ and $\mathcal{C}_2(f,1/2) > 4/5$ for all distributions that satisfy Assumption 2. By Proposition 1, when $f(\underline{t}) < 1/\underline{t}$ and condition (6) holds, $V^{\mathcal{F}} > V^{\mathcal{L}}$ for $n = 2$. Moreover, by Proposition 3, $V^{\mathcal{L}} > V^{\mathcal{F}}$ whenever $n \geq 3$, irrespective of the distribution $f(\cdot)$.*

Setting $\kappa = 1/2$, Corollary 2 establishes that condition (10) is satisfied, which ensures that loyalty-based pricing $\mathcal{L}$ maximizes consumer welfare broadly whenever $n \geq 3$. Intuitively, a weakly decreasing $g(\cdot)$ implies that when additional firms are introduced to the market, their realized values are more likely to remain closer to that of the second most preferred firm; consequently, the event $[x^{(2)} - x^{(3)}]/[x^{(1)} - x^{(2)}] \leq \kappa$ is more likely.

Two remarks are in order. First, condition (10) is sufficient but not necessary. As discussed above, the analysis focuses only on the price effect caused by additional firms, without accounting for the choice effect. Second, the distribution of x_i influences the comparison between $V^{\mathcal{F}}$ and $V^{\mathcal{L}}$ when $n \geq 3$, as indicated by Proposition 3 and Corollary 2. This contrasts with the duopoly case: By Proposition 1, the comparison between $V^{\mathcal{F}}$ and $V^{\mathcal{L}}$ under duopoly holds pointwisely for every realization of (x_1, x_2) and is thus independent of the (marginal) density function $g(\cdot)$: The conditions in Proposition 1—which govern the consumer welfare comparison between $V^{\mathcal{L}}$ and $V^{\mathcal{F}}$ in the duopoly case—depend solely on the distribution of consumers' choosiness $f(t)$ and not on that of their brand loyalty $\tilde{g}(\boldsymbol{x})$. In the duopoly case, the difference between firms' equilibrium prices across the two pricing regimes is proportional to $|x_1 - x_2|$ (see Lemma 1(iii) and Corollary 1). However, this property no longer holds when the market consists of three or more firms, in which case the consumer welfare comparison may depend on the entire profile $(x_1,\ldots,x_n)$ (see Lemma 2).

As in the duopoly case, industry profit can be unambiguously ranked. The first two inequalities are obvious and follow immediately from Lemmas 3 and 4. The last inequality can again be understood in light of the rationale outlined above: Additional firms influence market outcome under loyalty-based pricing $\mathcal{L}$ more significantly than fully personalized pricing $\mathcal{F}$. An additional firm may step up competition under loyalty-based pricing $\mathcal{L}$ even if it is least preferred by consumers. In contrast, under fully personalized pricing $\mathcal{F}$, it is irrelevant unless it is one of the two most preferred firms. These forces favor consumers and harm firms more under loyalty-based pricing $\mathcal{L}$ than under fully personalized pricing $\mathcal{F}$, leading to $\Pi^{\mathcal{F}} > \Pi^{\mathcal{L}}$.

5 Partial Market Coverage

We now relax the assumption of full market coverage in equilibrium. A higher marginal cost c raises equilibrium prices and potentially leads some consumers to opt for their outside option with zero surplus. By varying the marginal cost, we can adjust the degree of market coverage in equilibrium and examine its implications for consumer welfare and industry profit across different pricing regimes.[12]

A consumer purchases from firm i if and only if

$$v + tx_i - p_i \geq \max_{j \neq i} \left\{ v + tx_j - p_j, 0 \right\}.$$

A symmetric pure-strategy equilibrium under uniform pricing requires $p_j = p^{\mathcal{U}}$ for each $j \neq i$ in equilibrium. This condition can be rewritten as

$$p_i - p^{\mathcal{U}} \leq t \left[x_i - \max_{j \neq i} \left\{ x_j, \frac{p^{\mathcal{U}} - v}{t} \right\} \right].$$

With slight abuse of notation, we define $\hat{x}_i(\hat{y}) := x_i - \max_{j \neq i}\{x_j, \hat{y}\}$ and $z(y) := tx_i - \max_{j \neq i}\{tx_j, y\}$. The CDF and PDF of $\hat{x}_i(\hat{y})$ are denoted by $\widehat{G}(\cdot;\hat{y})$ and $\hat{g}(\cdot;\hat{y})$, respectively; those of $z(y)$ are similarly denoted by $H(\cdot;y)$ and $h(\cdot;y)$. We impose the following regularity conditions in parallel with Assumptions 1 and 3.

Assumption 1′ *$1 - \widehat{G}(\cdot;\hat{y})$ is log-concave for each $\hat{y} \in \mathbb{R}$. Moreover, $\frac{1-\widehat{G}(0,\hat{y})}{\hat{g}(0;\hat{y})}$ is weakly decreasing in $\hat{y}$.*

Assumption 3′ *$1 - H(\cdot;y)$ is log-concave for each $y \in \mathbb{R}$. Moreover, $\frac{1-H(0;y)}{h(0;y)}$ is weakly decreasing in y.*

The following result can be obtained.

[12] Equivalently, we can vary the market coverage by changing the value of the base utility v. Decreasing the base utility v and increasing the marginal cost c are isomorphic in our setting.

Proposition 5 ***(Welfare Comparison under Partial Market Coverage)*** *Suppose that Assumptions 1′, 2, and 3′ hold. There exists a unique symmetric equilibrium under $\mathcal{U}$ and $\mathcal{C}$ and a unique equilibrium outcome under $\mathcal{F}$ and $\mathcal{L}$.*[13] *Moreover, if $f(\bar{t}) > 0$ and $g(\overline{x}) > 0$, there exist a threshold $\underline{c} < v + \bar{t}\overline{x}$ such that the following results hold:*

(i) Uniform pricing maximizes consumer welfare when $c > \underline{c}$. Moreover,

$$\lim_{c \nearrow v+\bar{t}\overline{x}} V^{\mathcal{U}} : V^{\mathcal{C}} : V^{\mathcal{L}} : V^{\mathcal{F}} = 32 : 27 : 27 : 0.$$

(ii) Fully personalized pricing maximizes industry profit when $c > \underline{c}$. Moreover,

$$\lim_{c \nearrow v+\bar{t}\overline{x}} \Pi^{\mathcal{U}} : \Pi^{\mathcal{C}} : \Pi^{\mathcal{L}} : \Pi^{\mathcal{F}} = 48 : 54 : 54 : 108.$$

With a large marginal cost c, Proposition 5 states that, in the limiting case, uniform pricing $\mathcal{U}$ generates the highest consumer welfare, while fully personalized $\mathcal{F}$ pricing yields the highest industry profit. These contrast with the rankings under full market coverage. The intuition aligns with that of Rhodes and Zhou (2024). A large c effectively filters out competition and renders each firm a local monopolist: Conditional on a consumer whose value exceeds the large cost—i.e., $v + tx_i > c$ for some $i \in \mathcal{N}$—it is highly unlikely that this consumer values another product more than the cost threshold. Consequently, for each firm, competition with other products is overshadowed by competition with the outside option. The conventional wisdom of monopolistic first-degree price discrimination (Pigou, 1920) is reinstated in this setting, which suggests that finer consumer information benefits firms while harming consumers. In the absence of significant inter-firm competition, lacking either type of information would prevent a firm from perfectly profiling its consumers and fully extracting their surplus.

It is worth noting that when c is sufficiently large, information about consumers' choosiness plays a role analogous to that of brand loyalty. By Proposition 5, consumer welfare and industry profit under loyalty-based pricing $\mathcal{L}$ converge to those under choosiness-based pricing $\mathcal{C}$—i.e., $\lim_{c \nearrow v+\bar{t}\overline{x}} V^{\mathcal{C}}/V^{\mathcal{L}} = \lim_{c \nearrow v+\bar{t}\overline{x}} \Pi^{\mathcal{C}}/\Pi^{\mathcal{L}} = 1$. To see this, consider choosiness-based pricing $\mathcal{C}$ for a fixed t. Firms will not price below

[13] See the proof of Proposition 5 for detailed equilibrium characterization.

c, and consumers with $v + tx_i \geq p_i$ purchase from firm i. When $c \nearrow v + \bar{t}\overline{x}$, only consumers with x_i close to $\overline{x}$ consider a purchase. This causes firm i to face a *linear demand* in this limiting case, regardless of the marginal distribution $g(x_i)$. A similar rationale applies to loyalty-based pricing $\mathcal{L}$ for a fixed x_i. Each firm again faces linear demand in the limit, independent of the distribution $f(t)$. Consequently, both pricing regimes yield the same level of consumer welfare and industry profit.

6 Concluding Remarks

We analyze a general oligopoly model in which firms produce horizontally differentiated products, and consumers differ in two key dimensions: brand-dependent preferences (loyalty) and brand-independent preferences (choosiness). Subject to prevailing consumer privacy regulations and competition policies, firms can make either perfect or imperfect inferences about consumer preferences and set their prices accordingly. Four pricing regimes could emerge: (i) uniform pricing, (ii) choosiness-based pricing, (iii) loyalty-based pricing, and (iv) fully personalized pricing.

We show that if the market is fully covered, consumer welfare is maximized under either fully personalized pricing—in which firms set prices based on perfectly inferred consumer preferences—or loyalty-based pricing—in which firms set prices based solely on each consumer's loyalty rather than their choosiness. The latter regime is more likely to prevail in markets with a larger number of firms. In contrast, pricing based on consumers' choosiness always maximizes industry profit.

Our findings highlight the fundamentally different roles played by consumer information along different dimensions of preferences and the implications for pricing strategies. Specifically, learning about consumers' brand preferences (loyalty) renders individual consumers more contestable, whereas learning about their choosiness mitigates the marginal-inframarginal trade-off and enables more effective surplus extraction. This insight underscores the complexity of regulations regarding consumer data protection and firms' pricing behavior. Evaluating the implications of commercial surveillance and personalized pricing requires moving beyond the binary question of whether consumer information should be shared with firms. Instead, a more nuanced inquiry should focus on what types of data should be shared or subject to stricter regulation. Our results call for further research in this direction.

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Appendix: Proofs

We first state several intermediate results (the proofs can be found in the Supplemental Appendix).

Lemma A1 *Suppose that Assumption 2 holds. Then $f(t)$, $F(t)$, and $1-F(t)$ are all log-concave in t. Moreover, both $\frac{1-F(t)}{f(t)}-t$ and $\frac{1-F(t)}{f(t)}-tF(t)$ are strictly decreasing.*

Lemma A2 *Suppose that Assumption 2 holds and $f(\underline{t})<1/\underline{t}$. There exists a unique $\alpha^*\in(\underline{t},\overline{t})$ that solves $\alpha^* f(\alpha^*)=1-2F(\alpha^*)$.*

Lemma A3 *Fix $\delta_1,\delta_2>0$ and $\overline{t}\geq\alpha>\alpha'\geq\underline{t}$. The function*

$$\psi\left(\alpha,\alpha'\right):=\frac{F(\alpha)-F(\alpha')}{f(\alpha)/\delta_1+f(\alpha')/\delta_2}$$

is strictly increasing in α and is strictly decreasing in α'.

Proof of Lemma 1

Proof. We characterize firms' equilibrium pricing strategies under $\mathcal{U}$, $\mathcal{C}$, and $\mathcal{F}$, respectively.

Uniform pricing We focus on the symmetric equilibrium. Fixing $i\in\{1,\ldots,n\}$ and $p_j=p^{\mathcal{U}}$ for each $j\neq i$, firm i's profit for charging price p_i is

$$\pi_i(p_i;\boldsymbol{p}_{-i})\Big|_{p_j=p^{\mathcal{U}},\forall j\neq i}=(p_i-c)\Pr\left(p_i-p^{\mathcal{U}}\leq t\big(x_i-\max_{j\neq i}x_j\big)\right)=(p_i-c)\left[1-H(p_i-p^{\mathcal{U}})\right].$$

By Assumption 3, $1-H(p_i-p^{\mathcal{U}})$ is log-concave in p_i. This implies that $\pi_i(p_i;\boldsymbol{p}_{-i})$ is also log-concave in p_i, holding fixed $p_j=p^{\mathcal{U}}$ for all $j\neq i$. Therefore, the symmetric equilibrium is uniquely determined by the following first-order condition at $p_i=p^{\mathcal{U}}$:

$$0=\frac{\partial\pi_i(p_i,\boldsymbol{p}_{-i})}{\partial p_i}\Bigg|_{p_j=p^{\mathcal{U}},\forall j\in\mathcal{N}}=1-H(0)-\left(p^{\mathcal{U}}-c\right)h(0),$$

from which we can obtain that $p^{\mathcal{U}}=c+\frac{1-H(0)}{h(0)}=c+\frac{1}{nh(0)}$. The last equality follows from the assumption that the distribution of $\boldsymbol{x}$ is exchangeable, which implies that $1-H(0)=\Pr\left(x_i=\max_{j\in\mathcal{N}}x_j\right)=1/n$.

Recall that $h(\cdot)$ is the PDF of $z=t\hat{x}$. It holds that $h(0)=\int_0^1\frac{f(t)}{t}\hat{g}(0)dt=\hat{g}(0)\mathbb{E}\left[\frac{1}{t}\right]$, which implies that $p^{\mathcal{U}}=c+\frac{1}{nh(0)}=c+\frac{1}{n\hat{g}(0)\mathbb{E}[\frac{1}{t}]}$.

Choosiness-based pricing Fix t and suppose $p_j = p^{\mathcal{C}}(t)$ for each $j \neq i$. Similar to the case of uniform pricing, firm i's profit under Assumption 1 is log-concave. By the first-order condition for firm i, the equilibrium price can be solved as $p^{\mathcal{C}}(t) = c + \frac{t[1-\widehat{G}(0)]}{\hat{g}(0)} = c + \frac{t}{n\hat{g}(0)}$.

Fully personalized pricing Fixing $(\boldsymbol{x}, t)$, with $x_1 > \cdots > x_n$, the firms engage in standard asymmetric Bertrand competition and the equilibrium pricing schedules are $p_1^{\mathcal{F}}(\boldsymbol{x}, t) = c + t(x_1 - x_2)$, $p_2^{\mathcal{F}}(\boldsymbol{x}, t) = c$, and $p_i^{\mathcal{F}}(\boldsymbol{x}, t) \geq c$ for $i \in \{3, \ldots, n\}$. ■

Proof of Lemma 2

Proof. We show that firms' equilibrium strategies are as follows:

(i) When $f(\underline{t}) \geq 1/\underline{t}$, it holds that $k(\boldsymbol{x}) = 1$ and the equilibrium prices satisfy

$$p_1^{\mathcal{L}}(\boldsymbol{x}) = c + \underline{t}(x_1 - x_2), p_2^{\mathcal{L}}(\boldsymbol{x}) = c, \text{ and } p_i^{\mathcal{L}}(\boldsymbol{x}) \geq c \text{ for all } i \in \{3, \ldots, n\}. \tag{11}$$

(ii) When $f(\underline{t}) < 1/\underline{t}$, it holds that $k(\boldsymbol{x}) \geq 2$. In equilibrium, firms 1 to $k(\boldsymbol{x})$ have positive demand, while firms $k(\boldsymbol{x})+1$ to n have zero demand and thus earn zero profits. Without loss of generality, we set $p_{k(\boldsymbol{x})+1}^{\mathcal{L}}(\boldsymbol{x}) = \cdots = p_n^{\mathcal{L}}(\boldsymbol{x}) = c$. By arguments similar to the case of $f(\underline{t}) \geq 1/\underline{t}$, there may exist other equilibria. For example, when $k(\boldsymbol{x}) \leq n-2$, firms $k(\boldsymbol{x})+2$ to n can charge an arbitrary price above their marginal cost c. Again, all these equilibria are outcome equivalent.

For firms with positive demand, there exist a set of cutoffs $\big(\alpha_1(\boldsymbol{x}), \ldots, \alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\big)$, with $\alpha_0(\boldsymbol{x}) := \bar{t} > \alpha_1(\boldsymbol{x}) > \cdots > \alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x}) > \alpha_{k(\boldsymbol{x})}(\boldsymbol{x}) := \underline{t}$, such that a consumer purchases from firm i if and only if her choosiness level $t \in \big[\alpha_i(\boldsymbol{x}), \alpha_{i-1}(\boldsymbol{x})\big)$. Moreover, the equilibrium prices $\big(p_1^{\mathcal{L}}(\boldsymbol{x}), \ldots, p_{k(\boldsymbol{x})}^{\mathcal{L}}(\boldsymbol{x})\big)$, the set of cutoffs $\big(\alpha_1(\boldsymbol{x}), \ldots, \alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\big)$, and the number of firms with positive demand $k(\boldsymbol{x})$ are uniquely pinned down by the following conditions:

(a) First-order conditions for profit maximization for the first $k(\boldsymbol{x}) - 1$ firms:

$$p_1^{\mathcal{L}}(\boldsymbol{x}) = c + \frac{1 - F\left(\alpha_1(\boldsymbol{x})\right)}{\frac{f\left(\alpha_1(\boldsymbol{x})\right)}{x_2 - x_1}}, \text{ and} \tag{12}$$

$$p_i^{\mathcal{L}}(\boldsymbol{x}) = c + \frac{F\left(\alpha_{i-1}(\boldsymbol{x})\right) - F\left(\alpha_i(\boldsymbol{x})\right)}{\frac{f\left(\alpha_{i-1}(\boldsymbol{x})\right)}{x_{i-1} - x_i} + \frac{f\left(\alpha_i(\boldsymbol{x})\right)}{x_{i+1} - x_i}}, i \in \big\{2, \ldots, k(\boldsymbol{x}) - 1\big\}. \tag{13}$$

(b) Karush-Kuhn-Tucker (KKT) conditions for the $k(\boldsymbol{x})$-th firm:

$$p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x}) = c + \frac{F\left(\alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\right)}{\frac{f\left(\alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\right)}{x_{k(\boldsymbol{x})-1}-x_{k(\boldsymbol{x})}} + \xi f(\underline{t})}, \xi \in \left[0, \frac{1}{x_{k(\boldsymbol{x})} - x_{k(\boldsymbol{x})+1}}\right], \tag{14}$$

$$p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x}) \leq c + \underline{t}\left(x_{k(\boldsymbol{x})} - x_{k(\boldsymbol{x})+1}\right), \tag{15}$$

$$\xi \times \left[\underline{t}\left(x_{k(\boldsymbol{x})} - x_{k(\boldsymbol{x})+1}\right) - c - p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x})\right] = 0, \tag{16}$$

where $x_{n+1} := -\infty$ if $k(\boldsymbol{x}) = n$.

(c) Indifference conditions of consumers with pivotal choosiness levels:

$$p^{\mathcal{L}}_i(\boldsymbol{x}) = p^{\mathcal{L}}_{i+1}(\boldsymbol{x}) + \alpha_i(\boldsymbol{x})(x_i - x_{i+1}), i \in \left\{1, \ldots, k(\boldsymbol{x}) - 1\right\}. \tag{17}$$

Fix $\boldsymbol{x} = (x_1, \ldots, x_n)$ and suppose $x_1 > \cdots > x_n$ without loss of generality. The proof consists of six steps. First, we show that the $k(\boldsymbol{x})$ active firms are firms 1 through $k(\boldsymbol{x})$. Second, we show that there exists a set of cutoffs $\left(\alpha_1(\boldsymbol{x}), \ldots, \alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\right)$ such that a consumer purchases from firm i if and only if their choosiness level $t \in \left[\alpha_i(\boldsymbol{x}), \alpha_{i-1}(\boldsymbol{x})\right)$, and prove the indifference condition (17). Third, we show that the profit function of each active firm is log-concave. Fourth, we derive the first-order conditions (12) and (13) and the KKT conditions (14), (15), and (16). Fifth, we prove the equilibrium existence and uniqueness and show that $k(\boldsymbol{x}) = 1$ when $f(\underline{t}) \geq 1/\underline{t}$. Last, we prove the equilibrium existence and uniqueness when $f(\underline{t}) < 1/\underline{t}$.

Step I We show that the $k(\boldsymbol{x})$ active firms are exactly firms 1 through firm $k(\boldsymbol{x})$. Suppose, to the contrary, that there exists an active firm i and an inactive j with $i > j$. If firm j deviates to firm i's price, it can attract all consumers of firm i and earn positive profit. A contradiction.

Step II Fixing the number of active firms $k(\boldsymbol{x})$ and a price profile $(p^{\mathcal{L}}_1(\boldsymbol{x}), \ldots, p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x}))$, we show that there exists a set of cutoffs $(\alpha_1(\boldsymbol{x}), \ldots, \alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x}))$ such that a consumer purchases from firm i if and only if her choosiness level $t \in \left[\alpha_i(\boldsymbol{x}), \alpha_{i-1}(\boldsymbol{x})\right)$.

Let $\alpha_i(\boldsymbol{x}) := \frac{p^{\mathcal{L}}_i(\boldsymbol{x}) - p^{\mathcal{L}}_{i+1}(\boldsymbol{x})}{x_i - x_{i+1}}$, with $i \in \{1, \ldots, k(\boldsymbol{x}) - 1\}$. A consumer prefers firm i over firm $i+1$ if and only if $v + tx_i - p^{\mathcal{L}}_i(\boldsymbol{x}) \geq v + tx_{i+1} - p^{\mathcal{L}}_{i+1}(\boldsymbol{x})$, which is equivalent to $t \geq \alpha_i(\boldsymbol{x})$. Similarly, the consumer prefers firm i over firm $i-1$ if and only if $t < \alpha_{i-1}(\boldsymbol{x})$. Note that if $\alpha_{i-1}(\boldsymbol{x}) \leq \alpha_i(\boldsymbol{x})$, no consumers purchase from firm i. Therefore, we must have that $\alpha_{i-1}(\boldsymbol{x}) > \alpha_i(\boldsymbol{x})$ and $\alpha_0(\boldsymbol{x}) := \bar{t} > \alpha_1(\boldsymbol{x}) > \cdots > \alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x}) > \alpha_{k(\boldsymbol{x})}(\boldsymbol{x}) := \underline{t}$. Moreover, it is straightforward to verify that if a consumer's choosiness level $t \in [\alpha_i(\boldsymbol{x}), \alpha_{i-1}(\boldsymbol{x}))$, she buys from firm i.

Step III Prove the log-concavity of profit functions. For $i \leq k(\boldsymbol{x})$, firm i's profit is

$$\pi_i(p_i, \boldsymbol{p}_{-i}; \boldsymbol{x}) = (p_i - c) \times \Pr\left(\min_{j<i}\left\{\frac{p_j - p_i}{x_j - x_i}, \bar{t}\right\} > t \geq \max_{j>i}\left\{\frac{p_i - p_j}{x_i - x_j}, \underline{t}\right\}\right).$$

We focus on the case in which $\min_{j<i}\left\{\frac{p_j-p_i}{x_j-x_i}, \bar{t}\right\}$ and $\max_{j>i}\left\{\frac{p_i-p_j}{x_i-x_j}, \underline{t}\right\}$ are achieved by some unique $j_1 < i$ and $j_2 > i$, respectively. Firm i's profit can then be expressed as

$$\pi_i(p_i, \boldsymbol{p}_{-i}; \boldsymbol{x}) = (p_i - c) \times \underbrace{\left[F\left(\frac{p_{j_1} - p_i}{x_{j_1} - x_i}\right) - F\left(\frac{p_i - p_{j_2}}{x_i - x_{j_2}}\right)\right]}_{=:\mathcal{D}_i(p_i, \boldsymbol{p}_{-i}; \boldsymbol{x})},$$

which is twice differentiable at p_i. The analysis for other cases is similar and omitted for brevity. We state the following lemma (whose proof can be found in the Supplemental Appendix):

Lemma A4 *Under Assumption 2, $\mathcal{D}_i(p_i, \boldsymbol{p}_{-i}; \boldsymbol{x})$ is log-concave in p_i.*

By Lemma A4, $\pi_i(p_i, \boldsymbol{p}_{-i}; \boldsymbol{x}) = (p_i - c)\mathcal{D}_i(p_i, \boldsymbol{p}_{-i}; \boldsymbol{x})$ is log-concave in p_i.

Step IV Derive the first-order conditions and KKT conditions. Recall from Step II that $\alpha_1(\boldsymbol{x}) > \cdots > \alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})$. Therefore, the profit for firm $i \in \{2, \ldots, k(\boldsymbol{x}) - 1\}$ is differentiable at $p_i^{\mathcal{L}}(\boldsymbol{x})$ and the corresponding first-order condition is

$$\begin{aligned} 0 =& \left.\frac{\partial \pi_i\left(p_i, \boldsymbol{p}^{\mathcal{L}}_{-i}(\boldsymbol{x}); \boldsymbol{x}\right)}{\partial p_i}\right|_{p_i = p_i^{\mathcal{L}}(\boldsymbol{x})} \\ =& \left[F\left(\alpha_{i-1}(\boldsymbol{x})\right) - F\left(\alpha_i(\boldsymbol{x})\right)\right] - \left(p_i^{\mathcal{L}}(\boldsymbol{x}) - c\right) \times \left[\frac{f\left(\alpha_{i-1}(\boldsymbol{x})\right)}{x_{i-1} - x_i} + \frac{f\left(\alpha_i(\boldsymbol{x})\right)}{x_i - x_{i+1}}\right], \end{aligned}$$

which is equivalent to (13). Similarly, we can obtain (12) from the first-order condition for firm 1.

Next, consider firm $k(\boldsymbol{x})$ and prove the KKT conditions (14), (15), and (16). We first consider (15). Suppose, to the contrary, that $p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x}) > c + \underline{t}(x_{k(\boldsymbol{x})} - x_{k(\boldsymbol{x})+1})$.

Then firm $k(\boldsymbol{x})+1$ can deviate to $p_{k(\boldsymbol{x})+1} \in \left(c, p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x}) - \underline{t}(x_{k(\boldsymbol{x})} - x_{k(\boldsymbol{x})+1})\right)$ and obtain positive profit. A contradiction.

It remains to prove (14) and (16). We consider the following two cases:

Case I: $\boldsymbol{p^{\mathcal{L}}_{k(x)}(x) = c + \underline{t}(x_{k(x)} - x_{k(x)+1})}$. That is, (15) holds with equality. This implies (16) and it remains to prove (14). Firm $k(\boldsymbol{x})$'s profit is

$$\pi_{k(\boldsymbol{x})}\left(\tilde{p}_{k(\boldsymbol{x})}, \boldsymbol{p}^{\mathcal{L}}_{-k(\boldsymbol{x})}(\boldsymbol{x}); \boldsymbol{x}\right)$$
$$= \begin{cases} \left(\tilde{p}_{k(\boldsymbol{x})} - c\right) \times F\left(\frac{p^{\mathcal{L}}_{k(\boldsymbol{x})-1}(\boldsymbol{x}) - \tilde{p}_{k(\boldsymbol{x})}}{x_{k(\boldsymbol{x})-1} - x_{k(\boldsymbol{x})}}\right), & \tilde{p}_{k(\boldsymbol{x})} \in \left(p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x}) - \delta, p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x})\right), \\ \left(\tilde{p}_{k(\boldsymbol{x})} - c\right) \times \left[F\left(\frac{p^{\mathcal{L}}_{k(\boldsymbol{x})-1}(\boldsymbol{x}) - \tilde{p}_{k(\boldsymbol{x})}}{x_{k(\boldsymbol{x})-1} - x_{k(\boldsymbol{x})}}\right) - F\left(\frac{\tilde{p}_{k(\boldsymbol{x})}}{x_{k(\boldsymbol{x})} - x_{k(\boldsymbol{x})+1}}\right)\right], & \tilde{p}_{k(\boldsymbol{x})} \in \left(p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x}), p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x}) + \delta\right). \end{cases}$$

Note that $p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x})$ must satisfy

$$\frac{\partial}{\partial \tilde{p}_{k(\boldsymbol{x})}} \pi_{k(\boldsymbol{x})}\left(\tilde{p}_{k(\boldsymbol{x})}, \boldsymbol{p}^{\mathcal{L}}_{-k(\boldsymbol{x})}(\boldsymbol{x}); \boldsymbol{x}\right)\Big|_{\tilde{p}_{k(\boldsymbol{x})} = p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x}) - 0} \geq 0, \tag{18}$$

and

$$\frac{\partial}{\partial \tilde{p}_{k(\boldsymbol{x})}} \pi_{k(\boldsymbol{x})}\left(\tilde{p}_{k(\boldsymbol{x})}, \boldsymbol{p}^{\mathcal{L}}_{-k(\boldsymbol{x})}(\boldsymbol{x}); \boldsymbol{x}\right)\Big|_{\tilde{p}_{k(\boldsymbol{x})} = p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x}) + 0} \leq 0. \tag{19}$$

Simple algebra would verify that (18) and (19) are equivalent to (14).

Case II: $\boldsymbol{p^{\mathcal{L}}_{k(x)}(x) < c + \underline{t}(x_{k(x)} - x_{k(x)+1})}$. That is, strict inequality holds in (15). Therefore, firm k's profit is

$$\pi_{k(\boldsymbol{x})}\left(\tilde{p}_{k(\boldsymbol{x})}, \boldsymbol{p}^{\mathcal{L}}_{-k(\boldsymbol{x})}(\boldsymbol{x}); \boldsymbol{x}\right) = (\tilde{p}_{k(\boldsymbol{x})} - c) \times F\left(\frac{p^{\mathcal{L}}_{k(\boldsymbol{x})-1}(\boldsymbol{x}) - \tilde{p}_{k(\boldsymbol{x})}}{x_{k(\boldsymbol{x})-1} - x_{k(\boldsymbol{x})}}\right),$$

and the first-order condition with respect to $\tilde{p}_{k(\boldsymbol{x})}$ is

$$-\left(p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x}) - c\right) \times \frac{f\left(\alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\right)}{x_{k(\boldsymbol{x})-1} - x_{k(\boldsymbol{x})}} + F\left(\alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\right) = 0.$$

Note that setting $\xi = 0$ in (14)—which implies (16)—gives the above condition.

Step V Suppose $f(\underline{t}) \geq 1/\underline{t}$. It suffices to show that there exists no equilibrium that satisfies $k(\boldsymbol{x}) \geq 2$. Suppose, to the contrary, that such an equilibrium exists. Combining (12) and (17) yields that

$$c + \frac{1 - F(\alpha_1(\boldsymbol{x}))}{f(\alpha_1(\boldsymbol{x}))} \times (x_1 - x_2) = p_1^{\mathcal{L}}(\boldsymbol{x}) = p_2^{\mathcal{L}}(\boldsymbol{x}) + \alpha_1(\boldsymbol{x})(x_1 - x_2) \geq c + \alpha_1(\boldsymbol{x})(x_1 - x_2),$$

which implies that $\frac{1-F(\alpha_1(\boldsymbol{x}))}{f(\alpha_1(\boldsymbol{x}))} - \alpha_1(\boldsymbol{x}) \geq 0$. However, by Lemma A1 and the postulated $f(\underline{t}) \geq 1/\underline{t}$, we can conclude that $\frac{1-F(\alpha_1(\boldsymbol{x}))}{f(\alpha_1(\boldsymbol{x}))} - \alpha_1(\boldsymbol{x}) < \frac{1-F(\underline{t})}{f(\underline{t})} - \underline{t} \leq 0$. A contradiction.

Step VI Suppose $f(\underline{t}) < 1/\underline{t}$. Fixing $\boldsymbol{x}$, for each $\alpha_1 \in [\underline{t}, \overline{t}]$, define the following functions recursively:

$$\acute{p}_1(\alpha_1) := c + \frac{1 - F(\alpha_1)}{f(\alpha_1)} \times (x_1 - x_2) \text{ and } \acute{\alpha}_1(\alpha_1) := \alpha_1. \tag{20}$$

Define

$$\acute{p}_{j+1}(\alpha_1) := \acute{p}_j(\alpha_1) - \acute{\alpha}_j(\alpha_1)(x_j - x_{j+1}), \tag{21}$$

and

$$\acute{\alpha}_{j+1}(\alpha_1) := \begin{cases} \text{the unique solution to} & \\ \acute{p}_{j+1}(\alpha_1) = c + \frac{F(\acute{\alpha}_j(\alpha_1)) - F(\alpha_{j+1})}{\frac{f(\acute{\alpha}_j(\alpha_1))}{x_j - x_{j+1}} + \frac{f(\alpha_{j+1})}{x_{j+1} - x_{j+2}}}, & \acute{p}_{j+1}(\alpha_1) < c + \frac{F(\acute{\alpha}_j(\alpha_1))}{\frac{f(\acute{\alpha}_j(\alpha_1))}{x_j - x_{j+1}} + \frac{f(\underline{t})}{x_{j+1} - x_{j+2}}}, \\ \underline{t}, & \text{otherwise.} \end{cases} \tag{22}$$

By Lemma A3, $\acute{\alpha}_{j+1}(\alpha_1)$ is well defined. Recursion stops when either of the following conditions is satisfied:

$$\acute{p}_j(\alpha_1) - \acute{\alpha}_j(\alpha_1)(x_j - x_{j+1}) \leq c, \tag{23}$$

$$\acute{p}_j(\alpha_1) \geq c + \frac{F\left(\acute{\alpha}_{j-1}(\alpha_1)\right)}{\frac{f\left(\acute{\alpha}_{j-1}(\alpha_1)\right)}{x_{j-1} - x_j} + \frac{f(\underline{t})}{x_j - x_{j+1}}}, \tag{24}$$

in which case we define $k(\alpha_1) := j \geq 2$.

Fixing $\alpha_1 \in [\underline{t}, \overline{t}]$, it can be verified that the set of prices $\{\acute{p}_i(\alpha_1)\}_{i=1}^{k(\alpha_1)}$ and the set of cutoffs $\{\acute{\alpha}_i(\alpha_1)\}_{i=1}^{k(\alpha_1)-1}$ satisfy (12), (13), and (17). Therefore, $\{\acute{p}_i(\alpha_1)\}_{i=1}^{k(\alpha_1)}$ and $\{\acute{\alpha}_i(\alpha_1)\}_{i=1}^{k(\alpha_1)-1}$ constitute an equilibrium if and only if the KKT conditions (14), (15), and (16) are satisfied. It suffices to show that there exists a unique $\alpha_1 \in [\underline{t}, \overline{t}]$ such that (14), (15) and (16) hold.

We state the following lemma (whose proof can be found in the Supplemental Appendix).

Lemma A5 *Fixing $i \in \{1, \ldots, k(\alpha_1)\}$, $\acute{p}_i(\alpha_1)$ is decreasing in α_1 and $\acute{\alpha}_i(\alpha_1)$ is increasing in α_1.*

Let $k^* := \max_{\alpha_1 \in [\underline{t}, \overline{t}]} k(\alpha_1)$ and $\mathcal{X} := \{\alpha_1 | \, k(\alpha_1) = k^*\}$. It suffices to show that (i) there exists a unique $\alpha_1 \in \mathcal{X}$ that constitutes an equilibrium; and (ii) every $\alpha_1 \notin \mathcal{X}$ cannot constitute an equilibrium.

We first show that $\mathcal{X}$ is an interval. Let $\underline{\alpha}_1 := \inf \mathcal{X}$ and $\overline{\alpha}_1 := \sup \mathcal{X}$. Suppose $k^* = 2$. Then $k(\alpha_1) = 2$ for all $\alpha_1 \in [\underline{t}, \overline{t}]$, which implies that $\mathcal{X}$ is an interval and $\mathcal{X} = [\underline{t}, \overline{t}]$. Suppose $k^* \geq 3$. Because $k(\underline{\alpha}_1) = k(\overline{\alpha}_1) = k^*$, neither (23) nor (24) does not hold for $j = k^* - 1$. Therefore, for each $\alpha_1 \in \{\underline{\alpha}_1, \overline{\alpha}_1\}$, it holds that

$$\acute{p}_{k^*-1}(\alpha_1) \geq c + \acute{\alpha}_{k^*-1}(\alpha_1)(x_{k^*-1} - x_{k^*}), \tag{25}$$

$$\acute{p}_{k^*-1}(\alpha_1) \leq c + \frac{F\left(\acute{\alpha}_{k^*-2}(\alpha_1)\right)}{\frac{f\left(\acute{\alpha}_{k^*-2}(\alpha_1)\right)}{x_{k^*-2} - x_{k^*-1}} + \frac{f(\underline{t})}{x_{k^*-1} - x_{k^*}}}. \tag{26}$$

By Lemma A5, (25) and (26) hold for each $\alpha_1 \in [\underline{\alpha}_1, \overline{\alpha}_1]$, which implies that $\mathcal{X}$ is an interval (i.e., $\mathcal{X} = [\underline{\alpha}_1, \overline{\alpha}_1]$). Moreover, (25) holds with equality at $\alpha_1 = \overline{\alpha}_1$, and (26) holds with equality at $\alpha_1 = \underline{\alpha}_1$.

By Lemma A5, $\acute{\alpha}_{k^*}$ is increasing in α_1. Therefore, there exists a unique threshold $\hat{\alpha}_1 \in \mathcal{X}$ such that $\acute{\alpha}_{k^*}(\alpha_1) = \underline{t}$ if and only if $\alpha_1 \in [\underline{\alpha}_1, \hat{\alpha}_1]$, where $\hat{\alpha}_1$ solves (see (22))

$$\acute{p}_{k^*}(\hat{\alpha}_1) = c + \frac{F\left(\acute{\alpha}_{k^*-1}(\hat{\alpha}_1)\right)}{\frac{f\left(\acute{\alpha}_{k^*-1}(\hat{\alpha}_1)\right)}{x_{k^*-1} - x_{k^*}} + \frac{f(\underline{t})}{x_{k^*} - x_{k^*+1}}}. \tag{27}$$

Note that $\alpha_{k(\boldsymbol{x})}(\boldsymbol{x}) \equiv \underline{t}$ in the equilibrium. Therefore, the equilibrium cutoff $\alpha_1(\boldsymbol{x}) \notin (\hat{\alpha}_1, \overline{\alpha}_1]$. In addition, $\alpha_1(\boldsymbol{x}) \in [\underline{\alpha}_1, \hat{\alpha}_1]$ if $\alpha_1(\boldsymbol{x}) \in \mathcal{X}$. Next, we show that there exists a unique α_1 to satisfy (14), (15), and (16). We consider the following two cases:

Case (a): $\boldsymbol{\acute{p}_{k^*}(\underline{\alpha}_1) < c + \underline{t}(x_{k^*} - x_{k^*+1})}$. By Lemma A5, for each $\alpha_1 \in [\underline{\alpha}_1, \hat{\alpha}_1]$, $\acute{p}_{k^*}(\alpha_1) < c + \underline{t}(x_{k^*} - x_{k^*+1})$. Therefore, (15) is satisfied and $\xi = 0$ from (16). Define the following auxiliary function

$$\psi_2(\alpha_1) := \acute{p}_{k^*}(\alpha_1) - \frac{F\left(\acute{\alpha}_{k^*-1}(\alpha_1)\right)}{\frac{f\left(\acute{\alpha}_{k^*-1}(\alpha_1)\right)}{x_{k^*-1} - x_{k^*}}} - c, \alpha_1 \in [\underline{\alpha}_1, \hat{\alpha}_1].$$

To show that there exists a unique solution to satisfy (14), it suffices to show that there exists a unique solution to $\psi_2(\alpha_1) = 0$. By Lemma A5, $\psi_2(\alpha_1)$ is decreasing in α_1. First, recall that (26) holds with equality at $\alpha_1 = \underline{\alpha}_1$. Therefore, $\acute{\alpha}_{k^*-1}(\underline{\alpha}_1) = \underline{t}$ and thus $\psi_2(\underline{\alpha}_1) \geq 0$. Further, by (27), $\psi_2(\hat{\alpha}_1) \leq 0$. Therefore, there exists a unique $\alpha_1^* \in [\underline{\alpha}_1, \hat{\alpha}_1]$ such that $\psi_2(\alpha_1^*) = 0$.

Case (b): $\boldsymbol{\acute{p}_{k^*}(\underline{\alpha}_1) \geq c + \underline{t}(x_{k^*} - x_{k^*+1})}$. By Lemma A5, there exists a unique $\tilde{\alpha}_1 \in [\underline{\alpha}_1, \hat{\alpha}_1]$ such that $\acute{p}_{k^*}(\tilde{\alpha}_1) = c + \underline{t}(x_{k^*} - x_{k^*+1})$. If $\psi_2(\tilde{\alpha}_1) < 0$, we set $\alpha_1^* = \tilde{\alpha}_1$. Otherwise, let α_1^* be the unique solution to $\psi_2(\alpha_1^*) = 0$. It can be verified that (14), (15), and (16) are satisfied uniquely at $\alpha_1 = \alpha_1^*$.

Thus far, we have shown that there exists a unique $\alpha_1 = \alpha_1^* \in \mathcal{X}$ to satisfy (14), (15), and (16). To prove equilibrium uniqueness, it remains to show that, for an arbitrary $\alpha_1' \notin \mathcal{X} = [\underline{\alpha}_1, \overline{\alpha}_1]$, the KKT conditions (14), (15), and (16) cannot be satisfied simultaneously.

Suppose $\alpha_1' < \underline{\alpha}_1$. By definition of $\mathcal{X}$ and k^*, $k(\underline{\alpha}_1) = k^* > k(\alpha_1') =: k'$. Therefore, (23) does not hold for $j = k'$ at $\alpha_1 = \underline{\alpha}_1$, which implies that $\acute{p}_{k'}(\underline{\alpha}_1) - \acute{\alpha}_{k'}(\underline{\alpha}_1)(x_{k'} - x_{k'+1}) > c$. By Lemma A5, the left-hand side is decreasing in α_1, and thus $\acute{p}_{k'}(\alpha_1') - \acute{\alpha}_{k'}(\alpha_1')(x_{k'} - x_{k'+1}) > c$. Therefore, condition (15)—which must be satisfied in equilibrium—does not hold at $\alpha_1 = \alpha_1'$.

By similar arguments, we can show that condition (14) does not hold if $\alpha_1' > \overline{\alpha}_1$. This completes the whole proof. ■

Proof of Lemma 3

Proof. See main text. ■

Proof of Lemma 4

Proof. Applying Proposition 1 in Rhodes and Zhou (2024) yields $V^{\mathcal{U}} < V^{\mathcal{F}}$. It remains to show $V^{\mathcal{C}} < V^{\mathcal{U}}$. By Lemma 3, $W^{\mathcal{U}} = W^{\mathcal{C}}$. Moreover, note that

$$p^{\mathcal{U}} = c + \frac{1}{n\hat{g}(0)\mathbb{E}[\frac{1}{t}]} < c + \frac{\mathbb{E}(t)}{n\hat{g}(0)} = \mathbb{E}\big[p^{\mathcal{C}}(t)\big],$$

from which we can conclude that $V^{\mathcal{C}} < V^{\mathcal{U}}$. ■

Proof of Proposition 1

Proof. Without loss of generality, consider the case of $x_1 > x_2$. Recall from (7) and (8) that $V^{\mathcal{L}}(\boldsymbol{x})$ and $V^{\mathcal{F}}(\boldsymbol{x})$ are the interim consumer welfare. It suffices to show that the comparison between $V^{\mathcal{L}}(\boldsymbol{x})$ and $V^{\mathcal{F}}(\boldsymbol{x})$ holds pointwisely. Consider the following two cases:

Case (a): $\boldsymbol{f(\underline{t}) \geq 1/\underline{t}}$. By Lemma 1 and Corollary 1, fixing $x_1 > x_2$, consumers with (x_1, x_2) always buy from firm 1; moreover, we have that

$$p_1^{\mathcal{F}}(\boldsymbol{x}, t) = c + t(x_1 - x_2) \geq c + \underline{t}(x_1 - x_2) = p_1^{\mathcal{L}}(\boldsymbol{x}) \text{ and } p_2^{\mathcal{F}}(\boldsymbol{x}, t) = p_2^{\mathcal{L}}(\boldsymbol{x}) = c, \forall\, t \in [\underline{t}, \bar{t}].$$

Therefore, $V^{\mathcal{L}} > V^{\mathcal{F}}$.

Case (b): $\boldsymbol{f(\underline{t}) < 1/\underline{t}}$. By Corollary 1, under loyalty-based pricing, the consumer buys from firm 1 if and only if $t \geq \alpha^*$. The consumer welfare is

$$V^{\mathcal{L}}(\boldsymbol{x}) = \int_{\underline{t}}^{\alpha^*} \left[v + tx_2 - p_2^{\mathcal{L}}(\boldsymbol{x})\right] f(t)dt + \int_{\alpha^*}^{\bar{t}} \left[v + tx_1 - p_1^{\mathcal{L}}(\boldsymbol{x})\right] f(t)dt. \tag{28}$$

Similarly, consumers' welfare at $\boldsymbol{x}$ under fully personalized pricing amounts to

$$V^{\mathcal{F}}(\boldsymbol{x}) = \int_{\underline{t}}^{\bar{t}} \left[v + tx_1 - p_1^{\mathcal{F}}(\boldsymbol{x}, t)\right] f(t)dt = \int_{\underline{t}}^{\bar{t}} [v + tx_2 - c]\, f(t)dt. \tag{29}$$

Subtracting (29) from (28) and carrying out the algebra, we can obtain that

$$V^{\mathcal{L}}(\boldsymbol{x}) - V^{\mathcal{F}}(\boldsymbol{x}) = (x_1 - x_2)\left[\int_{\alpha^*}^{\bar{t}} [1 - F(t)]\, dt - \frac{F(\alpha^*)}{f(\alpha^*)}\right].$$

Recall the postulated $x_1 > x_2$. Therefore, $V^{\mathcal{L}}(\boldsymbol{x}) > V^{\mathcal{F}}(\boldsymbol{x})$ is equivalent to (5). This concludes the proof. ■

Proof of Proposition 2

Proof. By Lemma 3, $W^{\mathcal{F}} = W^{\mathcal{U}} = W^{\mathcal{C}}$. By Lemma 4, $V^{\mathcal{F}} > V^{\mathcal{U}} > V^{\mathcal{C}}$. Therefore, $\Pi^{\mathcal{F}} < \Pi^{\mathcal{U}} < \Pi^{\mathcal{C}}$ and it suffices to show $\Pi^{\mathcal{F}} > \Pi^{\mathcal{L}}$.

Denote firms' interim equilibrium profits at $\boldsymbol{x}$ under $\mathcal{L}$ and $\mathcal{F}$ by $\Pi^{\mathcal{L}}(\boldsymbol{x})$ and $\Pi^{\mathcal{F}}(\boldsymbol{x})$, respectively. It suffices to show that the comparison between $\Pi^{\mathcal{L}}(\boldsymbol{x})$ and $\Pi^{\mathcal{F}}(\boldsymbol{x})$ holds pointwisely. The proof for the case of $f(\underline{t}) \geq 1/\underline{t}$ is obvious and we focus on the case of $f(\underline{t}) < 1/\underline{t}$. $\Pi^{\mathcal{L}}(\boldsymbol{x})$ and $\Pi^{\mathcal{F}}(\boldsymbol{x})$ can be expressed as

$$\Pi^{\mathcal{L}}(\boldsymbol{x}) = |x_1 - x_2| \times \left(\frac{F(\alpha^*)}{f(\alpha^*)} + \alpha^* \left[1 - F(\alpha^*)\right] \right) - c,$$

and

$$\Pi^{\mathcal{F}}(\boldsymbol{x}) = \mathbb{E}_t \left[t|x_1 - x_2|\right] - c = |x_1 - x_2| \times \int_{\underline{t}}^{\bar{t}} tf(t)dt - c.$$

Therefore, it suffices to show that

$$\int_{\underline{t}}^{\bar{t}} tf(t)dt \geq \frac{F(\alpha^*)}{f(\alpha^*)} + \alpha^* \left[1 - F(\alpha^*)\right]. \tag{30}$$

It is useful to state the following two lemmas (whose proofs can be found in the Supplemental Appendix).

Lemma A6 *Fixing $f(\cdot)$, there exists a distribution supported on $[0, \bar{t}^\dagger]$ with CDF $F^\dagger(\cdot)$ and PDF $f^\dagger(\cdot)$, such that $\ln \frac{f^\dagger(t)}{t}$ is linear on $[0, \bar{t}^\dagger]$ and satisfies*

$$f^\dagger(\alpha^*) = f(\alpha^*), F^\dagger(\alpha^*) = F(\alpha^*), \tag{31}$$

and

$$\int_{\underline{t}}^{\bar{t}} tf(t)dt \geq \int_0^{\bar{t}^\dagger} tf^\dagger(t)dt. \tag{32}$$

Lemma A7 *Let $F^\dagger$ be the distribution constructed in Lemma A6. Then*

$$\int_0^{\bar{t}^\dagger} tf^\dagger(t)dt \geq \frac{F^\dagger(\alpha^*)}{f^\dagger(\alpha^*)} + \alpha^* \left[1 - F^\dagger(\alpha^*)\right]. \tag{33}$$

By Lemma A6, the left-hand side of (30) decreases and the right-hand side remains unchanged if we replace $F(\cdot)$ and $f(\cdot)$ with $F^\dagger(\cdot)$ and $f^\dagger(\cdot)$, respectively. By Lemma A7, (30) holds for the constructed distribution $F^\dagger(\cdot)$. Therefore, (30) holds for the distribution $F(\cdot)$. This completes the proof. ■

Proof of Lemmas 5 and 6

Proof. Fix $\boldsymbol{x}_n \equiv (x_1, \ldots, x_n)$, with $x_1 > \cdots > x_n$. By Lemma 1, $p_1^{\mathcal{F}}(\boldsymbol{x}_n, t) = t(x_1 - x_2) = p_1^{\mathcal{F}}(\boldsymbol{x}_{n+1}, t)$ and consumers buy from firm 1 under fully personalized pricing, implying that $\mathcal{W}^{\mathcal{F}}(\boldsymbol{x}_n) = \mathcal{W}^{\mathcal{F}}(\boldsymbol{x}_{n+1})$. To prove Lemma 5, it remains to show that $V^{\mathcal{L}}(\boldsymbol{x}_{n+1}) \geq V^{\mathcal{L}}(\boldsymbol{x}_n)$. Note that the inequality follows immediately from Lemma 6 and it remains to prove the lemma.

First, consider the choice effect. By (7) and (9), we have that

$$\begin{aligned}&V^{\mathcal{L}}(\boldsymbol{x}_{n+1}) - \widetilde{V}^{\mathcal{L}}(\boldsymbol{x}_{n+1})\\ =&\mathbb{E}_t\left[\max_{i\in\{1,\ldots,n+1\}}\left\{v+tx_i-p_i^{\mathcal{L}}(\boldsymbol{x}_{n+1})\right\} - \max_{i\in\{1,\ldots,n\}}\left\{v+tx_i-p_i^{\mathcal{L}}(\boldsymbol{x}_{n+1})\right\}\right] \geq 0.\end{aligned}$$

To see the nonmonotonicity of the choice effect, note that $v+tx_{n+1}-p_{n+1}^{\mathcal{L}}(\boldsymbol{x}_{n+1}) = -\infty$ for $x_{n+1} = -\infty$, which implies that $V^{\mathcal{L}}(\boldsymbol{x}_{n+1}) - \widetilde{V}^{\mathcal{L}}(\boldsymbol{x}_{n+1}) = 0$. Further, $v + tx_{n+1} - p_{n+1}^{\mathcal{L}}(\boldsymbol{x}_{n+1}) = v + tx_n - p_n^{\mathcal{L}}(\boldsymbol{x}_{n+1})$ for $x_{n+1} = x_n$, which also implies that $V^{\mathcal{L}}(\boldsymbol{x}_{n+1}) - \widetilde{V}^{\mathcal{L}}(\boldsymbol{x}_{n+1})$. In summary, $V^{\mathcal{L}}(\boldsymbol{x}_{n+1}) - \widetilde{V}^{\mathcal{L}}(\boldsymbol{x}_{n+1}) = 0$ at $x_{n+1} = -\infty$ and at $x_{n+1} = x_n$, which implies that $V^{\mathcal{L}}(\boldsymbol{x}_{n+1}) - \widetilde{V}^{\mathcal{L}}(\boldsymbol{x}_{n+1})$ is nonmonotone with respect to x_{n+1}.

Second, consider the price effect. Combining (7) and (9) yields that

$$\widetilde{V}^{\mathcal{L}}(\boldsymbol{x}_{n+1}) - V^{\mathcal{L}}(\boldsymbol{x}_n) = \mathbb{E}_t\left[\max_{i\in\{1,\ldots,n\}}\left\{v+tx_i-p_i^{\mathcal{L}}(\boldsymbol{x}_{n+1})\right\} - \max_{i\in\{1,\ldots,n\}}\left\{v+tx_i-p_i^{\mathcal{L}}(\boldsymbol{x}_n)\right\}\right].$$

Note that $p_i^{\mathcal{L}}(\boldsymbol{x}_n) = p_i^{\mathcal{L}}(x_1, \ldots, x_n, -\infty)$. The monotonicity of price effect follows immediately from the following lemma (whose proof can be found in the Supplemental Appendix).

Lemma A8 *$p_i^{\mathcal{L}}(\boldsymbol{x}_{n+1})$ is weakly decreasing in x_{n+1} for each $i \in \{1, \ldots, n\}$.*

This completes the proof. ■

Proof of Proposition 3

Proof. We first prove part (i) of the proposition. Fix $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$ and assume $x_1 > \cdots > x_n$ without loss of generality. By Lemma 5, we have that

$$V^{\mathcal{L}}(\boldsymbol{x}) \geq V^{\mathcal{L}}(\boldsymbol{x}_{n-1}) \geq \cdots \geq V^{\mathcal{L}}(x_1, x_2) \text{ and } V^{\mathcal{F}}(\boldsymbol{x}) = V^{\mathcal{F}}(\boldsymbol{x}_{n-1}) = \cdots = V^{\mathcal{F}}(x_1, x_2).$$

Recall from Proposition 1 that the consumer welfare comparison between $\mathcal{L}$ and $\mathcal{F}$ holds pointwisely for the case of $n = 2$. Therefore, we have that $V^{\mathcal{L}}(x_1, x_2) > V^{\mathcal{F}}(x_1, x_2)$, which in turn implies that $V^{\mathcal{L}}(\boldsymbol{x}) > V^{\mathcal{F}}(\boldsymbol{x})$.

Next, we prove part (ii) of the proposition. We first state the following intermediate result (whose proof is obvious and is omitted for brevity):

Lemma A9 *The consumer welfare $V^{\mathcal{L}}(\boldsymbol{x})$, $\widetilde{V}^{\mathcal{L}}(\boldsymbol{x})$, and $V^{\mathcal{F}}(\boldsymbol{x})$ are homogeneous of degree one and translation invariant—i.e., for each $k > 0$, we have that*

$$V^{\mathcal{L}}(k\boldsymbol{x}) = kV^{\mathcal{L}}(\boldsymbol{x}), \widetilde{V}^{\mathcal{L}}(k\boldsymbol{x}) = kV^{\mathcal{F}}(\boldsymbol{x}), V^{\mathcal{F}}(k\boldsymbol{x}) = kV^{\mathcal{F}}(\boldsymbol{x}),$$
$$V^{\mathcal{L}}(\boldsymbol{x}+k) = V^{\mathcal{L}}(\boldsymbol{x}), \widetilde{V}^{\mathcal{L}}(\boldsymbol{x}+k) = V^{\mathcal{F}}(\boldsymbol{x}), \text{ and } V^{\mathcal{F}}(\boldsymbol{x}+k) = V^{\mathcal{F}}(\boldsymbol{x}).$$

For notational convenience, define $\boldsymbol{x}^{(3)} := (x^{(1)}, x^{(2)}, x^{(3)})$, $\boldsymbol{x}^{(2)} := (x^{(1)}, x^{(2)})$, and $r(\boldsymbol{x}) := \frac{x^{(2)}-x^{(3)}}{x^{(1)}-x^{(2)}}$. Carrying out the algebra, we have that

$$\begin{aligned}
V^{\mathcal{L}} - V^{\mathcal{F}} =& \mathbb{E}_{\boldsymbol{x}\sim\tilde{g}}\left[V^{\mathcal{L}}(\boldsymbol{x}) - V^{\mathcal{F}}(\boldsymbol{x})\right] \\
\geq& \mathbb{E}_{\boldsymbol{x}\sim\tilde{g}}\left[V^{\mathcal{L}}\left(\boldsymbol{x}^{(3)}\right) - V^{\mathcal{F}}\left(\boldsymbol{x}^{(3)}\right)\right] \\
=& \mathbb{E}_{\boldsymbol{x}\sim\tilde{g}}\left[\mathbb{1}\left(r(\boldsymbol{x}) \leq \kappa\right) \times \left[V^{\mathcal{L}}\left(\boldsymbol{x}^{(3)}\right) - V^{\mathcal{F}}\left(\boldsymbol{x}^{(3)}\right)\right]\right] \\
&+ \mathbb{E}_{\boldsymbol{x}\sim\tilde{g}}\left[\mathbb{1}\left(r(\boldsymbol{x}) \geq \kappa\right) \times \left[V^{\mathcal{L}}\left(\boldsymbol{x}^{(3)}\right) - V^{\mathcal{F}}\left(\boldsymbol{x}^{(3)}\right)\right]\right] \\
\geq& \mathbb{E}_{\boldsymbol{x}\sim\tilde{g}}\left[\mathbb{1}\left(r(\boldsymbol{x}) \leq \kappa\right) \times \left[\widetilde{V}^{\mathcal{L}}\left(\boldsymbol{x}^{(3)}\right) - V^{\mathcal{F}}\left(\boldsymbol{x}^{(3)}\right)\right]\right] \\
&+ \mathbb{E}_{\boldsymbol{x}\sim\tilde{g}}\left[\mathbb{1}\left(r(\boldsymbol{x}) \geq \kappa\right) \times \left[V^{\mathcal{L}}\left(\boldsymbol{x}^{(3)}\right) - V^{\mathcal{F}}\left(\boldsymbol{x}^{(3)}\right)\right]\right],
\end{aligned} \tag{34}$$

where the first inequality follows from Lemma 5 and the second inequality from Lemma 6.

By Lemma A9, we have that

$$\widetilde{V}^{\mathcal{L}}\left(\boldsymbol{x}^{(3)}\right) - V^{\mathcal{F}}\left(\boldsymbol{x}^{(3)}\right) = \left(x^{(1)} - x^{(2)}\right) \times \left[\widetilde{V}^{\mathcal{L}}\left(1, 0, -r(\boldsymbol{x})\right) - V^{\mathcal{F}}\left(1, 0, -r(\boldsymbol{x})\right)\right]. \tag{35}$$

By Lemmas 5 and 6, $\widetilde{V}^{\mathcal{L}}(\boldsymbol{x}^{(3)})$ is increasing in $x^{(3)}$ and $W^{\mathcal{F}}(\boldsymbol{x}^{(3)})$ is independent of $x^{(3)}$; together with the postulated $r(\boldsymbol{x}) \leq \kappa$, we have that

$$\widetilde{V}^{\mathcal{L}}\left(1, 0, -r(\boldsymbol{x})\right) - V^{\mathcal{F}}\left(1, 0, -r(\boldsymbol{x})\right) \geq \widetilde{V}^{\mathcal{L}}\left(1, 0, -\kappa\right) - V^{\mathcal{F}}\left(1, 0, -\kappa\right)$$

$$= \widetilde{V}^{\mathcal{L}}(\kappa+1,\kappa,0) - V^{\mathcal{F}}(\kappa+1,\kappa,0), \quad (36)$$

where the equality follows from Lemma A9. Further, note that

$$\begin{aligned} V^{\mathcal{L}}\left(\boldsymbol{x}^{(3)}\right) - V^{\mathcal{F}}\left(\boldsymbol{x}^{(3)}\right) \geq & V^{\mathcal{L}}\left(\boldsymbol{x}^{(2)}\right) - V^{\mathcal{F}}\left(\boldsymbol{x}^{(2)}\right) \\ &= \left(x^{(1)} - x^{(2)}\right) \times \left[V^{\mathcal{L}}(\kappa+1,\kappa) - V^{\mathcal{F}}(\kappa+1,\kappa)\right], \quad (37) \end{aligned}$$

where the equality follows from Lemma A9 and the inequality from Lemma 5.

Recall that $f(\underline{t}) < 1/\underline{t}$ and (6) holds by assumption. Therefore, $V^{\mathcal{F}}(\kappa+1,\kappa) - V^{\mathcal{L}}(\kappa+1,\kappa) > 0$; together with (34), (35), (36), and (37), we have that

$$\begin{aligned} V^{\mathcal{L}} - V^{\mathcal{F}} \geq & \mathbb{E}_{\boldsymbol{x}\sim\tilde{g}}\left[\mathbb{1}\left(r(\boldsymbol{x}) \leq \kappa\right) \times (x^{(1)} - x^{(2)})\right] \times \left[\widetilde{V}^{\mathcal{L}}(\kappa+1,\kappa,0) - V^{\mathcal{F}}(\kappa+1,\kappa,0)\right] \\ & + \mathbb{E}_{\boldsymbol{x}\sim\tilde{g}}\left[\mathbb{1}\left(r(\boldsymbol{x}) \geq \kappa\right) \times (x^{(1)} - x^{(2)})\right] \times \left[V^{\mathcal{L}}(\kappa+1,\kappa) - V^{\mathcal{F}}(\kappa+1,\kappa)\right] \\ = & \mathbb{E}_{\boldsymbol{x}\sim\tilde{g}}\left[\mathbb{1}\left(r(\boldsymbol{x}) \geq \kappa\right) \times (x^{(1)} - x^{(2)})\right] \times \left[V^{\mathcal{F}}(\kappa+1,\kappa) - V^{\mathcal{L}}(\kappa+1,\kappa)\right] \\ & \times \left[\mathcal{C}_1(\tilde{g},\kappa)\mathcal{C}_2(f,\kappa) - 1\right] > 0, \end{aligned}$$

which concludes the proof. ■

Proof of Proposition 4

Proof. Similar to the proof of Proposition 2, we focus on the case of $f(\underline{t}) < 1/\underline{t}$ and show that $\Pi^{\mathcal{F}}(\boldsymbol{x}) > \Pi^{\mathcal{L}}(\boldsymbol{x})$ holds pointwisely. Note that $\Pi^{\mathcal{L}}(\boldsymbol{x})$ and $\Pi^{\mathcal{F}}(\boldsymbol{x})$ can be expressed as

$$\Pi^{\mathcal{L}}(\boldsymbol{x}) = \sum_{i=1}^{k(\boldsymbol{x})} \left(p_i^{\mathcal{L}}(\boldsymbol{x}) - c\right) \left[F\left(\alpha_{i-1}(\boldsymbol{x})\right) - F\left(\alpha_i(\boldsymbol{x})\right)\right], \quad (38)$$

$$\Pi^{\mathcal{F}}(\boldsymbol{x}) = \mathbb{E}[t](x_1 - x_2) = (x_1 - x_2)\int_{\underline{t}}^{\bar{t}} t f(t) dt. \quad (39)$$

Carrying out the algebra, we can obtain that

$$\begin{aligned} \Pi^{\mathcal{L}}(\boldsymbol{x}) &= p_1^{\mathcal{L}}(\boldsymbol{x}) - c - \sum_{i=1}^{k(\boldsymbol{x})-1} (x_i - x_{i+1})\alpha_i(\boldsymbol{x}) F\left(\alpha_i(\boldsymbol{x})\right) \\ &\leq p_1^{\mathcal{L}}(\boldsymbol{x}) - c - (x_1 - x_2)\alpha_1(\boldsymbol{x}) F\left(\alpha_1(\boldsymbol{x})\right) \end{aligned}$$

$$= (x_1 - x_2) \times \left[\frac{1 - F\left(\alpha_1(\boldsymbol{x})\right)}{f\left(\alpha_1(\boldsymbol{x})\right)} - \alpha_1(\boldsymbol{x}) F\left(\alpha_1(\boldsymbol{x})\right) \right], \tag{40}$$

where the first equality follows from (38) and (17); the inequality follows from the fact that $(x_i - x_{i+1})\alpha_i(\boldsymbol{x})F(\alpha_i(\boldsymbol{x}))$ is nonnegative for each $i \in \{2, \ldots, k(\boldsymbol{x}) - 1\}$; and the last equality follows from (12).

Recall that α^* is the solution to (4). It is useful to state the following lemma (whose proof can be found in the Supplemental Appendix):

Lemma A10 *Suppose that $f(\underline{t}) < 1/\underline{t}$. Fixing $\boldsymbol{x} = (x_1, \ldots, x_n)$ with $x_1 > \cdots > x_n$, it holds that $\alpha_1(\boldsymbol{x}) \geq \alpha^* \geq \alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})$. Moreover, both inequalities are strict if $k(\boldsymbol{x}) = n$.*

By Lemma A10, we can obtain that

$$\frac{1 - F\left(\alpha_1(\boldsymbol{x})\right)}{f\left(\alpha_1(\boldsymbol{x})\right)} - \alpha_1(\boldsymbol{x}) F\left(\alpha_1(\boldsymbol{x})\right) \leq \frac{1 - F(\alpha^*)}{f(\alpha^*)} - \alpha^* F(\alpha^*) = \frac{F(\alpha^*)}{f(\alpha^*)} + \alpha^* \left(1 - F(\alpha^*)\right), \tag{41}$$

where the inequality follows from Lemmas A1 and A10, and the equality follows from (4). Combining (40) and (41) yields that

$$\Pi^{\mathcal{L}}(\boldsymbol{x}) \leq (x_1 - x_2) \times \left[\frac{F(\alpha^*)}{f(\alpha^*)} + \alpha^* \left(1 - F(\alpha^*)\right) \right] \leq (x_1 - x_2) \times \int_{\underline{t}}^{\bar{t}} t f(t) dt = \Pi^{\mathcal{F}}(\boldsymbol{x}),$$

where the second inequality follows from (30) and the equality from (39). This concludes the proof. ■

Proof of Proposition 5

Proof. Firms' equilibrium pricing schedules when the market can be partially covered are characterized as follows:

(i) *(Uniform pricing)* The equilibrium price satisfies

$$p^{\mathcal{U}} = c + \frac{1 - H\left(0; p^{\mathcal{U}} - v\right)}{h\left(0; p^{\mathcal{U}} - v\right)}.$$

(ii) *(Choosiness-based pricing)* The equilibrium price satisfies

$$p^{\mathcal{C}}(t) = c + \frac{1 - \widehat{G}\left(0; \frac{p^{\mathcal{C}}(t)-v}{t}\right)}{\hat{g}\left(0; \frac{p^{\mathcal{C}}(t)-v}{t}\right)}.$$

(iii) *(Loyalty-based pricing)* Fix $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$, with $x_1 > \cdots > x_n$. Denote the number of active firms by $k(\boldsymbol{x})$. There exists a set of cutoffs $\alpha_0(\boldsymbol{x}) := \bar{t} \geq \alpha_1(\boldsymbol{x}) > \cdots > \alpha_{k(\boldsymbol{x})}(\boldsymbol{x}) \geq \alpha_{k(\boldsymbol{x})+1}(\boldsymbol{x}) := \underline{t}$, such that a consumer purchases from firm i if and only if her choosiness level $t \in [\alpha_i(\boldsymbol{x}), \alpha_{i-1}(\boldsymbol{x}))$ and does not make a purchase if $t \in [\underline{t}, \alpha_{k(\boldsymbol{x})}(\boldsymbol{x}))$. The details of each firm i's equilibrium pricing strategy can be found in Lemma B3 in the Supplemental Appendix.

(iv) *(Fully personalized pricing)* The equilibrium pricing schedules are

$$p_1^{\mathcal{F}}(\boldsymbol{x}, t) = \max\Big\{c, \min\big\{t(x_1 - x_2), v + tx_1\big\}\Big\} \text{ and } p_i^{\mathcal{F}}(\boldsymbol{x}, t) = c, i \in \{2, \ldots, n\}.$$

The equilibrium characterization under uniform pricing and that under choosiness pricing follow from Lemma 1 in Rhodes and Zhou (2024). The characterization under fully personalized pricing is obvious, and that under loyalty-based pricing can be found in the Supplemental Appendix.

Consider the consumer welfare comparison as $c \nearrow v + \bar{t}\overline{x}$. We first consider uniform pricing. The equilibrium consumer welfare under uniform pricing amounts to $V^{\mathcal{U}} = \mathbb{E}\big[\max_{i \in \mathcal{N}} \big\{v + tx_i - p^{\mathcal{U}}, 0\big\}\big]$. In what follows, we show that

$$V^{\mathcal{U}} = \frac{4n}{81} \times \frac{f(\bar{t})g(\overline{x})}{\overline{x}\bar{t}} \times \big(\overline{x}\bar{t} + v - c\big)^3 \times \big(1 + o(1)\big). \tag{42}$$

For notational convenience, define

$$\mathcal{A}_1 := \sum_{i \in \mathcal{N}} \mathbb{E}\left[\max\Big\{v + tx_i - p^{\mathcal{U}}, 0\Big\}\right] \text{ and}$$

$$\mathcal{A}_2 := \sum_{i,j \in \mathcal{N}, i<j} \mathbb{E}\left[\max\Big\{v + tx_i - p^{\mathcal{U}}, v + tx_j - p^{\mathcal{U}}\Big\} \mathbb{1}\Big(v + tx_i - p^{\mathcal{U}} \geq 0, v + tx_j - p^{\mathcal{U}} \geq 0\Big)\right].$$

By the inclusion-exclusion principle, we can conclude that

$$\mathcal{A}_1 - \mathcal{A}_2 \leq V^{\mathcal{U}} \leq \mathcal{A}_1. \tag{43}$$

It can be verified that $p^{\mathcal{U}} - c = \frac{\overline{x}\overline{t}+v-c}{3} \times (1+o(1))$ (see Lemma B4 in the Supplemental Appendix for more details). Therefore, $\mathcal{A}_1$ and $\mathcal{A}_2$ can be expressed as follows when c approaches $v + \overline{t}\overline{x}$:

$$\mathcal{A}_1 = \frac{4n}{81} \times \frac{f(\overline{t})g(\overline{x})}{\overline{x}\overline{t}} \times \left(\overline{x}\overline{t} + v - c\right)^3 \times \left(1 + o(1)\right) \text{ and } \mathcal{A}_2 = o\left((\overline{x}\overline{t} + v - c)^3\right). \tag{44}$$

Plugging (44) into (43) yields that (42). Similarly, the consumer welfare under the other three pricing regimes when c approaches $v + \overline{t}\overline{x}$ can be expressed as follows:

$$V^{\mathcal{L}} = V^{\mathcal{C}} = \frac{n}{24} \times \frac{f(\overline{t})g(\overline{x})}{\overline{x}\overline{t}} \times \left(\overline{x}\overline{t} + v - c\right)^3 \times \left(1 + o(1)\right) \tag{45}$$

and

$$V^{\mathcal{F}} = o\left((v + \overline{x}\overline{t} - c)^3\right). \tag{46}$$

Combining (42), (45), and (46) yields that

$$\lim_{c \nearrow v + \overline{t}\overline{x}} V^{\mathcal{U}} : V^{\mathcal{C}} : V^{\mathcal{L}} : V^{\mathcal{F}} = 32 : 27 : 27 : 0.$$

The ratios between industry profits under different pricing regimes can be similarly established. This concludes the proof. ■

Supplemental Appendix: Omitted Proofs

Proof of Lemma A1

Proof. Clearly, $\ln f(t) = \ln \frac{f(t)}{t} + \ln t$ is concave. The log-concavity of $F(t)$ and $1 - F(t)$ follows from the Prékopa-Borell theorem. Further, the log-concavity of $1 - F(t)$ implies that $\frac{1-F(t)}{f(t)}$ is weakly decreasing. Therefore, both $\frac{1-F(t)}{f(t)} - t$ and $\frac{1-F(t)}{f(t)} - tF(t)$ are strictly decreasing. ■

Proof of Lemma A2

Proof. Let $\mathcal{Q}(\alpha) := \frac{1-F(\alpha)}{f(\alpha)} - \frac{F(\alpha)}{f(\alpha)} - \alpha$. By Lemma A1, both $F(t)$ and $1 - F(t)$ are log-concave, which implies that $\frac{F(\alpha)}{f(\alpha)}$ is increasing in α and $\frac{1-F(\alpha)}{f(\alpha)}$ is decreasing in α. Therefore, $\mathcal{Q}$ is strictly decreasing in α. It follows from $f(\underline{t}) < 1/\underline{t}$ that $\mathcal{Q}(\underline{t}) = \frac{1}{f(\underline{t})} - \underline{t} > 0$; moreover, $\mathcal{Q}(\overline{t}) = -\frac{1}{f(\overline{t})} - \overline{t} < 0$. Therefore, there exists a unique $\alpha^* \in (\underline{t}, \overline{t})$ such that $\mathcal{Q}(\alpha^*) = 0$. This concludes the proof. ■

Proof of Lemma A3

Proof. Carrying out the algebra, we have that

$$\begin{aligned}
\frac{\partial \psi}{\partial \alpha} &= \psi(\alpha, \alpha') \times \left[\frac{f(\alpha)}{F(\alpha) - F(\alpha')} - \frac{f'(\alpha)}{f(\alpha) + \frac{f(\alpha')\delta_1}{\delta_2}} \right] \\
&> \psi(\alpha, \alpha') \times \left[\frac{f(\alpha)}{F(\alpha)} - \frac{f'(\alpha)}{f(\alpha) + \frac{f(\alpha')\delta_1}{\delta_2}} \right] \\
&= \frac{\psi(\alpha, \alpha')}{F(\alpha) \times \left[f(\alpha) + \frac{f(\alpha')\delta_1}{\delta_2} \right]} \times \left\{ f(\alpha) \left[f(\alpha) + \frac{f(\alpha')\delta_1}{\delta_2} \right] - f'(\alpha)F(\alpha) \right\} \\
&\geq \frac{\psi(\alpha, \alpha')}{F(\alpha) \times \left[f(\alpha) + \frac{f(\alpha')\delta_1}{\delta_2} \right]} \times \left\{ \left[f(\alpha) \right]^2 - f'(\alpha)F(\alpha) \right\}.
\end{aligned}$$

By Lemma A1, $F(\cdot)$ is log-concave, which implies that $\left[f(\alpha) \right]^2 - f'(\alpha)F(\alpha) \geq 0$ and thus $\frac{\partial \psi}{\partial \alpha} > 0$. Similarly, we have that

$$\frac{\partial \psi}{\partial \alpha'} = -\, \psi(\alpha, \alpha') \times \left[\frac{f(\alpha')}{F(\alpha) - F(\alpha')} + \frac{f'(\alpha')}{\frac{f(\alpha)\delta_2}{\delta_1} + f(\alpha')} \right]$$

$$
\begin{aligned}
&< -\psi(\alpha,\alpha') \times \left[\frac{f(\alpha')}{1-F(\alpha')} + \frac{f'(\alpha')}{\frac{f(\alpha)\delta_2}{\delta_1} + f(\alpha')}\right] \\
&= -\frac{\psi(\alpha,\alpha')}{\left[1-F(\alpha')\right] \times \left[\frac{f(\alpha)\delta_2}{\delta_1} + f(\alpha')\right]} \times \left\{ f(\alpha') \times \left[\frac{f(\alpha)\delta_2}{\delta_1} + f(\alpha')\right] + f'(\alpha') \times \left[1-F(\alpha')\right] \right\} \\
&\leq -\frac{\psi(\alpha,\alpha')}{\left[1-F(\alpha')\right] \times \left[\frac{f(\alpha)\delta_2}{\delta_1} + f(\alpha')\right]} \times \left\{ f(\alpha')^2 + f'(\alpha') \times \left[1-F(\alpha')\right] \right\}.
\end{aligned}
$$

By Lemma A1, $1-F(\cdot)$ is log-concave, which implies that $f(\alpha')^2+f'(\alpha')\times\left[1-F(\alpha')\right] \geq 0$ and thus $\frac{\partial\psi}{\partial\alpha'} < 0$. This concludes the proof. ■

Proof of Lemma A4

Proof. For notational convenience, denote $\alpha_+ := \frac{p_{j_1}-p_i}{x_{j_1}-x_i}$ and $\alpha_- := \frac{p_i-p_{j_2}}{x_i-x_{j_2}}$. It suffices to show that $\frac{\mathcal{D}_i(p_i,\boldsymbol{p}_{-i};\boldsymbol{x})}{\partial\mathcal{D}_i(p_i,\boldsymbol{p}_{-i};\boldsymbol{x})/\partial p_i}$ is increasing in p_i. Set $\delta_1 = x_{j_1} - x_i$ and $\delta_2 = x_i - x_{j_2}$ in the function $\psi\left(\alpha,\alpha'\right)$ defined in Lemma A3. Simple algebra would verify that

$$
\frac{\mathcal{D}_i(p_i,\boldsymbol{p}_{-i};\boldsymbol{x})}{\partial\mathcal{D}_i(p_i,\boldsymbol{p}_{-i};\boldsymbol{x})/\partial p_i} = -\psi(\alpha_+,\alpha_-).
$$

By Lemma A3, $\psi(\alpha_+,\alpha_-)$ is increasing in α_+ and decreasing in α_-. This, together with the fact that α_+ is decreasing in p_i and α_- is increasing in p_i, implies that $\frac{\mathcal{D}_i(p_i,\boldsymbol{p}_{-i};\boldsymbol{x})}{\partial\mathcal{D}_i(p_i,\boldsymbol{p}_{-i};\boldsymbol{x})/\partial p_i}$ increases with p_i. This completes the proof. ■

Proof of Lemma A5

Proof. We prove the lemma by induction.

Base case: $\acute{\alpha}_1(\alpha_1)$ is obviously increasing in α_1. By Lemma A3, $\acute{p}_1(\alpha_1)$—which we define in (20)—is decreasing in α_1.

Inductive step: Suppose that $\acute{p}_i(\alpha_1)$ is decreasing in α_1 and $\acute{\alpha}_i(\alpha_1)$ is increasing in α_1. By (21), $\acute{p}_{i+1}(\alpha_1)$ is decreasing in α_1. Set $\delta_1 = x_{i-1} - x_i$ and $\delta_2 = x_i - x_{i+1}$ in the function $\psi\left(\alpha,\alpha'\right)$ defined in Lemma A3. It follows from (22) that

$$
\acute{p}_{i+1}(\alpha_1) = c + \psi\left(\acute{\alpha}_i(\alpha_1), \acute{\alpha}_{i+1}(\alpha_1)\right).
$$

Recall that $\acute{p}_{i+1}(\alpha_1)$ is decreasing in α_1 and $\acute{\alpha}_i(\alpha_1)$ is increasing in α_1 by assumption. Therefore,

$$0 > \frac{d\acute{p}_{i+1}(\alpha_1)}{d\alpha_1} = \underbrace{\frac{\partial\psi}{\partial\alpha}}_{\geq 0} \times \underbrace{\frac{d\acute{\alpha}_i}{d\alpha_1}}_{>0} + \underbrace{\frac{\partial\psi}{\partial\alpha'}}_{\leq 0} \times \frac{d\acute{\alpha}_{i+1}}{d\alpha_1},$$

from which we can conclude that $\frac{d\acute{\alpha}_{i+1}}{d\alpha_1} > 0$.

Conclusion: By the principle of induction, $\acute{p}_i(\alpha_1)$ is decreasing in α_1 and $\acute{\alpha}_i(\alpha_1)$ is increasing in α_1 for all $i \in \{1, \ldots, k(\alpha_1)\}$. This concludes the proof. ∎

Proof of Lemma A6

Proof. The proof consists of two steps. In the first step, we construct the PDF $f^\dagger$ and the CDF $F^\dagger$ such that (31) holds. In the second step, we prove that the constructed $f^\dagger$ satisfies (32).

Step I Define $f^\dagger(t) := \mathcal{M}te^{\beta t}$ and

$$F^\dagger(t) = \int_0^t f^\dagger(s)ds = \mathcal{M}\frac{e^{\beta t}(\beta t - 1) + 1}{\beta^2}, \forall\, t \in \left[0, \bar{t}^\dagger\right],$$

where the parameters $(\mathcal{M}, \beta, \bar{t}^\dagger)$ are to be constructed to satisfy (31) and $F^\dagger(\bar{t}^\dagger) = 1$.

We first construct $(\mathcal{M}, \beta)$. Note that

$$\frac{F^\dagger(\alpha^*)}{f^\dagger(\alpha^*)} = \frac{e^{\beta\alpha^*}(\beta\alpha^* - 1) + 1}{\beta^2\alpha^* e^{\beta\alpha^*}} = \alpha^* \times \underbrace{\frac{e^u(u-1)+1}{u^2 e^u}}_{=:\phi(u)},$$

where $u := \beta\alpha^*$. Further, we have that $\phi(+\infty) = 0$, $\phi(-\infty) = +\infty$, and $\phi'(\cdot) < 0$ on $\mathbb{R}$. Therefore, there exists a unique u to satisfy

$$\phi(u) = \frac{F^\dagger(\alpha^*)}{\alpha^* f^\dagger(\alpha^*)}.$$

It can be verified that $(\mathcal{M}, \beta) = \left(\frac{f(\alpha^*)}{\alpha^* e^{\beta\alpha^*}}, \frac{u}{\alpha^*}\right)$ satisfies (31). By construction, β can be positive or negative. Fixing $\mathcal{M}$ and β, it can be verified that $\mathcal{M}\frac{e^{u'}(u'-1)+1}{\beta^2}$ is strictly decreasing in u' for $u' < 0$ and increasing in u' for $u' > 0$, and thus there exist two solutions to $\mathcal{M}\frac{e^{u'}(u'-1)+1}{\beta^2} = 1$. Pick the solution that has the same sign as β and denote it as u' with slight abuse of notation. To complete the construction, we set $\bar{t}^\dagger = u'/\beta$.

Step II We prove (32). Let

$$\beta' := \frac{d}{dt}\left[\ln \frac{f(t)}{t}\right]\bigg|_{t=\alpha^*}.$$

We first show that $\beta' \leq \beta$. Suppose, to the contrary, that $\beta' > \beta$. From the log-concavity of $f(t)/t$, we have that

$$\ln \frac{f(t)}{t} \leq \ln \frac{f(\alpha^*)}{\alpha^*} + \beta'(t-\alpha^*) < \ln \frac{f(\alpha^*)}{\alpha^*} + \beta(t-\alpha^*), \forall\, t \in [\underline{t}, \alpha^*),$$

which in turn implies that

$$f(t) < t\frac{f(\alpha^*)}{\alpha^*}e^{\beta(t-\alpha^*)} = f^\dagger(t), \forall\, t \in [\underline{t}, \alpha^*).$$

It follows from the above inequality that

$$F(\alpha^*) = \int_{\underline{t}}^{\alpha^*} f(t)dt < \int_0^{\alpha^*} f^\dagger(t)dt = F^\dagger(\alpha^*) = F(\alpha^*),$$

which is a contradiction. Therefore, we must have $\beta' \leq \beta$.

Next, recall that $\ln \frac{f(t)}{t}$ is concave in t and $\ln \frac{f^\dagger(t)}{t}$ is linear in t by construction. Further, $f(\alpha^*) = f^\dagger(\alpha^*)$ and $\beta' \leq \beta$. Therefore, there exists $\alpha^\dagger < \alpha^*$ such that

$$f(t) \leq f^\dagger(t), \forall\, t \in [0, \alpha^\dagger], \text{ and } f(t) \geq f^\dagger(t), \forall\, t \in [\alpha^\dagger, \alpha^*].$$

The above condition, together with $F(\alpha^*) = F^\dagger(\alpha^*)$, implies that $F(\cdot|t \leq \alpha^*)$ first-order stochastically dominates $F^\dagger(\cdot|t \leq \alpha^*)$—i.e., $F(\cdot|t \leq \alpha^*) \geq_{FOSD} F^\dagger(\cdot|t \leq \alpha^*)$—from which we can conclude that

$$\int_{\underline{t}}^{\alpha^*} tf(t)dt \geq \int_0^{\alpha^*} tf^\dagger(t)dt. \tag{A1}$$

Similarly, it follows from $\beta' \leq \beta$ that

$$\ln \frac{f(t)}{t} \leq \ln \frac{f(\alpha^*)}{\alpha^*} + \beta'(t-\alpha^*) \leq \ln \frac{f(\alpha^*)}{\alpha^*} + \beta(t-\alpha^*), \forall\, t \in \left[\alpha^*, \bar{t}^\dagger\right],$$

which implies that

$$f(t) < t\frac{f(\alpha^*)}{\alpha^*}e^{\beta(t-\alpha^*)} \leq f^\dagger(t), \forall t \in \left[\alpha^*, \bar{t}^\dagger\right].$$

Because $F(\alpha^*) = F^\dagger(\alpha^*)$, we have that $\bar{t}^\dagger \leq \bar{t}$. Moreover, $f(t) \leq f^\dagger(t), \forall t \in \left[\alpha^*, \bar{t}^\dagger\right]$ and $f(t) \geq f^\dagger(t), \forall t \in \left[\bar{t}^\dagger, \bar{t}\right]$. Therefore, $F(\cdot|t \geq \alpha^*) \geq_{FOSD} F^\dagger(\cdot|t \geq \alpha^*)$, which implies that

$$\int_{\alpha^*}^{\bar{t}} tf(t)dt \geq \int_{\alpha^*}^{\bar{t}^\dagger} tf^\dagger(t)dt. \tag{A2}$$

Summing (A1) and (A2) completes the proof. ■

Proof of Lemma A7

Proof. Recall from the construction in Lemma A6 that $u' = \beta\bar{t}^\dagger$ and $u = \beta\alpha^*$. Further, we have that

$$\int_0^{\bar{t}^\dagger} tf^\dagger(t)dt = \int_0^{\bar{t}^\dagger} \mathcal{M}t^2e^{\beta t}dt = \mathcal{M}\frac{e^{u'}\left[(u')^2 - 2u' + 2\right] - 2}{\beta^3}, \tag{A3}$$

$$F(\alpha^*) = F^\dagger(\alpha^*) = \mathcal{M}\frac{e^u(u-1)+1}{\beta^2}, \text{ and } f(\alpha^*) = f^\dagger(\alpha^*) = \mathcal{M}\frac{ue^u}{\beta}. \tag{A4}$$

Substituting $\alpha^* = u/\beta$, (A3), and (A4) into (33), it remains to prove

$$\mathcal{M}\frac{e^{u'}\left[(u')^2 - 2u' + 2\right] - 2}{\beta^2} \geq \frac{e^u(u-1)+1}{ue^u} + u\left[1 - \mathcal{M}\frac{e^u(u-1)+1}{\beta^2}\right]. \tag{A5}$$

Further, from (4), we have that

$$1 = \alpha^* f^\dagger(\alpha^*) + 2F^\dagger(\alpha^*) = \frac{\mathcal{M}}{\beta^2}\left[u^2e^u + 2e^u(u-1) + 2\right]. \tag{A6}$$

Therefore, (A5) can be rewritten as

$$\frac{e^{u'}\left[(u')^2 - 2u' + 2\right] - 2}{u^2e^u + 2e^u(u-1) + 2} \geq \frac{e^u(u-1)+1}{ue^u} + u\frac{u^2e^u + e^u(u-1)+1}{u^2e^u + 2e^u(u-1) + 2}. \tag{A7}$$

Recall that u' is the solution to $\mathcal{M}\frac{e^{u'}(u'-1)+1}{\beta^2} = 1$; together with (A6), u and u' must satisfy

$$u^2 e^u + 2e^u(u-1) + 2 = \frac{\beta^2}{\mathcal{M}} = e^{u'}(u'-1) + 1. \tag{A8}$$

It can be verified that (A7) holds for all (u, u') satisfying (A8). This completes the proof. ■

Proof of Lemma A8

Proof. By Lemma 2, the equilibrium is independent of x_{n+1} when $k(\boldsymbol{x}_{n+1}) \leq n-1$. Therefore, $k(\boldsymbol{x}_{n+1}) \in \{n, n+1\}$. We consider the following two cases.

Case I: $\boldsymbol{k(x_{n+1}) = n}$. If $\xi = 0$, then the equilibrium price $p_i^{\mathcal{L}}(\boldsymbol{x}_{n+1})$ is independent of x_{n+1} and is thus weakly decreasing in x_{n+1} for $i \in \{1, \ldots, n\}$. If otherwise $\xi > 0$, by (16), $p_n^{\mathcal{L}}(\boldsymbol{x}_{n+1}) = c + \underline{t}(x_n - x_{n+1})$. For each i, we slightly abuse notation and denote the functions defined in (21) and (22) by $\acute{p}_{i+1}(\alpha_1; \boldsymbol{x}_{i+1})$ and $\acute{\alpha}_{i+1}(\alpha_1; \boldsymbol{x}_{i+2})$, respectively, where $\boldsymbol{x}_i := (x_1, \ldots, x_i)$. Similarly, denote $\acute{p}_1(\alpha_1)$ by $\acute{p}_1(\alpha_1; \boldsymbol{x}_2)$.

Note that $\acute{p}_n(\alpha_1(\boldsymbol{x}_{n+1}); \boldsymbol{x}_n) = p_n^{\mathcal{L}}(\boldsymbol{x}_{n+1}) = c + \underline{t}(x_n - x_{n+1})$. By the implicit function theorem, we have that $0 > -\underline{t} = \frac{dp_n^{\mathcal{L}}(\boldsymbol{x}_{n+1})}{dx_{n+1}} = \frac{\partial \acute{p}_n}{\partial \alpha_1} \times \frac{d\alpha_1(\boldsymbol{x}_{n+1})}{dx_{n+1}}$. Recall from Lemma A5 that $\frac{\partial \acute{p}_n}{\partial \alpha_1} < 0$. We can conclude that

$$\frac{d\alpha_1(\boldsymbol{x}_{n+1})}{dx_{n+1}} > 0. \tag{A9}$$

Therefore, for each $i \in \{1, \ldots, n-1\}$, we have that

$$\frac{dp_i^{\mathcal{L}}(\boldsymbol{x}_{n+1})}{dx_{n+1}} = \left.\frac{\partial \acute{p}_i}{\partial \alpha_1}\right|_{\alpha_1 = \alpha_1(\boldsymbol{x}_{n+1})} \times \frac{d\alpha_1(\boldsymbol{x}_{n+1})}{dx_{n+1}}. \tag{A10}$$

By Lemma A5 and (A9), the right-hand side of (A10) is negative, which implies that $p_i^{\mathcal{L}}(\boldsymbol{x}_{n+1})$ is decreasing in x_{n+1}.

Case II: $\boldsymbol{k(x_{n+1}) = n+1}$. By Lemma A5 and (A10), it suffices to prove (A9). The KKT condition (14) becomes

$$p_{n+1}^{\mathcal{L}}(\boldsymbol{x}_{n+1}) = c + \frac{F\big(\alpha_n(\boldsymbol{x}_{n+1})\big)}{\frac{f\left(\alpha_n(\boldsymbol{x}_{n+1})\right)}{x_n - x_{n+1}}} = c + (x_n - x_{n+1}) \times \frac{F\big(\alpha_n(\boldsymbol{x}_{n+1})\big)}{f\left(\alpha_n(\boldsymbol{x}_{n+1})\right)}. \tag{A11}$$

Setting $i = n$ in (17), we can obtain that

$$p_n^{\mathcal{L}}(\boldsymbol{x}_{n+1}) = p_{n+1}^{\mathcal{L}}(\boldsymbol{x}_{n+1}) + \alpha_n(\boldsymbol{x}_{n+1})(x_n - x_{n+1}). \tag{A12}$$

Combining (A11) and (A12) yields that

$$p_n^{\mathcal{L}}(\boldsymbol{x}_{n+1}) = c + \left[\frac{F\left(\alpha_n(\boldsymbol{x}_{n+1})\right)}{f\left(\alpha_n(\boldsymbol{x}_{n+1})\right)} + \alpha_n(\boldsymbol{x}_{n+1})\right] \times (x_n - x_{n+1}). \tag{A13}$$

Setting $j = n - 1$ in (22) yields that

$$\acute{p}_n(\alpha_1, \boldsymbol{x}_n) = c + \frac{F\left(\acute{\alpha}_{n-1}(\alpha_1; \boldsymbol{x}_n)\right) - F\left(\acute{\alpha}_n(\alpha_1; \boldsymbol{x}_{n+1})\right)}{\frac{f\left(\acute{\alpha}_{n-1}(\alpha_1; \boldsymbol{x}_n)\right)}{x_{n-1} - x_n} + \frac{f\left(\acute{\alpha}_n(\alpha_1; \boldsymbol{x}_{n+1})\right)}{x_n - x_{n+1}}}.$$

Taking the derivative of $\acute{p}_n(\alpha_1; \boldsymbol{x}_n)$ with respect to x_{n+1}, we have that

$$\underbrace{\frac{\partial}{\partial \alpha_n}\left[\frac{F\left(\acute{\alpha}_{n-1}(\alpha_1; \boldsymbol{x}_n)\right) - F(\alpha_n)}{\frac{f\left(\acute{\alpha}_{n-1}(\alpha_1; \boldsymbol{x}_n)\right)}{x_{n-1} - x_n} + \frac{f(\alpha_n)}{x_n - x_{n+1}}}\right]\Bigg|_{\alpha_n = \acute{\alpha}_n(\alpha_1; \boldsymbol{x}_{n+1})}}_{<0} \times \frac{\partial \acute{\alpha}_n}{\partial x_{n+1}}$$
$$+ \underbrace{\frac{\partial}{\partial x_{n+1}}\left[\frac{F\left(\acute{\alpha}_{n-1}(\alpha_1; \boldsymbol{x}_n)\right) - F(\alpha_n)}{\frac{f\left(\acute{\alpha}_{n-1}(\alpha_1; \boldsymbol{x}_n)\right)}{x_{n-1} - x_n} + \frac{f(\alpha_n)}{x_n - x_{n+1}}}\right]\Bigg|_{\alpha_n = \acute{\alpha}_n(\alpha_1; \boldsymbol{x}_{n+1})}}_{<0} = 0,$$

where the equality follows from the implicit function theorem. The above equation implies that $\frac{\partial \acute{\alpha}_n}{\partial x_{n+1}} < 0$.

Recall that $p_n^{\mathcal{L}}(\boldsymbol{x}_{n+1}) = \acute{p}_n(\alpha_1(\boldsymbol{x}_{n+1}); \boldsymbol{x}_n)$ and $\alpha_n(\boldsymbol{x}_{n+1}) = \acute{\alpha}_n(\alpha_1(\boldsymbol{x}_{n+1}); \boldsymbol{x}_{n+1})$; together with (A13), we have that

$$\acute{p}_n(\alpha_1(\boldsymbol{x}_{n+1}); \boldsymbol{x}_n) = c + \left[\frac{F\left(\acute{\alpha}_n(\alpha_1(\boldsymbol{x}_{n+1}); \boldsymbol{x}_{n+1})\right)}{f\left(\acute{\alpha}_n(\alpha_1(\boldsymbol{x}_{n+1}); \boldsymbol{x}_{n+1})\right)} + \acute{\alpha}_n\left(\alpha_1(\boldsymbol{x}_{n+1}); \boldsymbol{x}_{n+1}\right)\right] \times (x_n - x_{n+1}).$$

Taking the derivative of $\acute{p}_n(\alpha_1(\boldsymbol{x}_{n+1}); \boldsymbol{x}_n)$ with respect to x_{n+1}, we can obtain that

$$
\begin{aligned}
&\underbrace{\frac{\partial \acute{p}_n}{\partial \alpha_1}}_{<0} \times \frac{d\alpha_1(\boldsymbol{x}_{n+1})}{dx_{n+1}} \\
= -& \underbrace{\left[\frac{F\left(\acute{\alpha}_n(\alpha_1(\boldsymbol{x}_{n+1}); \boldsymbol{x}_{n+1})\right)}{f\left(\acute{\alpha}_n(\alpha_1(\boldsymbol{x}_{n+1}); \boldsymbol{x}_{n+1})\right)} + \acute{\alpha}_n\left(\alpha_1(\boldsymbol{x}_{n+1}); \boldsymbol{x}_{n+1}\right)\right]}_{>0} \\
&+ \underbrace{\frac{\partial}{\partial \alpha_n}\left[\frac{F(\alpha_n)}{f(\alpha_n)} + \alpha_n\right]\Bigg|_{\alpha_n = \acute{\alpha}_n\left(\alpha_1(\boldsymbol{x}_{n+1}); \boldsymbol{x}_{n+1}\right)}}_{>0} \times \left[\underbrace{\frac{\partial \acute{\alpha}_n}{\partial \alpha_1}}_{>0} \times \frac{d\alpha_1(\boldsymbol{x}_{n+1})}{dx_{n+1}} + \underbrace{\frac{\partial \acute{\alpha}_n}{\partial x_{n+1}}}_{<0}\right],
\end{aligned}
$$

where the equality again follows from the implicit function theorem. This implies that $\frac{d\alpha_1(\boldsymbol{x}_{n+1})}{dx_{n+1}} > 0$ and completes the proof. ■

Proof of Corollary 2

Proof. We first show that $\mathcal{C}_1(\tilde{g}, \kappa) \geq \kappa^2 + 2\kappa$. Note that the PDF of the conditional distribution $x^{(2)}|(x^{(1)}, x^{(3)})$ is $\frac{g(x^{(2)})}{G(x^{(1)}) - G(x^{(3)})}$ and is weakly decreasing in $x^{(2)}$. Therefore,

$$
\begin{aligned}
&\mathbb{E}_{\boldsymbol{x}\sim\tilde{g}}\left[\mathbb{1}\left(\frac{x^{(2)} - x^{(3)}}{x^{(1)} - x^{(2)}} \geq \kappa\right) \times (x^{(1)} - x^{(2)}) \middle| \left(x^{(1)}, x^{(3)}\right)\right] \\
&= \left(x^{(1)} - x^{(3)}\right) \times \mathbb{E}_{\boldsymbol{x}\sim\tilde{g}}\left[\mathbb{1}\left(\frac{x^{(2)} - x^{(3)}}{x^{(1)} - x^{(2)}} \geq \kappa\right) \times \frac{x^{(1)} - x^{(2)}}{x^{(1)} - x^{(3)}} \middle| \left(x^{(1)}, x^{(3)}\right)\right] \\
&= \left(x^{(1)} - x^{(3)}\right)^2 \times \int_{\frac{\kappa}{\kappa+1}}^{1} (1 - s)\frac{g\left(sx^{(1)} + (1 - s)x^{(3)}\right)}{G(x^{(1)}) - G(x^{(3)})} ds \\
&\leq \left(x^{(1)} - x^{(3)}\right)^2 \times \frac{g\left(\frac{\kappa x^{(1)} + x^{(3)}}{\kappa+1}\right)}{G(x^{(1)}) - G(x^{(3)})} \times \int_{\frac{\kappa}{\kappa+1}}^{1} (1 - s) ds \\
&= \left(x^{(1)} - x^{(3)}\right)^2 \times \frac{g\left(\frac{\kappa x^{(1)} + x^{(3)}}{\kappa+1}\right)}{G(x^{(1)}) - G(x^{(3)})} \times \frac{1}{2(\kappa + 1)^2}. \qquad \text{(A14)}
\end{aligned}
$$

Similarly, we have that

$$\begin{aligned}
&\mathbb{E}_{\boldsymbol{x}\sim\tilde{g}}\left[\mathbb{1}\left(\frac{x^{(2)}-x^{(3)}}{x^{(1)}-x^{(2)}}\leq\kappa\right)\times\left(x^{(1)}-x^{(2)}\right)\middle|\left(x^{(1)},x^{(3)}\right)\right]\\
&\geq\left(x^{(1)}-x^{(3)}\right)^2\times\frac{g\left(\frac{\kappa x^{(1)}+x^{(3)}}{\kappa+1}\right)}{G(x^{(1)})-G(x^{(3)})}\times\int_0^{\frac{\kappa}{\kappa+1}}(1-s)ds\\
&=\left(x^{(1)}-x^{(3)}\right)^2\times\frac{g\left(\frac{\kappa x^{(1)}+x^{(3)}}{\kappa+1}\right)}{G(x^{(1)})-G(x^{(3)})}\times\frac{\kappa^2+2\kappa}{2(\kappa+1)^2}.
\end{aligned}\tag{A15}$$

Combining (A14) and (A15) yields that

$$\begin{aligned}
&\mathbb{E}_{\boldsymbol{x}\sim\tilde{g}}\left[\mathbb{1}\left(\frac{x^{(2)}-x^{(3)}}{x^{(1)}-x^{(2)}}\leq\kappa\right)\times(x^{(1)}-x^{(2)})\middle|\left(x^{(1)},x^{(3)}\right)\right]\\
&\geq(\kappa^2+2\kappa)\times\mathbb{E}_{\boldsymbol{x}\sim\tilde{g}}\left[\mathbb{1}\left(\frac{x^{(2)}-x^{(3)}}{x^{(1)}-x^{(2)}}\geq\kappa\right)\times(x^{(1)}-x^{(2)})\middle|(x^{(1)},x^{(3)})\right].
\end{aligned}$$

By the law of iterated expectations, we can obtain that

$$\begin{aligned}
&\mathbb{E}_{\boldsymbol{x}\sim\tilde{g}}\left[\mathbb{1}\left(\frac{x^{(2)}-x^{(3)}}{x^{(1)}-x^{(2)}}\leq\kappa\right)\times(x^{(1)}-x^{(2)})\right]\\
&\geq(\kappa^2+2\kappa)\times\mathbb{E}_{\boldsymbol{x}\sim\tilde{g}}\left[\mathbb{1}\left(\frac{x^{(2)}-x^{(3)}}{x^{(1)}-x^{(2)}}\geq\kappa\right)\times(x^{(1)}-x^{(2)})\right],
\end{aligned}$$

which implies that $\mathcal{C}_1(\tilde{g},\kappa)\geq\kappa^2+2\kappa$. Setting $\kappa=\frac{1}{2}$ yields that $\mathcal{C}_1(\tilde{g},\frac{1}{2})\geq\frac{5}{4}$.

Next, we show that $\min_f\mathcal{C}_2(f,\frac{1}{2})>\frac{4}{5}$. It is useful to state the following lemma:

Lemma B1 *Fix κ and suppose that $f^\ddagger$ minimizes $\mathcal{C}_2(f,\kappa)$ among all density functions f that satisfy Assumption 2, $f(\underline{t})<1/\underline{t}$, and (6). Then $\ln\frac{f^\ddagger(t)}{t}$ is piecewise linear with at most three segments.*

Proof. Fix $\boldsymbol{x}=(\kappa+1,\kappa,0)$. For each PDF $f(\cdot)$ that satisfies Assumption 2, $f(\underline{t})<1/\underline{t}$, and (6), we construct an auxiliary distribution with PDF $f^\ddagger(\cdot)$ that takes the following form:

$$f^{\ddagger}(t) := \begin{cases} 0, & t < t_0^{\ddagger} \\ \mathcal{M}_1 t e^{\beta_1 (t-\alpha_1(\boldsymbol{x}))}, & t_0^{\ddagger} \le t \le t_1^{\ddagger}, \\ \mathcal{M}_2 t e^{\beta_2 (t-\alpha^*)}, & t_1^{\ddagger} \le t \le t_2^{\ddagger}, \\ \mathcal{M}_3 t e^{\beta_3 (t-\alpha_2(\boldsymbol{x}))}, & t_2^{\ddagger} \le t \le t_3^{\ddagger}, \\ 0, & t > t_3^{\ddagger}. \end{cases} \tag{A16}$$

Further, let $F^{\ddagger}(t) := \int_0^t f^{\ddagger}(s)ds$.

The set of parameters $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \beta_1, \beta_2, \beta_3, t_0^{\ddagger}, t_1^{\ddagger}, t_2^{\ddagger}, t_3^{\ddagger})$ in (A16) are to be constructed to satisfy

$$f^{\ddagger}(t) = f(t), t \in \left\{\alpha^*, \alpha_1(\boldsymbol{x}), \alpha_2(\boldsymbol{x})\right\}, \tag{A17}$$

$$F^{\ddagger}(t) = F(t), t \in \left\{\alpha^*, \alpha_1(\boldsymbol{x}), \alpha_2(\boldsymbol{x})\right\}, \tag{A18}$$

$$f^{\ddagger\prime}(\alpha^*) = f'(\alpha^*), \tag{A19}$$

$$F^{\ddagger}(t_3^{\ddagger}) = 1, \tag{A20}$$

and ensure that $f^{\ddagger}(t)$ is continuous at $t \in \{t_1^{\ddagger}, t_2^{\ddagger}\}$. Moreover, we require that $t_0^{\ddagger} \in [\underline{t}, \alpha_2(\boldsymbol{x})]$, $t_1^{\ddagger} \in [\alpha_2(\boldsymbol{x}), \alpha^*]$, $t_2^{\ddagger} \in [\alpha^*, \alpha_1(\boldsymbol{x})]$, and $t_3^{\ddagger} \in [\alpha_1(\boldsymbol{x}), \bar{t}\,]$. As will be clear later, under loyalty-based pricing, the equilibrium pricing schedules—i.e., $\left(p_i^{\mathcal{L}}(\boldsymbol{x})\right)_{i=1,2,3}$ and $\left(p_i^{\mathcal{L}}(\kappa+1, \kappa)\right)_{i=1,2}$—and the equilibrium cutoffs—i.e., $\left(\alpha_1(\boldsymbol{x}), \alpha_2(\boldsymbol{x})\right)$ under triopoly and α^* under duopoly with density f—are the same as those with the constructed density $f^{\ddagger}(t)$. Moreover, $\mathcal{C}_2(f^{\ddagger}, \kappa) \le \mathcal{C}_2(f, \kappa)$.

Step I We first prove the existence of the set of parameters. First, fixing $\left(\alpha_1(\boldsymbol{x}), \alpha^*, \alpha_2(\boldsymbol{x})\right)$, we set $\mathcal{M}_1 = f(\alpha_2(\boldsymbol{x}))/\alpha_2(\boldsymbol{x})$, $\mathcal{M}_2 = f(\alpha^*)/\alpha^*$, and $\mathcal{M}_3 = f(\alpha_1(\boldsymbol{x}))/\alpha_1(\boldsymbol{x})$. It is straightforward to verify that the constructed $(\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3)$ satisfies (A17).

Second, we construct β_2 such that (A19) is satisfied. By (A16), we have that $\frac{d}{dt}\ln\frac{f^{\ddagger}(t)}{t} = \beta_2$. Evidently, (A19) is satisfied when we set $\beta_2 = \frac{d}{dt}\ln\frac{f(t)}{t}|_{t=\alpha^*}$.

Third, we construct $(t_1^{\ddagger}, t_2^{\ddagger})$—which depends on β_1 and β_3—such that $f^{\ddagger}$ is continuous at $t_1^{\ddagger}$ and $t_2^{\ddagger}$. Again, by (A16), the continuity of $f^{\ddagger}$ at $t \in \{t_1^{\ddagger}, t_2^{\ddagger}\}$ is equivalent to

$$\beta_1\left(t_1^{\ddagger} - \alpha_2(\boldsymbol{x})\right) + \ln \mathcal{M}_1 = \beta_2\left(t_1^{\ddagger} - \alpha^*\right) + \ln \mathcal{M}_2, \text{ and}$$

$$\beta_2\left(t_2^\ddagger-\alpha^*\right)+\ln\mathcal{M}_2=\beta_3\left(t_2^\ddagger-\alpha_1(\boldsymbol{x})\right)+\ln\mathcal{M}_3,$$

from which we can solve $(t_1^\ddagger, t_2^\ddagger)$ as follows:

$$\begin{aligned}
t_1^\ddagger =& \frac{1}{\beta_1-\beta_2}\times\left[-\beta_2\alpha^*+\ln\mathcal{M}_2+\beta_1\alpha_2(\boldsymbol{x})-\ln\mathcal{M}_1\right], \text{ and}\\
t_2^\ddagger =& \frac{1}{\beta_2-\beta_3}\times\left[\beta_2\alpha^*-\ln\mathcal{M}_2-\beta_3\alpha_1(\boldsymbol{x})+\ln\mathcal{M}_3\right].
\end{aligned}$$

Note that our construction requires that $t_1^\ddagger\in\left[\alpha_2(\boldsymbol{x}),\alpha^*\right]$ and $t_2^\ddagger\in\left[\alpha^*,\alpha_1(\boldsymbol{x})\right]$, which will be verified after β_1 and β_3 are pinned down later.

From the above analysis, it remains to construct $(\beta_1,\beta_3,t_0^\ddagger,t_3^\ddagger)$ to satisfy (A18) and (A20), which are equivalent to the following equations:

$$\begin{aligned}
F^\ddagger(\alpha^*)-F^\ddagger\left(\alpha_2(\boldsymbol{x})\right) =& F(\alpha^*)-F\left(\alpha_2(\boldsymbol{x})\right), && \text{(A21)}\\
F^\ddagger\left(\alpha_1(\boldsymbol{x})\right)-F^\ddagger(\alpha^*) =& F\left(\alpha_1(\boldsymbol{x})\right)-F(\alpha^*), && \text{(A22)}\\
F\left(\alpha_2(\boldsymbol{x})\right) =& F^\ddagger\left(\alpha_2(\boldsymbol{x})\right), \text{ and} && \text{(A23)}\\
1-F\left(\alpha_1(\boldsymbol{x})\right) =& F^\ddagger(t_3^\ddagger)-F^\ddagger\left(\alpha_1(\boldsymbol{x})\right). && \text{(A24)}
\end{aligned}$$

We first construct β_1 and β_3 to satisfy (A21) and (A22), respectively. By Lemma A10, we have that $\alpha^*>\alpha_2(\boldsymbol{x})$. Therefore, (A21) can be expressed as

$$\int_{\alpha_2(\boldsymbol{x})}^{\alpha^*}f(t)dt=\int_{\alpha_2(\boldsymbol{x})}^{\alpha^*}f^\ddagger(t)dt=\int_{\alpha_2(\boldsymbol{x})}^{t_1^\ddagger}\mathcal{M}_1te^{\beta_1(t-\alpha_1(\boldsymbol{x}))}dt+\int_{t_1^\ddagger}^{\alpha^*}\mathcal{M}_2te^{\beta_2(t-\alpha^*)}dt=:\psi_3(\beta_1),$$

where the second equality follows from (A16). It suffices to show that there exists β_1 such that $\psi_3(\beta_1)=\int_{\alpha_2(\boldsymbol{x})}^{\alpha^*}f(t)dt$.

Define $\underline{\beta}_1:=\frac{\ln\frac{f(\alpha^*)}{\alpha^*}-\ln\frac{f(\alpha_2(\boldsymbol{x}))}{\alpha_2(\boldsymbol{x})}}{\alpha^*-\alpha_2(\boldsymbol{x})}$ and $\overline{\beta}_1:=\frac{d}{dt}\ln\frac{f(t)}{t}\Big|_{t=\alpha_2(\boldsymbol{x})}$. Fix $\beta_1=\underline{\beta}_1$. By the concavity of the function $\ln\frac{f(t)}{t}$ and the constructed $(\mathcal{M}_1,\mathcal{M}_2,t_1^\ddagger,\beta_2)$, we can verify that $f^\ddagger(t)\le f(t)$ for all $t\in[\alpha_2(\boldsymbol{x}),\alpha^*]$, which implies that $\psi_3(\underline{\beta}_1)=\int_{\alpha_2(\boldsymbol{x})}^{\alpha^*}f^\ddagger(t)dt\le\int_{\alpha_2(\boldsymbol{x})}^{\alpha^*}f(t)dt$. Similarly, fixing $\beta_1=\overline{\beta}_1$, we can verify that $f^\ddagger(t)\ge f(t)$ for $t\in[\alpha_2(\boldsymbol{x}),\alpha^*]$, which implies that $\psi_3(\overline{\beta}_1)\ge\int_{\alpha_2(\boldsymbol{x})}^{\alpha^*}f(t)dt$. Therefore, there exists β_1 with

$$\frac{\ln\frac{f(\alpha^*)}{\alpha^*}-\ln\frac{f\left(\alpha_2(\boldsymbol{x})\right)}{\alpha_2(\boldsymbol{x})}}{\alpha^*-\alpha_2(\boldsymbol{x})}\equiv\underline{\beta_1}\le\beta_1\le\overline{\beta}_1\equiv\frac{d}{dt}\ln\frac{f(t)}{t}\Big|_{t=\alpha_2(\boldsymbol{x})}, \tag{A25}$$

such that (A21) is satisfied. From (A25) and the construction of $t_1^\ddagger$, it can be verified that $t_1^\ddagger \in [\alpha_2(\boldsymbol{x}), \alpha^*]$. Similarly, we can show that there exists β_3 such that (A22) is satisfied; moreover, $t_2^\ddagger \in [\alpha^*, \alpha_1(\boldsymbol{x})]$.

Next, we construct $t_0^\ddagger$ and $t_3^\ddagger$ to satisfy (A23) and (A24), respectively. By (A16), we have that

$$F^\ddagger\left(\alpha_2(\boldsymbol{x})\right) = \int_{t_0^\ddagger}^{\alpha_2(\boldsymbol{x})} f^\ddagger(t)dt = \int_{t_0^\ddagger}^{\alpha_2(\boldsymbol{x})} \mathcal{M}_1 t e^{\beta_1\left(t-\alpha_2(\boldsymbol{x})\right)}dt.$$

Clearly, $F^\ddagger(\alpha_2(\boldsymbol{x}))$ is decreasing in $t_0^\ddagger$ and $F^\ddagger(\alpha_2(\boldsymbol{x})) = 0 < F(\alpha_2(\boldsymbol{x}))$ at $t_0^\ddagger = \alpha_2(\boldsymbol{x})$. Therefore, it suffices to show that $F^\ddagger\left(\alpha_2(\boldsymbol{x})\right) \geq F\left(\alpha_2(\boldsymbol{x})\right)$ at $t_0^\ddagger = \underline{t}$, which is equivalent to

$$\int_{\underline{t}}^{\alpha_2(\boldsymbol{x})} \mathcal{M}_1 t e^{\beta_1\left(t-\alpha_2(\boldsymbol{x})\right)}dt \geq \int_{\underline{t}}^{\alpha_2(\boldsymbol{x})} f(t)dt. \tag{A26}$$

In fact, from the concavity of $\ln \frac{f(t)}{t}$, $f\left(\alpha_2(\boldsymbol{x})\right) = f^\ddagger\left(\alpha_2(\boldsymbol{x})\right)$, and (A25), we can conclude that

$$\mathcal{M}_1 t e^{\beta_1\left(t-\alpha_2(\boldsymbol{x})\right)} \geq f(t), \forall\, t \in \left[\underline{t}, \alpha_2(\boldsymbol{x})\right], \tag{A27}$$

which implies (A26). Therefore, there exists $t_0^\ddagger \in [\underline{t}, \alpha_2^*(\boldsymbol{x})]$ to satisfy (A23). Similarly, we can show that there exists $t_3^\ddagger \in [\alpha_1^*(\boldsymbol{x}), \bar{t}\,]$ to satisfy (A24). This completes the construction.

Step II We show that $\mathcal{C}_2(f^\ddagger, \kappa) \leq \mathcal{C}_2(f, \kappa)$. It is useful to prove the following lemma:

Lemma B2 *Suppose that* $f^\ddagger$ *is defined in* (A16) *such that* (A17)-(A20) *hold. The following statements hold:*

(i) $\frac{f^\ddagger(t)}{t}$ *is piecewise linear with at most three segments.*

(ii) $F^\ddagger(\cdot|t \in [\underline{t}, \alpha_2(\boldsymbol{x})]) \geq_{FOSD} F(\cdot|t \in [\underline{t}, \alpha_2(\boldsymbol{x})])$ *and* $F(\cdot|t \in [\alpha_1(\boldsymbol{x}), \bar{t}]) \geq_{FOSD} F^\ddagger(\cdot|t \in [\alpha_1(\boldsymbol{x}), \bar{t}])$.

(iii) $F^\ddagger(\cdot|t \in [\underline{t}, \alpha^*]) \geq_{FOSD} F(\cdot|t \in [\underline{t}, \alpha^*])$ *and* $F(\cdot|t \in [\alpha^*, \bar{t}]) \geq_{FOSD} F^\ddagger(\cdot|t \in [\alpha^*, \bar{t}])$.

Proof. Part (i) is obvious. For part (ii), $F^\ddagger(\cdot|t \in [\underline{t}, \alpha_2(\boldsymbol{x})]) \geq_{FOSD} F(\cdot|t \in [\underline{t}, \alpha_2(\boldsymbol{x})])$ follows immediately from (A27). Similarly, we can show that $F(\cdot|t \in [\alpha_1(\boldsymbol{x}), \bar{t}\,]) \geq_{FOSD} F^\ddagger(\cdot|t \in [\alpha_1(\boldsymbol{x}), \bar{t}\,])$. It remains to prove part (iii). Next, we prove

$F^{\ddagger}(\cdot|t \in [\underline{t}, \alpha^*]) \geq_{FOSD} F(\cdot|t \in [\underline{t}, \alpha^*])$. The proof of $F(\cdot|t \in [\alpha^*, \bar{t}\,]) \geq_{FOSD} F^{\ddagger}(\cdot|t \in [\alpha^*, \bar{t}\,])$ is similar.

Note that $F^{\ddagger}(\alpha_2(\boldsymbol{x})) = F(\alpha_2(\boldsymbol{x}))$, $F^{\ddagger}(\alpha^*) = F(\alpha^*)$, and $F^{\ddagger}(\cdot|t \in [\underline{t}, \alpha_2(\boldsymbol{x})]) \geq_{FOSD} F(\cdot|t \in [\underline{t}, \alpha_2(\boldsymbol{x})])$. It suffices to show that $F^{\ddagger}(\cdot|t \in [\alpha_2(\boldsymbol{x}), \alpha^*]) \geq_{FOSD} F(\cdot|t \in [\alpha_2(\boldsymbol{x}), \alpha^*])$, which holds if we can show that there exists $\xi \in [\alpha_2(\boldsymbol{x}), \alpha^*]$ such that $f^{\ddagger}(t) \leq f(t)$ for $t \in [\alpha_2(\boldsymbol{x}), \xi]$ and $f^{\ddagger}(t) \geq f(t)$ for $t \in [\xi, \alpha^*]$.

Recall $t_1^{\ddagger} \in [\alpha_2(\boldsymbol{x}), \alpha^*]$. By (A19) and the concavity of $\ln \frac{f(t)}{t}$, we have that

$$f^{\ddagger}(t) = \mathcal{M}_2 t e^{\beta_2(t-\alpha^*)} \geq f(t), \forall\, t \in [t_1^{\ddagger}, \alpha^*].$$

Therefore, it suffices to show that there exists $\xi \in [\alpha_2(\boldsymbol{x}), t_1^{\ddagger}]$ such that $f^{\ddagger}(t) \leq f(t)$ for $t \in [\alpha_2(\boldsymbol{x}), \xi]$ and $f^{\ddagger}(t) \geq f(t)$ for $t \in [\xi, t_1^{\ddagger}]$.

Note that $\psi_4(t) := \ln \frac{f(t)}{t} - \ln \frac{f^{\ddagger}(t)}{t}$ is concave in t for $t \in [\alpha_2(\boldsymbol{x}), t_1^{\ddagger}]$. Moreover, by (A17) and (A25), we have that $\psi_4(\alpha_2(\boldsymbol{x})) = 0$ and $\psi_4'(\alpha_2(\boldsymbol{x})) > 0$. Therefore, there exists $\xi \in [\alpha_2(\boldsymbol{x}), t_1^{\ddagger}]$ such that $\psi_4(t) \geq 0$—or equivalently, $f^{\ddagger}(t) \leq f(t)$—for $t \in [\alpha_2(\boldsymbol{x}), \xi]$ and $\psi_4(t) \leq 0$—or equivalently, $f^{\ddagger}(t) \geq f(t)$—for $t \in [\xi, t_1^{\ddagger}]$. This concludes the proof. ■

Recall $\mathcal{C}_2(f, \kappa)$. It suffices to show that

$$\left[\widetilde{V}^{\mathcal{L}}(\boldsymbol{x}) - V^{\mathcal{F}}(\boldsymbol{x})\right]\Big|_{t\sim F^{\ddagger}} \leq \left[\widetilde{V}^{\mathcal{L}}(\boldsymbol{x}) - V^{\mathcal{F}}(\boldsymbol{x})\right]\Big|_{t\sim F} \text{ and} \tag{A28}$$

$$\left[V^{\mathcal{F}}(\kappa+1, \kappa) - \widetilde{V}^{\mathcal{L}}(\kappa+1, \kappa)\right]\Big|_{t\sim F^{\ddagger}} \geq \left[V^{\mathcal{F}}(\kappa+1, \kappa) - \widetilde{V}^{\mathcal{L}}(\kappa+1, \kappa)\right]\Big|_{t\sim F}. \tag{A29}$$

We first prove (A28). Carrying out the algebra, we have that

$$\begin{aligned}
&\left[\widetilde{V}^{\mathcal{L}}(\boldsymbol{x}) - V^{\mathcal{F}}(\boldsymbol{x})\right]\Big|_{t\sim F} \\
=&\int_{\alpha_1(\boldsymbol{x})}^{\bar{t}} \left[v + tx_1 - p_1^{\mathcal{L}}(\boldsymbol{x})\right] f(t)dt + \int_{\underline{t}}^{\alpha_1(\boldsymbol{x})} \left[v + tx_2 - p_2^{\mathcal{L}}(\boldsymbol{x})\right] f(t)dt \\
&\quad - \int_{\underline{t}}^{\bar{t}} \left[v + tx_1 - t(x_1 - x_2)\right] f(t)dt \\
=&(x_1 - x_2) \int_{\alpha_1(\boldsymbol{x})}^{\bar{t}} tf(t)dt - p_1^{\mathcal{L}}(\boldsymbol{x}) \left[1 - F(\alpha_1(\boldsymbol{x}))\right] - p_2^{\mathcal{L}}(\boldsymbol{x}) F\left(\alpha_1(\boldsymbol{x})\right).
\end{aligned} \tag{A30}$$

By (A17) and (A18), under loyalty-based pricing, the equilibrium pricing schedules

$\left(p_i^{\mathcal{L}}(\boldsymbol{x})\right)_{i=1,2,3}$ and the cutoffs $\left(\alpha_1(\boldsymbol{x}), \alpha_2(\boldsymbol{x})\right)$ with density f are the same as those with density $f^{\ddagger}$. Therefore, we have that

$$\left[\widetilde{V}^{\mathcal{L}}(\boldsymbol{x}) - V^{\mathcal{F}}(\boldsymbol{x})\right]\Bigg|_{t\sim F^{\ddagger}} = (x_1 - x_2)\int_{\alpha_1(\boldsymbol{x})}^{\bar{t}} tf^{\ddagger}(t)dt - p_1^{\mathcal{L}}(\boldsymbol{x})\left[1 - F(\alpha_1(\boldsymbol{x}))\right] - p_2^{\mathcal{L}}(\boldsymbol{x})F\left(\alpha_1(\boldsymbol{x})\right). \tag{A31}$$

By (A30) and (A31), it suffices to show that $\int_{\alpha_1(\boldsymbol{x})}^{\bar{t}} tf(t)dt \geq \int_{\alpha_1(\boldsymbol{x})}^{\bar{t}} tf^{\ddagger}(t)dt$, which follows immediately from $F(\cdot|t \in [\alpha_1(\boldsymbol{x}), \bar{t}]) \geq_{FOSD} F^{\ddagger}(\cdot|t \in [\alpha_1(\boldsymbol{x}), \bar{t}])$, as shown in Lemma B2.

Next, we prove (A29). Fix $(x_1, x_2) = (\kappa+1, \kappa)$. Note that

$$\begin{aligned}
&\left[V^{\mathcal{F}}(\kappa+1,\kappa) - \widetilde{V}^{\mathcal{L}}(\kappa+1,\kappa)\right]\Bigg|_{t\sim F} \\
=& -\int_{\alpha^*}^{\bar{t}} \left[v + t(\kappa+1) - p_1^{\mathcal{L}}(\kappa+1,\kappa)\right] f(t)dt \\
&\quad - \int_{\underline{t}}^{\alpha^*} \left[v + t\kappa - p_2^{\mathcal{L}}(\kappa+1,\kappa)\right] f(t)dt + \int_{\underline{t}}^{\bar{t}} \left[v + t\kappa\right] f(t)dt \\
=& -\int_{\alpha^*}^{\bar{t}} tf(t)dt + p_1^{\mathcal{L}}(\kappa+1,\kappa)\left[1 - F(\alpha^*)\right] + p_2^{\mathcal{L}}(\kappa+1,\kappa)F(\alpha^*).
\end{aligned} \tag{A32}$$

By (A17) and (A18), under loyalty-based pricing, the equilibrium pricing schedules $\left(p_i^{\mathcal{L}}(\kappa+1,\kappa)\right)_{i=1,2}$ and the cutoff α^* with density f are the same as those with density $f^{\ddagger}$. Therefore,

$$\begin{aligned}
&\left[V^{\mathcal{F}}(\kappa+1,\kappa) - \widetilde{V}^{\mathcal{L}}(\kappa+1,\kappa)\right]\Bigg|_{t\sim F^{\ddagger}} \\
=& -\int_{\alpha^*}^{\bar{t}} tf^{\ddagger}(t)dt + p_1^{\mathcal{L}}(\kappa+1,\kappa)\left[1 - F(\alpha^*)\right] + p_2^{\mathcal{L}}(\kappa+1,\kappa)F(\alpha^*).
\end{aligned} \tag{A33}$$

By (A32) and (A33), it suffices to show that $\int_{\alpha^*}^{\bar{t}} tf(t)dt \geq \int_{\alpha^*}^{\bar{t}} tf^{\ddagger}(t)dt$, which follows immediately from $F(\cdot|t \in [\alpha^*, \bar{t}\,]) \geq_{FOSD} F^{\ddagger}(\cdot|t \in [\alpha^*, \bar{t}\,])$, as shown in Lemma B2. This completes the proof of Lemma B1. ■

By Lemma B1, to search for f that minimizes $\mathcal{C}_2(f, \kappa)$, it suffices to look over density functions that satisfy Assumption 2 and are piecewise linear with at most three segments. Note that these functions can be parameterized by seven parameters and it can be verified that $\min_f \mathcal{C}_2(f, \frac{1}{2}) > \frac{4}{5}$. This completes the whole proof of

Corollary 2. ■

Proof of Lemma A10

Proof. By $f(\underline{t}) < 1/\underline{t}$, we have that $k(\boldsymbol{x}) \geq 2$. If $k(\boldsymbol{x}) = 2$, then we have that $\alpha_1(\boldsymbol{x}) = \alpha^* > \alpha_2(\boldsymbol{x}) = \underline{t}$. It remains to consider the case of $k(\boldsymbol{x}) \geq 3$.

From the proof of Lemma A2, we can conclude that $tf(t) + 2F(t) > 1$ if and only if $t > \alpha^*$. Therefore, it suffices to show that

$$\alpha_1(\boldsymbol{x}) f\left(\alpha_1(\boldsymbol{x})\right) + 2F\left(\alpha_1(\boldsymbol{x})\right) > 1, \tag{A34}$$

and

$$\alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x}) f\left(\alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\right) + 2F\left(\alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\right) < 1. \tag{A35}$$

In what follows, we prove (A34). The proof of (A35) is similar and omitted for brevity. Carrying out the algebra, we can obtain that

$$\begin{aligned}
\frac{1 - F\left(\alpha_1(\boldsymbol{x})\right)}{\frac{f\left(\alpha_1(\boldsymbol{x})\right)}{x_1 - x_2}} = p_1^{\mathcal{L}}(\boldsymbol{x}) - c &= p_2^{\mathcal{L}}(\boldsymbol{x}) + \alpha_1(\boldsymbol{x})(x_1 - x_2) - c \\
&= \frac{F\left(\alpha_1(\boldsymbol{x})\right) - F\left(\alpha_2(\boldsymbol{x})\right)}{\frac{f\left(\alpha_1(\boldsymbol{x})\right)}{x_1 - x_2} + \frac{f\left(\alpha_2(\boldsymbol{x})\right)}{x_2 - x_3}} + \alpha_1(\boldsymbol{x})(x_1 - x_2) \\
&< \frac{F\left(\alpha_1(\boldsymbol{x})\right)}{\frac{f\left(\alpha_1(\boldsymbol{x})\right)}{x_1 - x_2}} + \alpha_1(\boldsymbol{x})(x_1 - x_2),
\end{aligned}$$

where the first equality follows from (12); the second equality follows from (17); the third equality follows from (13); and the inequality follows from the fact that $F(\alpha_2(\boldsymbol{x})) \geq 0$ and $f(\alpha_2(\boldsymbol{x})) > 0$. Simplifying the above condition gives (A34). This concludes the proof. ■

Lemma B3 *When the market is partially covered, the equilibrium pricing strategies* $\left(p_1^{\mathcal{L}}(\boldsymbol{x}), \ldots, p_n^{\mathcal{L}}(\boldsymbol{x})\right)$ *under loyalty-based pricing can be characterized as follows:*

(a) If $v + \underline{t}x_{k(\boldsymbol{x})+1} > 0$*, then* $\alpha_{k(\boldsymbol{x})}(\boldsymbol{x}) = 0$*, and the equilibrium pricing schedule* $\left(p_1^{\mathcal{L}}(\boldsymbol{x}), \ldots, p_n^{\mathcal{L}}(\boldsymbol{x})\right)$*, the set of cutoffs* $\left(\alpha_1(\boldsymbol{x}), \ldots, \alpha_{k(\boldsymbol{x})}(\boldsymbol{x})\right)$*, and the number of active firms* $k(\boldsymbol{x})$ *can be uniquely pinned down by the same conditions* (12)-(17) *as provided in Lemma 2.*

(b) Otherwise, the KKT conditions (14), (15), *and* (16) *are replaced by the following conditions:*

$$p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x}) = c + \frac{F\left(\alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\right) - F\left(\alpha_{k(\boldsymbol{x})}(\boldsymbol{x})\right)}{\frac{f\left(\alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\right)}{x_{k(\boldsymbol{x})-1} - x_{k(\boldsymbol{x})}} + \xi f\left(\alpha_{k(\boldsymbol{x})}(\boldsymbol{x})\right)}, \xi \in \left[0, \frac{1}{x_{k(\boldsymbol{x})}}\right], \tag{A36}$$

$$p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x}) \leq \alpha_{k(\boldsymbol{x})} x_{k(\boldsymbol{x})} + v, \tag{A37}$$

$$\xi \times \left(\alpha_{k(\boldsymbol{x})} x_{k(\boldsymbol{x})} + v - p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x})\right) = 0, \tag{A38}$$

$$\left(\frac{1}{x_{k(\boldsymbol{x})}} - \xi\right) \times \left(\alpha_{k(\boldsymbol{x})}(\boldsymbol{x}) - \underline{t}\right) = 0. \tag{A39}$$

Proof. We characterize the equilibrium under loyalty-based pricing when the market can be partially covered. Suppose $v + \underline{t} x_{k(\boldsymbol{x})+1} > 0$. Then for each consumer, making no purchase is strictly dominated by buying from firm $k(\boldsymbol{x}) + 1$, and thus the equilibrium is the same as that characterized in Lemma 2.

Now suppose $v + \underline{t} x_{k(\boldsymbol{x})+1} \leq 0$. If $\alpha_{k(\boldsymbol{x})}(\boldsymbol{x}) > \underline{t}$, the marginal consumer with $t = \alpha_{k(\boldsymbol{x})}(\boldsymbol{x})$ is indifferent between buying from firm k and making no purchase, which implies that $p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x}) = v + \alpha_{k(\boldsymbol{x})}(\boldsymbol{x}) x_{k(\boldsymbol{x})}$ (this condition is equivalent to (A37) holding with equality). Moreover, the first-order condition for firm $k(\boldsymbol{x})$ is

$$p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x}) = c + \frac{F\left(\alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\right) - F\left(\alpha_{k(\boldsymbol{x})}(\boldsymbol{x})\right)}{\frac{f\left(\alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\right)}{x_{k(\boldsymbol{x})-1} - x_{k(\boldsymbol{x})}} + \frac{f\left(\alpha_{k(\boldsymbol{x})}(\boldsymbol{x})\right)}{x_{k(\boldsymbol{x})}}},$$

which is equivalent to (A36) at $\xi = \frac{1}{x_{k(\boldsymbol{x})}}$. Note that $\xi = \frac{1}{x_{k(\boldsymbol{x})}}$ implies the following: (i) (A39) is satisfied and (ii) (A38) is satisfied if and only if (A37) holds with equality.

Similarly, if $\alpha_{k(\boldsymbol{x})}(\boldsymbol{x}) = \underline{t}$, then (A39) is satisfied. A consumer with $t = \underline{t}$ buys from firm $k(\boldsymbol{x})$, which implies (A37). If (A37) holds with strict inequality, then firm $k(\boldsymbol{x})$'s price is pinned down by the following first-order condition:

$$p^{\mathcal{L}}_{k(\boldsymbol{x})}(\boldsymbol{x}) = c + \frac{F\left(\alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\right) - F\left(\alpha_{k(\boldsymbol{x})}(\boldsymbol{x})\right)}{\frac{f\left(\alpha_{k(\boldsymbol{x})-1}(\boldsymbol{x})\right)}{x_{k(\boldsymbol{x})-1} - x_{k(\boldsymbol{x})}}},$$

which is equivalent to (A36) at $\xi = 0$. Note that $\xi = 0$ implies that (A38) holds. Otherwise, if (A37) holds with equality, then (A38) holds and the first-order condition for firm $k(\boldsymbol{x})$ is (A36).

From the above analysis, we can conclude that the KKT conditions (14)-(16) are replaced by (A36)-(A39) if the market is partially covered. Proofs of equilibrium existence and uniqueness are analogous to Lemma 2, and omitted for brevity. ■

Lemma B4 *Suppose that Assumptions 1′, 2, and 3′ hold. Moreover,* $f(\bar{t}) > 0$ *and* $g(\overline{x}) > 0$*. Then we have that*

$$\lim_{c \nearrow v+\bar{t}\overline{x}} \frac{p^{\mathcal{U}} - c}{v + \bar{t}\overline{x} - c} = \frac{1}{3}, \tag{A40}$$

$$\lim_{c \nearrow v+\bar{t}\overline{x}, t \nearrow \bar{t}, v+t\overline{x} > c} \frac{p^{\mathcal{C}}(t) - c}{v + t\overline{x} - c} = \frac{1}{2}, \tag{A41}$$

$$\lim_{c \nearrow v+\bar{t}\overline{x}, x^{(1)} \nearrow \overline{x}, v+\bar{t}x^{(1)} > c > v+\bar{t}x^{(2)}} \frac{p^{\mathcal{L}}(\boldsymbol{x}) - c}{v + \bar{t}x - c} = \frac{1}{2}. \tag{A42}$$

Proof. In what follows, we prove (A40). The proofs of (A41) and (A42) are similar and omitted for brevity. Under uniform pricing, a firm's expected profit is

$$\pi^{\mathcal{U}}(p_i)\Big|_{p_j = p^{\mathcal{U}}} = (p_i - c) \times \Pr\left[v + tx_i - p_i \geq 0, v + tx_i - p_i \geq \max_{j \neq i}\left\{v + tx_j - p^{\mathcal{U}}\right\}\right]. \tag{A43}$$

First, note that

$$\Pr\left[v + tx_i - p_i \geq 0, v + tx_i - p_i \geq \max_{j \neq i}\left\{v + tx_j - p^{\mathcal{U}}\right\}\right] \leq \Pr\left[v + tx_i - p_i \geq 0\right], \tag{A44}$$

and

$$\begin{aligned}
&\Pr\left[v + tx_i - p_i \geq 0, v + tx_i - p_i \geq \max_{j \neq i}\left\{v + tx_j - p^{\mathcal{U}}\right\}\right] \\
\geq &\Pr\left[v + tx_i - p_i \geq 0, 0 \geq \max_{j \neq i}\{v + tx_j\}\right] \\
= &\Pr\left[v + tx_i - p_i \geq 0\right] \times \Pr\left[\max_{j \neq i}\{v + tx_j\} \leq 0 \,\middle|\, v + tx_i - p_i \geq 0\right] \\
= &\Pr\left[v + tx_i - p_i \geq 0\right] \times \big(1 + o(1)\big).
\end{aligned} \tag{A45}$$

Combining (A44) and (A45), we can conclude that

$$\Pr\left[v + tx_i - p_i \geq 0, v + tx_i - p_i \geq \max_{j\neq i}\left\{v + tx_j - p^{\mathcal{U}}\right\}\right] = \Pr\left[v + tx_i - p_i \geq 0\right]\times\big(1 + o(1)\big)\,. \tag{A46}$$

Second, note that t and x_i are independent. Therefore, we have that

$$\begin{aligned}
\Pr\left[v + tx_i - p_i \geq 0\right] &= \int_{t=-\frac{v-p_i}{\overline{x}}}^{\overline{t}} \int_{x_i=-\frac{v-p_i}{t}}^{\overline{x}} f(t)g(x_i)dx_i dt \\
&= f(\overline{t})g(\overline{x}) \int_{t=-\frac{v-p_i}{\overline{x}}}^{\overline{t}} \int_{x_i=-\frac{v-p_i}{t}}^{\overline{x}} 1 dx_i dt \times \big(1 + o(1)\big) \\
&= f(\overline{t})g(\overline{x}) \times \left[\overline{x}\overline{t} + (v - p_i) + (v - p_i)\ln\left(-\frac{\overline{x}\overline{t}}{v - p_i}\right)\right] \times \big(1 + o(1)\big) \\
&= f(\overline{t})g(\overline{x}) \times \frac{(v + \overline{x}\overline{t} - p_i)^2}{2\overline{x}\overline{t}} \times \big(1 + o(1)\big)\,.
\end{aligned} \tag{A47}$$

Combining (A43), (A46), and (A47) yields that

$$\pi^{\mathcal{U}}(p_i)\Big|_{p_j=p^{\mathcal{U}}} = (p_i - c) \times \big(v + \overline{x}\overline{t} - p_i\big)^2 \times \frac{f(\overline{t})g(\overline{x})}{2\overline{x}\overline{t}} \times \big(1 + o(1)\big)\,.$$

The above equation, together with equilibrium existence and uniqueness, implies that $p^{\mathcal{U}} = c + \frac{v+\overline{t}\overline{x}-c}{3} \times \big(1 + o(1)\big)$. This concludes the proof. ■