Orchestrating the Distributive Process: Baron and Ferejohn Meet Tullock*

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Abstract

This paper examines the process of dividing a fixed surplus as a multilateral bargaining game with costly recognition. We establish the existence of equilibrium and explore the roles of the institutional rules that govern this process. Specifically, we consider a design problem in which the designer maximizes a general objective function by determining both the voting rule—i.e., the minimum number of votes required to approve a proposal—and the mechanism for proposer recognition, modeled as a biased generalized lottery contest. We demonstrate that any feasible outcome regarding equilibrium efforts and recognition probabilities can be implemented using a rule that incorporates a dictatorial voting rule. In other words, the designer can maximize her interests by resolving the distributive process through a simple static biased contest.

Keywords: Multilateral Bargaining; Contest Design

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1 Introduction

Economic agents often negotiate to divide scarce resources. Different units within a firm split advertising or R&D budgets; academic departments within a school deliberate over the allocation of hiring lines; and electoral candidates bargain over the distribution of campaign funding from a political party. The seminal study by Baron and Ferejohn (1989) provides a framework for the analysis of distributive politics.

A canonical multilateral bargaining model typically involves an agent who proposes a plan to share a fixed surplus, seeking to secure a minimum number of favorable votes from peers for its approval. Conventional wisdom in the literature holds that the proposer enjoys a disproportionately large share thanks to their ability to set the agenda, while the majority of studies assume that the proposer is chosen at random. The rent associated with the proposing right arguably tempts agents to engage in costly influencing activities to acquire this privilege, and rent-seeking contests à la Tullock (1980, 1988) naturally emerge. Such dynamics are indeed commonplace in reality. For instance, an academic department might promote its faculty members' publications as leverage for priority in future hiring, and disputing parties may lobby mediators for more favorable bargaining positions. Executives also strive to ascend hierarchical ladders to positions that offer significant authority over corporate agendas. Yildirim (2007) endogenizes the recognition of the proposer and integrates it into the distributive process. He considers a contest prior to the bargaining, in which agents expend effort to vie for the proposing right, thereby generating a bargaining game with costly recognition.

In this paper, we endogenize the rules that govern the entire process and examine this scenario from a design perspective. Agents' incentives to exert influencing efforts, as well as the distributive outcome of recognition opportunities, depend sensitively on the prevailing institutional environment. This creates room for setting the rules strategically to promote certain interests.

Our model generalizes that of Yildirim (2007) and allows agents to be heterogeneous in contest technologies, effort cost functions, and patience levels. A designer sets (i) the bargaining protocol and (ii) the recognition mechanism, i.e., the rules that regulate the contest for the proposing right. The former is defined by a k-majority rule, whereby k denotes the minimum number of favorable votes (including the proposer) required for approval. This specifies the limit of the power or influence the proposer enjoys. The latter converts agents' efforts into their recognition probabilities. By varying the recognition mechanism, the designer can effectively bias the competition in favor of certain contenders, and thereby tilt the playing field and reshape agents' incentives. For instance, a preferred candidate in a company's succession process is often assigned a significant position—e.g., president or COO—that enhances their visibility to board members; a politician competing for party leadership can be endorsed by powerful party elites (Cross, 2013); alternatively, the dean of a school might prioritize one department over others when evaluating their performance. We assume that the designer's utility depends on both agents' efforts and the ex ante distributive outcome, i.e., the probability of each agent's winning the proposing right. Our analysis allows for a general objective that accommodates the designer's diverse interests.

The game is intuitive by nature, but its analysis poses a technical challenge and the optimum is unclear a priori. The game can be viewed as a contest with an endogenously determined prize. To vie for recognition, agents weigh their potential payoffs from winning the proposing right against those from losing the competition. This payoff differential effectively functions as the prize spread that motivates their efforts in the contest. The winner offers a subset of peers (i.e., agents in his winning coalition) their equilibrium continuation values in the dynamic bargaining process in order to secure their votes. A loser, on the other hand, receives his equilibrium continuation value in exchange for his approval if he is included in a winning coalition or nothing if he is excluded. The prize spreads, which depend on agents' continuation values, motivate their efforts, while agents' efforts determine their recognition probabilities, the formation of each agent's winning coalition, and, ultimately, their equilibrium continuation values. The endogeneity, together with agents' heterogeneity, complicates the analysis and differentiates the game from traditional bargaining or contest models. We demonstrate in Section 3.2 that conventional comparative statics with respect to k, the minimum number of votes required to approve a proposal, are no longer feasible, since equilibrium total effort may vary nonmonotonically with k. Similarly, a change in the recognition mechanism (contest rules) reshapes the equilibrium, which not only rebalances the competition but also varies agents' equilibrium continuation values and thereby the endogenously determined prize spreads. This undermines established results derived from previously studied settings. We elaborate on these nuances in Section 3.2.

Despite these challenges, our analysis delivers an unambiguous message: When the designer can set both k and the contest rules, the general objective function can always be maximized by a dictatorial voting rule with $k = 1$, although the specific form of the contest rules depends on the particular environmental factors. Notably, we establish that any feasible outcome of the game regarding agents' efforts and recognition probabilities can be implemented through a rule with $k = 1$. In this case, a proposal is accepted with the consent of only the proposer; an agent captures the entire surplus once recognized and receives nothing otherwise. The game thus reduces to a standard static contest with a fixed prize. In other words, the designer can achieve any feasible outcome by resolving the distributive process through a properly biased contest. To maximize the designer's objective, it is thus without loss of generality to search for the optimum within the set of possible rules with $k = 1$.

The optimality of the dictatorial voting rule naturally leads to a broader inquiry: Suppose the designer faces a constraint in setting the minimum k (e.g., the standard democratic procedural requirement), while retaining full flexibility to adjust the recognition mechanism for any given k. Does adopting a less inclusive voting rule—i.e., a smaller k—necessarily improve the value of her objective function? A monotonic relationship does not generally hold. However, our analysis reveals that when agents are relatively impatient, the optimal value of her objective function weakly decreases in k, provided that the recognition mechanism is recalibrated as k varies. Monotonicity is restored: The designer always prefers a less inclusive voting rule.

Link to the Literature Our paper adds to the literature on multilateral bargaining by endogenizing the bargaining protocol and the proposer recognition mechanism. An extensive body of studies has been built on the framework established by Baron and Ferejohn (1989)—e.g., Merlo and Wilson (1995, 1998); Banks and Duggan (2000); Eraslan (2002); Eraslan and Merlo (2002, 2017); Diermeier and Fong (2011); Diermeier, Prato, and Vlaicu (2015, 2016); Ali, Bernheim, and Fan (2019); and Evdokimov (2023). The majority of this literature assumes that the proposer is exogenously and randomly selected from the agents.

A small strand of the literature considers the selection of the proposer to be an integral part of the political process and examines the endogenous formation of bargaining protocols. Yildirim (2007) models the process of selecting proposers as a contest in which agents exert costly effort to gain power and pioneers the integration of a contest model (generalized Tullock contest) with a multilateral bargaining game to endogenize the recognition mechanism. Yildirim (2010) compares total effort and distributive outcomes between persistent and transitory recognition procedures, and Ali (2015) models the recognition process as an all-pay auction. Relatedly, Lee and Sethi (2024) examine how recognition probabilities affect agents' payoffs; they allow subjects to pay money in exchange for preferred recognition probabilities. Recent experimental work by Baranski and Reuben (2026) studies costly competition for proposal rights under majority and unanimity rules. They find that unanimity reduces contest effort but increases bargaining delay and disagreement, which leaves neither rule with a statistically significant efficiency advantage.

Our paper models the recognition process as an influence competition and is closely related to Yildirim (2007). Similar to Yildirim, we adopt a generalized Tullock contest, but we introduce heterogeneous production technologies with fewer restrictions as well as nonlinear effort cost functions. Yildirim conducts comparative statics of the prevailing voting rule for homogeneous agents and shows that a more inclusive voting rule—i.e., a larger minimum number of required votes—always leads to lower total effort. In contrast, we explore optimal rule design in a setting that allows for a general design objective, heterogeneous agents, and multiple design instruments (voting rule and recognition mechanism). Agents' heterogeneity catalyzes complex effects under varying voting rules or contest rules, which prevents standard comparative static analysis and differentiates our game from conventional bargaining or contest models.

There has been growing scholarly effort to examine the role played by institutional elements of the legislative bargaining process; a survey of this strand of the literature has been provided by Eraslan, Evdokimov, and Zápal (2022). Diermeier, Egorov, and Sonin (2017) explore how the allocation of veto power affects distributional outcomes. Nunnari (2021) assesses the consequences of veto power in a dynamic bargaining setting and demonstrates that reducing the recognition probability of the veto player can effectively limit his rent appropriation. Sethi and Verriest (2025) study an infinitely repeated legislative bargaining process with a dynamically evolving status quo. They show that veto power and recognition probability are substitutes instead of complements to each other; that is, the veto player may benefit from a smaller recognition probability because non-veto players may endogenously prevent his expropriation.

Several papers model the endogenous formation of bargaining protocols without using a contest approach. Diermeier, Prato, and Vlaicu (2015, 2016) employ a pre-bargaining process to determine proposal power in bargaining over policy. In McKelvey and Riezman (1992, 1993); Muthoo and Shepsle (2014); and Eguia and Shepsle (2015), the recognition probability is determined by seniority, which is endogenously voted on at the end of each session. Kim (2019) assumes that current and past proposers are excluded from the pool of eligible candidates when a round of bargaining fails to reach consensus. Jeon and Hwang (2022) assume that an agent's recognition probability and bargaining power depend on the previous bargaining outcome in a dynamic legislative bargaining model, which leads to an oligopolistic outcome as the result of an evolutionary process. Agranov, Cotton, and Tergiman (2020) examine, both theoretically and experimentally, a repeated multilateral bargaining model in which the agenda setter can retain his power with the majoritarian support of other committee members.

Our paper is closely related to Jeon and Hwang (2022) and Ali, Bernheim, and Fan (2019), who demonstrate that power concentration could arise in equilibrium. The former study attributes the endogenous formation of oligopoly to the influence of past bargaining outcomes. The latter shows that the prevailing information structure could lead to extreme power in terms of distributive outcomes when the voting rule is not unanimous. Neither involves a contest for proposing rights or an endogenously set voting rule, which is the focus of this paper.

Our paper is also naturally linked to the literature on contest design and, particularly, that on optimally biased contests (see, e.g., Epstein, Mealem, and Nitzan, 2011; Franke, Kanzow, Leininger, and Schwartz, 2013, 2014; Franke, Leininger, and Wasser, 2018). We develop a technique similar to that of Fu and Wu (2020) and Fu, Wu, and Zhu (2023), who characterize the optimum without explicitly solving for the equilibrium. Our analysis complements these studies by embedding the contest in a multilateral sequential bargaining framework, which generates an endogenous prize spread.

The rest of the paper is structured as follows. Section 2 sets up the model and characterizes the equilibrium. Section 3 allows a designer to set the voting rule and the recognition mechanism and solves the design problem, and Section 4 concludes. The appendix collects the proofs and derivations for examples that are not provided in the main text.

2 Preliminaries

In this section, we first lay out the modeling details, then present the equilibrium fundamentals of the game and set up the design problem.

2.1 Model: Multilateral Bargaining with Costly Recognition

A set of $n \geq 2$ agents, indexed by $\mathcal{N} := \{1, 2, \ldots, n\}$, decide how to divide a dollar. In each period $t = 0, 1, 2, \ldots$, one agent (proposer) makes a proposal $\boldsymbol{s}_t \in \triangle^{n-1} := \{(s_{1,t}, \ldots, s_{n,t}) : 0 \leq s_{i,t} \leq 1, \sum_{i \in \mathcal{N}} s_{i,t} = 1\}$, where $s_{i,t}$ denotes the share of the dollar each agent $i \in \mathcal{N}$ is to receive under this proposal. Agents simultaneously vote in favor of or against the proposal. We assume a "k-majority" voting rule for this sequential bargaining process: The proposal is approved if at least k agents accept it (including the proposer), with $1 \leq k \leq n$. Specifically, $k = n$ implies a unanimity rule wherein the proposal can be vetoed by any single dissident; $k = \lfloor n/2 \rfloor + 1$ refers to a simple majority rule; with $k = 1$, the proposer dictates the decision process.

At the beginning of each period t, a contest takes place in which each agent exerts effort $x_{i,t} \geq 0$ to vie for the proposing right, which incurs a cost $c_i(x_{i,t})$. We assume that $c_i(\cdot)$ is twice differentiable and satisfies $c_i(0) = 0$, $c_i'(\cdot) > 0$, and $c_i''(\cdot) \geq 0$.

The game ends if agents agree on a proposal, and each agent collects the share specified in the proposal. The game continues to the next period and the process repeats, if the proposer is unable to secure enough favorable votes.

Proposer Recognition Given an effort profile $\boldsymbol{x}_t := (x_{1,t}, \ldots, x_{n,t})$, an agent i is recognized as the proposer for period t with probability

$$p_i(\boldsymbol{x}_t) = \begin{cases} \dfrac{f_i(x_{i,t})}{\sum_{j\in\mathcal{N}} f_j(x_{j,t})}, & \sum_{j\in\mathcal{N}} f_j(x_{j,t}) > 0, \\ \dfrac{1}{n}, & \sum_{j\in\mathcal{N}} f_j(x_{j,t}) = 0, \end{cases} \tag{1}$$

where $f_i(\cdot)$ is called the impact function in the contest literature. This function indicates each agent's technology for converting effort into output in the competition. We assume that $f_i(\cdot)$ is twice differentiable and satisfies $f_i(\cdot) \geq 0$, $f_i'(\cdot) > 0$, and $f_i''(\cdot) \leq 0$.

Preferences and Payoffs Each agent is risk neutral and has a discount factor $\delta_i \in (0, 1)$, which measures the degree of his patience. If a proposal is approved in period τ, agent i's discounted payoff is

$$\Pi_i := \delta_i^\tau s_{i,\tau} - \sum_{t=0}^{\tau} \delta_i^t c_i(x_{i,t}),$$

where $s_{i,\tau}$ is the share he receives under the approved proposal and $c_i(x_{i,t})$ the effort cost incurred in each period $t \in \{0, \ldots, \tau\}$.[1]

Solution Concept The bargaining game with costly recognition can be described as $\langle (f_i(\cdot))_{i\in\mathcal{N}}, (c_i(\cdot))_{i\in\mathcal{N}}, \boldsymbol{\delta}, k \rangle$, where $(f_i(\cdot))_{i\in\mathcal{N}}$ denotes the set of impact functions, $(c_i(\cdot))_{i\in\mathcal{N}}$ the set of effort cost functions, $\boldsymbol{\delta} := (\delta_1, \ldots, \delta_n)$ the set of discount factors, and k the voting rule.

We assume that agents use stationary strategies whereby, for each period t, agents' actions are independent of the history (see Theorem 1 for details of the strategies). Following the literature, we adopt the solution concept of the stationary subgame perfect equilibrium (SSPE) and drop the time subscript t throughout. A strategy profile is an

[1] If no agreement is reached, agent i's discounted payoff is $\Pi_i = -\sum_{t=0}^{+\infty} \delta_i^t c_i(x_{i,t})$.

SSPE if it is stationary and constitutes a subgame perfect equilibrium.[2]

2.2 Equilibrium

Let $\boldsymbol{v} := (v_1, \ldots, v_n)$ be the set of agents' equilibrium expected payoffs and consider stage-undominated voting strategies, such that agents vote as if they were pivotal. Suppose that an agent is not recognized as the proposer. He accepts a proposal if his share exceeds the discounted continuation value (i.e., $s_i \geq \delta_i v_i$) and rejects it otherwise. The proposer needs to select $k-1$ agents to form the least costly winning coalition and offers them their continuation values. His expected vote-buying cost is

$$w_i = \sum_{j \neq i} \psi_{ij} \delta_j v_j,$$

where ψ_{ij} gives the probability of agent i's including another agent j in his winning coalition. For each $j \in \mathcal{N}$, we further define $\mu_j := \sum_{i \neq j} \psi_{ij} p_i$ as agent j's probability of being included in others' winning coalitions before a proposer is recognized. The following ensues.

Theorem 1 *For each game $\langle (f_i(\cdot))_{i \in \mathcal{N}}, (c_i(\cdot))_{i \in \mathcal{N}}, \boldsymbol{\delta}, k \rangle$, there exists an SSPE characterized by $(\boldsymbol{x}, \boldsymbol{v})$ and $\{\psi_{ij}\}_{i \neq j}$. In equilibrium, each agent $i \in \mathcal{N}$ exerts effort x_i in each period. If selected as the proposer, he forms a winning coalition of $k-1$ agents such that agent j is included with probability ψ_{ij} and offers the agent $\delta_j v_j$. Otherwise, he accepts a proposer's offer if and only if his share is no less than $\delta_i v_i$. In an equilibrium, an agreement is always reached in the first period. The equilibrium is unique when $k = 1$.*

Two remarks are in order. First, the game stands in contrast to a standard contest or a standard multilateral sequential bargaining game. The game can be viewed as a contest with an endogenous prize, since w_i, p_i, μ_i, and v_i all depend on agents' effort profile $\boldsymbol{x} = (x_1, \ldots, x_n)$ and vice versa. The equilibria may not be unique.[3] However, uniqueness is restored when $k = 1$: The game reduces to a standard contest with a fixed prize, since an agent secures the entire surplus for being the proposer while receiving nothing once he loses the competition. Equilibrium uniqueness follows from the standard rent-seeking

[2]Note that when establishing the equilibrium, our analysis allows for arbitrary—i.e., nonstationary—deviations from the equilibrium path.

[3]Fixing a recognition probability profile—i.e., fixing an effort profile—the literature on multilateral bargaining has noticed that there exist multiple equilibria that differ in $\{\psi_{ij}\}$, but they result in the same profile of $(\mu_1, \ldots, \mu_n)$. An additional layer of equilibrium multiplicity may arise within our context, in the sense that agents' effort profile may differ across equilibria.

result of Szidarovszky and Okuguchi (1997); see also Fu and Wu (2020, Theorem 1), whose Supplemental Material extends the result to heterogeneous convex effort costs. Second, our game differs from the setting of Yildirim (2007) in several respects. Yildirim assumes a linear cost function and a homogeneous impact function $f(\cdot)$, with $f(0) = 0$, and weakly decreasing elasticity $xf'(x)/f(x)$. All of these restrictions are relaxed in our setting.

2.3 Rule Design

A designer sets both the voting rule k and the proposer recognition mechanism (i.e., contest rules) that determines each agent's recognition probability for any effort profile.

Design Instruments The impact function $f_i(\cdot)$ in (1) takes the form

$$f_i(\cdot) = \alpha_i \cdot h_i(\cdot) + \beta_i, \forall\, i \in \mathcal{N}, \tag{2}$$

where $h_i(\cdot)$ is twice differentiable and satisfies $h_i(0) = 0$, $h_i'(\cdot) > 0$, and $h_i''(\cdot) \leq 0$. The designer imposes the multiplicative weights $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}_+^n \setminus \{(0, \ldots, 0)\}$, which scale one's output up or down, as well as additive headstarts $\boldsymbol{\beta} := (\beta_1, \ldots, \beta_n) \in \mathbb{R}_+^n$. We can view $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ as nominal scoring rules that manipulate agents' relative competitiveness and tilt the balance of the competition. Alternatively, they can be viewed as resources assigned to agents that alter their productivity or influence (see, e.g., Fu and Wu, 2022).

Both multiplicative weights $\boldsymbol{\alpha}$ and additive headstarts $\boldsymbol{\beta}$ are broadly adopted in modeling biased contests: Epstein, Mealem, and Nitzan (2011); Franke, Kanzow, Leininger, and Schwartz (2014); and Deng, Fu, and Wu (2021), for instance, consider the former; Konrad (2002); Siegel (2009, 2014); and Kirkegaard (2012) focus on the latter; and Franke, Leininger, and Wasser (2018) and Fu and Wu (2020) allow for both.[4]

Design Objective The designer's utility depends on agents' equilibrium efforts and recognition probabilities, i.e., the ex ante distribution of recognition opportunities. She sets $(\boldsymbol{\alpha}, \boldsymbol{\beta}, k)$ to maximize an objective function $\Lambda(\boldsymbol{x}, \boldsymbol{p})$, where $\boldsymbol{x} := (x_1, \ldots, x_n)$ and $\boldsymbol{p} := (p_1, \ldots, p_n)$ denote the profiles of equilibrium efforts and agents' recognition probabilities, respectively.[5] The objective function accommodates a diverse array of preferences.

[4] It is noteworthy that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ play different roles in impacting the contest's outcome: $\boldsymbol{\alpha}$ alter the marginal returns of agents' efforts, while $\boldsymbol{\beta}$ directly add to their effective output regardless of their efforts.

[5] As stated in Theorem 1, the bargaining game always settles with an approved proposal in the first period. Thus, $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$ and $\boldsymbol{p} \equiv (p_1, \ldots, p_n)$ can effectively be viewed as the agents' equilibrium efforts and recognition probabilities in the single-period equilibrium. However, our objective function can also be interpreted and applied in settings involving delayed agreement in equilibrium. We can view x_i and p_i as each agent's average effort and average recognition probability in the dynamics.

Consider, for example, $\Lambda(\boldsymbol{x},\boldsymbol{p}) = \sum_{i\in\mathcal{N}} x_i - \lambda \sum_{i\in\mathcal{N}}\left|p_i - \frac{1}{n}\right|$, with $\lambda \geq 0$. When $\lambda = 0$, this objective boils down to maximizing equilibrium total effort, which is widely assumed in the contest design literature. The term $\sum_{i\in\mathcal{N}}\left|p_i - \frac{1}{n}\right|$ (i.e., the mean absolute deviation of $\boldsymbol{p}$) increases in the dispersion of $\boldsymbol{p}$. When $\lambda > 0$, the function depicts a preference for a less dispersed ex ante distribution of recognition opportunities, which compels the designer to reduce $\sum_{i\in\mathcal{N}}\left|p_i - \frac{1}{n}\right|$.[6,7]

It is noteworthy that the designer is allowed to dislike an agent's effort, so $\Lambda(\boldsymbol{x},\boldsymbol{p})$ may not increase with each x_i. This differs from conventional studies of optimally biased contests, in that the literature typically assumes that contestants' efforts accrue to the designer's benefit.

3 Optimal Rules

We now present our analysis of optimal rules and discuss the logic.

We first examine the nature of the game $\left\langle (f_i(\cdot))_{i\in\mathcal{N}}, (c_i(\cdot))_{i\in\mathcal{N}}, \boldsymbol{\delta}, k\right\rangle$ for given $(\boldsymbol{\alpha},\boldsymbol{\beta},k)$. For each agent $i \in \mathcal{N}$, the expected gross payoff conditional on winning the competition and being the proposer is $1 - w_i$, and that when not being selected is $\frac{\mu_i}{1-p_i}\delta_i v_i$. The payoff differential between winning the competition and losing it, $1 - w_i - \frac{\mu_i}{1-p_i}\delta_i v_i$, is the effective prize spread that motivates his effort. He chooses effort x_i to solve the maximization problem on the right-hand side of the following Bellman equation:

$$v_i = \max_{x_i \geq 0}\left\{p_i(x_i,\boldsymbol{x}_{-i})(1-w_i) + [1-p_i(x_i,\boldsymbol{x}_{-i})] \times \frac{\mu_i}{1-p_i(x_i,\boldsymbol{x}_{-i})}\delta_i v_i - c_i(x_i)\right\}. \tag{3}$$

Note that the objective function on the right-hand side is concave in x_i.[8] The first-order

[6] Eraslan and Merlo (2017) examine the distributive implications of voting rules. They show that unanimity may paradoxically lead to a more unequal distributive outcome. It is noteworthy that in our context, the designer's fairness concern refers to her preference for the ex ante distribution of bargaining power among agents—i.e., the recognition probability profile—instead of the ex post distribution of the surplus.

[7] Our model assumes that the designer's utility depends on agents' effort profile and recognition probability profile. This setting accommodates concerns about the ex ante distribution of bargaining power among agents, but not about the ex post distributive outcome: The game is inherently stochastic due to the nature of a contest for recognition, which makes the ex post outcome difficult to evaluate within the current framework. We leave this to future research.

[8] Although μ_i itself may vary with x_i through recognition probabilities, the conditional inclusion probability $\mu_i/[1-p_i(x_i,\boldsymbol{x}_{-i})]$ does not. Holding other agents' efforts and equilibrium coalition-formation probabilities fixed, $\mu_i/(1-p_i) = \sum_{j\neq i}\frac{p_j}{1-p_i}\psi_{ji} = \sum_{j\neq i}\frac{f_j(x_j)}{\sum_{\ell\neq i} f_\ell(x_\ell)}\psi_{ji}$, which is independent of x_i. Hence, up to a constant, the objective is the effective prize spread times $p_i(x_i,\boldsymbol{x}_{-i})$ minus $c_i(x_i)$. Its concavity follows from the concavity of p_i and the convexity of c_i.

condition ensues:

$$\underbrace{c_i'(x_i)}_{\text{marginal cost of effort}} \geq \underbrace{\frac{f_i'(x_i)}{f_i(x_i)} \times p_i(1-p_i) \times \overbrace{\left(1 - w_i - \frac{\mu_i}{1-p_i}\delta_i v_i\right)}^{\text{effective prize spread}}}_{\text{marginal benefit of effort}}, \text{ with equality if } x_i > 0. \tag{4}$$

The game can be viewed as a contest with an endogenous prize, since w_i, p_i, μ_i, and v_i all depend on agents' effort profile $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$ and vice versa. These nuances dismiss the regularity typically present in standard contest games with fixed prizes or multilateral sequential bargaining games. For instance, equilibrium uniqueness cannot be guaranteed except for the case of $k = 1$. Conventional approaches for contest design do not apply.

3.1 Main Results

Our analysis first concludes the following when k and $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ are set all together.

Theorem 2 *Consider an arbitrary rule $(\boldsymbol{\alpha}, \boldsymbol{\beta}, k)$ with $k \geq 2$ that induces an equilibrium with an outcome $(\boldsymbol{x}, \boldsymbol{p})$, where $\boldsymbol{x} = (x_1, \ldots, x_n)$ is agents' effort profile and $\boldsymbol{p} = (p_1, \ldots, p_n)$ is the associated recognition probability profile. There always exists an alternative rule $(\widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}}, 1)$ that induces the same outcome $(\boldsymbol{x}, \boldsymbol{p})$ in the unique equilibrium. As a result, the objective function $\Lambda(\boldsymbol{x}, \boldsymbol{p})$ can be maximized by a set of rules that involve a dictatorial voting rule with $k = 1$.*

Recall that an agreement is always reached in period 1 in equilibrium, although the bargaining process can repeat if no proposal is approved by enough favorable votes. With $k = 1$, the bargaining process reduces the game to a standard static contest with a fixed prize of unit value, since both w_i and $\frac{\mu_i}{1-p_i}\delta_i v_i$ are zero. Note that multiple equilibria may emerge when $k = 2$. Our analysis verifies that the outcome under any given $(\boldsymbol{\alpha}, \boldsymbol{\beta}, k)$ with $k > 1$ can be weakly dominated by a rule with $k = 1$. As a result, the designer can implement any feasible outcome $(\boldsymbol{x}, \boldsymbol{p})$ by resolving the distributive process through a static contest, although the specific form of the associated recognition mechanism $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ depends on the particular context. It is without loss to search for the optimum within the subset of possible rules with $k = 1$, i.e., setting the optimal rules of the multilateral bargaining game as if it were a standard contest design problem.

Suppose that agents' efforts are productive and benefit the designer, such that $\Lambda(\boldsymbol{x}, \boldsymbol{p})$ weakly increases with x_i for every $i \in \mathcal{N}$. For instance, a mediator might value the lobbying efforts of disputing parties, or a school's dean might seek to encourage departments' research efforts in exchange for priority in future resource allocation. The following ensues.

Theorem 3 *Suppose that the objective function $\Lambda(\boldsymbol{x}, \boldsymbol{p})$ weakly increases with x_i for every $i \in \mathcal{N}$. When the designer can flexibly choose $(\boldsymbol{\alpha}, \boldsymbol{\beta}, k)$, $\Lambda(\boldsymbol{x}, \boldsymbol{p})$ can always be maximized by a rule $(\widehat{\boldsymbol{\alpha}}, \mathbf{0}, 1)$, i.e., a dictatorial voting rule and zero headstart for every agent.*

With $\Lambda(\boldsymbol{x}, \boldsymbol{p})$ weakly increasing in x_i for every $i \in \mathcal{N}$, the optimum requires that the designer properly set multiplicative biases, while setting headstarts to zero. Adjusting multiplicative biases alters the marginal returns to agents' efforts, which can incentivize efforts more effectively than resorting to $\boldsymbol{p}$.

3.2 Discussion

Next, we examine the respective roles played by the voting rule k and the contest rules $(\boldsymbol{\alpha}, \boldsymbol{\beta})$, which reveals the nature of these results. The two sets of instruments, k and $(\boldsymbol{\alpha}, \boldsymbol{\beta})$, serve distinct functions.

Role of Voting Rule We now explore how a change in k affects agents' effort incentives and total effort $\sum_{i=1}^{n} x_i$, which demonstrates how our setting differs from a conventional sequential bargaining game.

Recall that the game is a contest with an endogenous prize spread $1 - w_i - \frac{\mu_i}{1-p_i}\delta_i v_i$. Imagine an increase in k. This reshapes agents' prize spreads. A larger k changes both an agent's winning value (i.e., $1 - w_i$) and losing value (i.e., $\frac{\mu_i}{1-p_i}\delta_i v_i$). A proposer has to buy more votes, and he needs to buy votes from a different set of his peers; each peer demands a different offer, since their continuation values change. All of these factors change w_i. Further, a losing candidate expects a different continuation value because he is more likely to be included in some winning coalitions; the change in his continuation value further alters the minimum share he would demand, which changes the probability of his being included in other agents' winning coalitions, μ_i. These changes lead to a different losing value $\frac{\mu_i}{1-p_i}\delta_i v_i$.

Varying k triggers two effects that intertwine, leaving the overall effect ambiguous. First, there is a *prize effect*, as prize spreads adjust endogenously and change agents' incentives to compete for recognition. Second, there is a *rebalancing effect*: Changes in prize spreads are nonuniform across asymmetric agents, altering their relative strengths in the competition, tilting the playing field, and reshaping their effort incentives.

To see this, begin with $k = 1$, under which all agents have a prize spread of 1. When k increases to 2, however, the decrease in prize spreads is asymmetric. The most patient agent is least likely to be included in a winning coalition, since high patience elevates his continuation value and therefore the cost of buying his vote. His losing value, $\frac{\mu_i}{1-p_i}\delta_i v_i$, tends to rise less than that of others, which in turn limits the decrease in his prize spread compared with that of others. As a result, the most patient agent tends to retain a larger prize spread and stronger prize incentives than the others. This reshapes the competition.

We construct an example to illustrate this subtlety.

Example 1 *Suppose that* $n = 4$, $f_i(x_i) = \eta_i x_i$, *and* $c_i(x_i) = \eta_i x_i$, *with* $\boldsymbol{\eta} = (1, 0.2, 0.2, 0.2)$ *and* $\boldsymbol{\delta} = (0.1, 0.5, 0.5, 0.5)$. *The equilibria under different voting rules are summarized in Table 1. The equilibrium is unique for* $k = 1$, *and numerical verification confirms the uniqueness of the reported equilibrium outcomes for* $k \in \{2, 3, 4\}$.

	$k = 1$	$k = 2$	$k = 3$	$k = 4$
Winning probability of agent 1	0.2500	0.2322	0.2386	0.2500
Winning probability of agents 2-4	0.2500	0.2559	0.2538	0.2500
Equilibrium effort of agent 1	0.1875	0.1711	0.1656	0.1570
Equilibrium effort of agents 2-4	0.9375	0.9433	0.8805	0.7849
Total effort	3.0000	3.0011	2.8072	2.5116

Table 1: Equilibrium Outcomes in Example 1.

Table 1 demonstrates the complex effects of k: Equilibrium total effort varies nonmonotonically with k and is maximized at $k = 2$. There are two types of agents in this example: one impatient agent (player 1 with $\delta = 0.1$) and three patient agents (players 2-4 with $\delta = 0.5$). When $k = 1$, all agents have an effective prize spread of 1; their patience levels are irrelevant; and the heterogeneity in effort costs and that in impact functions perfectly offset each other. As a result, all agents win with equal probability.

When k increases to 2, all players expect smaller prize spreads. This prize effect tends to decrease effort. However, the decreases in prize spreads are uneven. The impatient agent 1 is always included in the winning coalition, regardless of which of the three patient agents wins; in contrast, each patient agent, if he loses, can be included in a winning coalition only if the single impatient agent wins. As a result, agent 1 expects a larger losing value and a smaller prize spread than his more patient counterparts; his motivation becomes relatively weaker after the change in k.

The playing field is thus tilted, and the rebalancing effect comes into play. The three patient agents expect to win with a larger probability. This incentivizes them to exert more effort, while disincentivizing agent 1. However, this change in relative positions is efficient: The three patient agents have lower marginal effort costs ($c_i = 0.2$), so the effort increase from the patient agents more than offsets the decrease from the impatient agent 1. As a result, the rebalancing effect caused by nonuniform decreases in prize spreads elevates overall effort supply and outweighs the prize effect caused by smaller prize spreads. Total effort increases when k increases from 1 to 2.

This observation sharply contrasts with the result of Yildirim (2007). With symmetric agents, Yildirim shows that total effort strictly decreases with k. In this symmetric baseline, an increase in k decreases agents' prize spreads and weakens their incentives, but the decrease is uniform across all agents. The rebalancing effect demonstrated above is entirely muted: Agents remain symmetric, and their relative strengths are unaltered. As shown in Table 1, the monotonicity established by Yildirim fails to hold with heterogeneous agents.

Role of Recognition Mechanism (Contest Rules) We now discuss the role played by the recognition mechanism (contest rules), which further illustrates how this exercise differs from conventional contest design.

Example 2 *Let the designer set $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ for a given k to maximize the objective function*

$$\Lambda(\boldsymbol{x}, \boldsymbol{p}) = \sum_{i \in \mathcal{N}} x_i - \lambda \sum_{i \in \mathcal{N}} \left| p_i - \frac{1}{n} \right|, \text{ with } \lambda > 0. \tag{5}$$

Suppose that $n = 3$, $k = 2$, $f_i(x_i) = x_i$, and $c_i(x_i) = c_i x_i$ with $(c_1, c_2, c_3) = (1, 1, c)$. Let $(\delta_1, \delta_2, \delta_3) = (\frac{3}{8}, \frac{1}{2}, \frac{12}{13})$. Assume that λ is sufficiently large and c is sufficiently small, such that $\lambda \gg \frac{1}{c} \gg 1$.

The objective function can be maximized by a recognition mechanism with $\boldsymbol{\alpha}^* = (\frac{62Y}{35}, \frac{62Y}{37}, \frac{62Yc}{39})$ and $\boldsymbol{\beta}^* = (0, \frac{17Y}{222}, 0)$, where $Y > 0$ is an arbitrary positive constant. The game yields an equilibrium outcome of $\boldsymbol{x} = (\frac{70}{372}, \frac{57}{372}, \frac{78}{372c})$ and $\boldsymbol{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. The designer's payoff is $\Lambda = \frac{127}{372} + \frac{78}{372c}$.

	Agent 1	Agent 2	Agent 3
Equilibrium effort	70/372	57/372	78/(372c)
Winning probability	1/3	1/3	1/3
Equilibrium payoff	56/372	72/372	39/372
Winning coalition	{1, 2}	{1, 2}	{1, 3}

Table 2: Equilibrium Outcomes in Example 2.

Notably, the designer awards a positive headstart to agent 2 in this example. This stands in contrast to findings in the literature on contest design. Fu and Wu (2020) formally establish the suboptimality of a headstart and show that adjusting $\boldsymbol{\alpha}$ suffices to achieve the optimum. This contrast reveals how the endogenous prize structure differentiates the game from a standard contest with a fixed prize.

With $c_1 = c_2 > c_3$ and $\delta_1 < \delta_2 < \delta_3$, agent 3 is ex ante the strongest contender, followed by agent 2 and then agent 1. The designer would benefit if agent 3 could be sufficiently incentivized given his low effort cost, which requires a larger prize spread for this agent. This would happen if the designer could reduce agent 1's continuation value, which decreases agent 3's vote-buying cost (i.e., w_3) given that, according to Table 2, agent 3 includes agent 1 in his winning coalition.

We then examine how the designer can decrease agent 1's continuation value. According to Table 2, agent 1 buys agent 2's vote upon becoming the proposer. The designer can increase agent 2's continuation value to render agent 1 worse off, which can be achieved by awarding agent 2 either a headstart $\beta_2 > 0$ or a larger α_2. The former is more effective in this context: While both increase agent 2's recognition probability and improve his payoff, a larger α_2 increases the marginal benefit of effort, which compels agent 2 to exert more effort; the associated effort cost partially offsets the payoff-improving effect of a larger α_2. In summary, a positive β_2 effectively increases agent 2's continuation value, which raises agent 1's vote-buying cost and decreases his continuation value; this, in turn, decreases agent 3's vote-buying cost, widens his effective prize spread, and motivates his effort.

These subtleties, as artifacts of the dynamic bargaining process, are absent in a static contest. The prize is fixed in a standard contest, so the contest rules $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ only rebalance the playing field and do not alter agents' prize incentives. Multiplicative biases $\boldsymbol{\alpha}$ can more effectively motivate effort due to their direct impact on the marginal benefit of effort, which renders headstarts $\boldsymbol{\beta}$ redundant, as Fu and Wu (2020) show. In contrast, a change in the contest rules $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ in our context not only tilts the playing field (rebalancing effect) but also alters players' prize spreads (prize effect) when $k \geq 2$. A headstart, as explained above, allows the designer to fully exploit the endogenous payoff structure of the game.

Optimal Rule Design: The Logic The intertwining prize and rebalancing effects triggered by changes in k or $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ pose a challenge to optimal rule design. The results obtained in conventional settings, such as those of Yildirim (2007) and Fu and Wu (2020), do not directly apply. However, Theorem 2 shows that adjusting $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ while setting $k = 1$ is sufficient to induce any feasible outcome $(\boldsymbol{x}, \boldsymbol{p})$.

Notably, by setting $k = 1$, the prize spread is now fixed and independent of agents' continuation values; therefore, a change in $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ no longer affects agents' effective prize spreads. This eliminates the dynamic linkage within the game. The designer can thus adjust $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ to optimally set the competitive balance of the contest for the proposing right without introducing the complications caused by nonuniform changes in agents' endogenously determined prize spreads.

The logic can be further illustrated for cases when agents' effort accrues to the designer's benefit, such that $\Lambda(\boldsymbol{x}, \boldsymbol{p})$ is weakly increasing in x_i for every $i \in \mathcal{N}$. Setting $k = 1$ maximizes the prize spread and provides the strongest prize incentive for all agents, which encourages them to strive for recognition. The designer can then adjust $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ accordingly to exploit agents' heterogeneity and optimally balance the playing field. The two sets of instruments complement each other in this optimum.

Two remarks are in order. First, Theorems 2 and 3 do not rely on comparative statics with respect to k. As shown above, the intertwining effects caused by varying k rule out the monotonicity established by Yildirim (2007) in our setting. Theorem 2 shows that a rule with $k = 1$ can weakly dominate any equilibrium outcome under a rule with $k \neq 1$, regardless of how the objective function $\Lambda(\boldsymbol{x}, \boldsymbol{p})$ depends on agents' efforts. However, our analysis does not rank the equilibrium outcomes under two sets of rules when both involve $k > 1$. Second, the results and approach of Fu and Wu (2020) do not directly apply because our setting involves endogenously determined prizes when $k > 1$ and the joint choice of k and $(\boldsymbol{\alpha}, \boldsymbol{\beta})$. Theorem 2 restores the relevance of Fu and Wu (2020) in our context: It allows us to limit the search for the optimum to rules with $k = 1$ without loss of generality.

Motivated by this discussion, we next consider an extension to our baseline analysis that sheds further light on the nature of our results.

3.3 Extension

Theorem 2 inspires us to further explore the general effect of varying k: Does there exist a monotonic relationship between the value of $\Lambda(\boldsymbol{x}, \boldsymbol{p})$ and the voting rule k when the designer can optimally adjust the recognition mechanism for every given k? In other words, does a less inclusive voting rule (i.e., a smaller k) consistently improve the value of the objective function when the mechanism is adjusted accordingly?

The answer is a priori unclear. As noted above, $k = 1$ eliminates the dynamic linkage and reduces the process to a static contest. For $k \neq 1$, the dynamic linkage persists, and the game remains a contest with endogenously determined prizes. Adjusting $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ alters agents' relative strengths in the contest and thereby changes their winning probabilities, continuation values, coalition-formation strategies, and the costs of purchasing votes upon becoming the proposer. These shifts further modify the effective prize for each agent. Section 3.2 demonstrates this intricacy.

Despite this complexity, our analysis yields the following result, which extends the scope of our main findings.

Theorem 4 *Suppose that $\delta_i \leq \frac{1}{2}$ for each $i \in \mathcal{N}$. Consider an arbitrary outcome $(\boldsymbol{x}, \boldsymbol{p})$ of the game that can be induced by a rule $(\boldsymbol{\alpha}, \boldsymbol{\beta}, k)$ with $k \in \{2, \ldots, n\}$. There always exists an alternative rule $(\widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}}, k-1)$ that induces the same equilibrium outcome $(\boldsymbol{x}, \boldsymbol{p})$. As a result, if the designer can flexibly adjust $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ for a given k, the optimal value of the objective function Λ is weakly decreasing in k.*

Theorem 4 confirms that with the additional restriction $\delta_i \leq 1/2$, any feasible outcome $(\boldsymbol{x}, \boldsymbol{p})$ can be implemented using a less inclusive voting rule (i.e., a smaller k), provided that the recognition mechanism $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is adjusted accordingly. The designer can further improve the outcome beyond $(\boldsymbol{x}, \boldsymbol{p})$ by fine-tuning the recognition mechanism. As a result, she always prefers a less inclusive voting rule when agents are relatively impatient.

To illustrate the logic, assume that the designer benefits from agents' effort (i.e., $\Lambda(\boldsymbol{x}, \boldsymbol{p})$ is weakly increasing in x_i). A smaller k enlarges prize spreads in the recognition contest and boosts effort, since a lower vote requirement increases the rent the proposer can secure and reduces losers' continuation values. However, the discussion in Section 3.2 reveals the rebalancing effect caused by nonuniform changes in prize spreads among asymmetric agents, which alter agents' relative strengths in the competition and reshape the equilibrium. A low patience level (i.e., a small δ) weakens the rebalancing effect and allows the prize effect to dominate. Similarly, the designer can adjust $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ to optimally balance the playing field without substantial complications from the endogenously determined prize spreads. Together, a smaller k unambiguously amplifies prize incentives.

4 Conclusion

In this paper, we explore a multilateral sequential bargaining game with costly recognition, in which a set of agents divide a fixed amount of resources and the proposer is recognized through a contest. We examine the roles of the institutional rules that govern the distributive process. We illustrate the complexity of the game and consider a general design problem in which a designer's payoff depends on agents' effort and their recognition probabilities. The designer can deploy two sets of design instruments: (i) the voting rule that governs how proposals are accepted or rejected and (ii) the recognition mechanism that determines how the proposer is selected based on agents' productive effort. We demonstrate that any feasible outcome regarding equilibrium effort and recognition probabilities can be implemented by a standard contest—i.e., a rule that involves a dictatorial voting rule. As a result, the designer can maximize her objective function by resolving the distributive process through a simple static contest.

Our analysis also leaves room for extensions. For instance, future studies can address even broader objectives in rule design. Our optimal rule leads to the most ex post unequal distributive outcome, which allows the proposer to secure the entire rent. As we noted in Footnote 7, our game is inherently stochastic, so a criterion based on ex post equity cannot be readily defined. Intuitively, however, any improvement in ex post equity would require a more inclusive voting rule with $k > 1$: This prevents any single agent from securing the entire rent, and the optimal rule under such a criterion could therefore depart from the $k = 1$ identified here under an objective defined over the ex ante distribution of bargaining power. A formal treatment of ex post equity concerns warrants serious future research.

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Appendix: Proofs

Proof of Theorem 1

Proof. We first characterize the SSPE assuming its existence, then prove equilibrium existence.

Equilibrium Characterization Denote by V^Δ the k-th lowest continuation value. Let $\mathcal{N}_1 := \{i \in \mathcal{N} : \delta_i v_i < V^\Delta\}$, $\mathcal{N}_2 := \{i \in \mathcal{N} : \delta_i v_i = V^\Delta\}$, and $\mathcal{N}_3 := \{i \in \mathcal{N} : \delta_i v_i > V^\Delta\}$. Thus, V^Δ is the reservation value of a marginal coalition member. Agents in $\mathcal{N}_1$ have continuation values strictly below this threshold. Their votes are strictly cheaper than the marginal vote, so they are included whenever another agent forms a winning coalition. Agents in $\mathcal{N}_2$ are marginal coalition members: Their continuation values equal V^Δ, and they may be included with equilibrium probabilities to complete the winning coalition. Agents in $\mathcal{N}_3$ have continuation values above V^Δ. Their votes are too costly and are never purchased.

Evidently, agent i, when becoming the proposer, buys out the votes of the cheapest "winning coalition"—i.e., $\mathcal{N}_1$ and a subset of $\mathcal{N}_2$, from which we can conclude

$$\psi_{ij} \begin{cases} = 1, & j \in \mathcal{N}_1, \\ \in [0,1], & j \in \mathcal{N}_2, \\ 0, & j \in \mathcal{N}_3, \end{cases} \text{ and } \mu_i \begin{cases} = 1 - p_i, & i \in \mathcal{N}_1, \\ \in [0, 1-p_i], & i \in \mathcal{N}_2, \\ = 0, & i \in \mathcal{N}_3. \end{cases} \tag{6}$$

Define

$$V_L := 1 - \sum_{j \in \mathcal{N}_1} \delta_j v_j - \left(k - |\mathcal{N}_1|\right) V^\Delta. \tag{7}$$

Agent i's expected cost is then

$$w_i = \begin{cases} 1 - V_L - \delta_i v_i, & i \in \mathcal{N}_1, \\ 1 - V_L - V^\Delta, & \text{otherwise.} \end{cases}$$

The effective prize spread $1 - w_i - \frac{\mu_i}{1-p_i}\delta_i v_i$ in (4) can be expressed as

$$1 - w_i - \frac{\mu_i}{1-p_i}\delta_i v_i = V_L + \frac{1-\mu_i-p_i}{1-p_i}V^\Delta = \begin{cases} V_L, & i \in \mathcal{N}_1, \\ V_L + \frac{1-p_i-\mu_i}{1-p_i}V^\Delta, & i \in \mathcal{N}_2, \\ V_L + V^\Delta, & i \in \mathcal{N}_3. \end{cases} \tag{8}$$

The term V_L is the residual surplus generated by the least costly winning coalition after accounting for its members' continuation values. It is also the minimum effective prize spread, attained by agents in $\mathcal{N}_1$. In contrast, agents in $\mathcal{N}_3$ have the maximum effective prize spread, $V_L + V^\Delta$. Hence, V^Δ measures the equilibrium gap between the maximum and minimum effective prize spreads, while the effective prize spreads of agents in $\mathcal{N}_2$ lie between these two bounds and depend on their coalition-inclusion probabilities.

We are ready to lay out the conditions for equilibrium characterization. An SSPE can be characterized by $(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{p}, \boldsymbol{\mu}, V_L, V^\Delta)$. Combining (4) and (8) yields

$$\frac{c_i'(x_i) f_i(x_i)}{f_i'(x_i)} \geq p_i(1-p_i)\left(V_L + \frac{(1-p_i-\mu_i)V^\Delta}{1-p_i}\right), \text{ with equality if } x_i > 0. \tag{9}$$

Next, consider the expected payoff v_i. By (3), we have

$$v_i = p_i(1-w_i) + \mu_i \delta_i v_i - c_i(x_i) = \begin{cases} \frac{1}{1-\delta_i}\left(p_i V_L - c_i(x_i)\right), & i \in \mathcal{N}_1, \\ \frac{V^\Delta}{\delta_i}, & i \in \mathcal{N}_2, \\ p_i(V_L + V^\Delta) - c_i(x_i), & i \in \mathcal{N}_3. \end{cases} \tag{10}$$

Combining (3), (6), and (10) yields

$$\mu_i \begin{cases} = 1 - p_i, & i \in \mathcal{N}_1, \\ \in [0, 1-p_i] \text{ solves } \frac{V^\Delta}{\delta_i} = p_i V_L + (\mu_i + p_i)V^\Delta - c_i(x_i), & i \in \mathcal{N}_2, \\ = 0, & i \in \mathcal{N}_3. \end{cases} \tag{11}$$

Each agent chooses exactly $k-1$ agents in his winning coalition—i.e., $\sum_{j \neq i} \psi_{ij} = k-1,\ \forall i \in \mathcal{N}$. Therefore,

$$\sum_{i \in \mathcal{N}} \mu_i = \sum_{i \in \mathcal{N}} \sum_{j \neq i} \psi_{ji} p_j = \sum_{j \in \mathcal{N}} p_j \sum_{i \neq j} \psi_{ji} = \sum_{j \in \mathcal{N}} (k-1) p_j = k-1. \tag{12}$$

Last, (7) can be rewritten as

$$V_L + \sum_{i \in \mathcal{N}_1} (\delta_i v_i) + \left(k - |\mathcal{N}_1|\right) V^\Delta = 1. \tag{13}$$

To characterize an SSPE, it suffices to find $(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{p}, \boldsymbol{\mu}, V^\Delta, V_L)$ that satisfies (9)-(13).

Equilibrium Existence Let $Y := \sum_{i \in \mathcal{N}} f_i(x_i)$. By (1), we have $p_i = f_i(x_i)/Y$, which implies that

$$x_i = f_i^{-1}(Y p_i), \text{ for } p_i \in \left[f_i(0)/Y, 1\right], \tag{14}$$

and

$$\sum_{i \in \mathcal{N}} p_i = 1. \tag{15}$$

Substituting (14) into (9) yields

$$\frac{Y c_i'\left(f_i^{-1}(Y p_i)\right)}{f_i'\left(f_i^{-1}(Y p_i)\right)} \geq (1 - p_i)(V_L + V^\Delta) - \mu_i V^\Delta, \text{with equality holding if } p_i > \frac{f_i(0)}{Y}. \tag{16}$$

Rewriting (11) and (12) and substituting (10) into (13) yield

$$\mu_i = \frac{1}{V^\Delta} \text{med} \left\{ 0, V^\Delta(1 - p_i), \frac{V^\Delta}{\delta_i} - p_i(V_L + V^\Delta) + c_i\left(f_i^{-1}(Y p_i)\right) \right\}, \tag{17}$$

$$\sum_{i \in \mathcal{N}} \mu_i = k - 1, \tag{18}$$

and

$$\sum_{i \in \mathcal{N}_1} \frac{\delta_i}{1 - \delta_i} \left[p_i V_L - c_i\left(f_i^{-1}(Y p_i)\right) \right] + \left(k - |\mathcal{N}_1|\right) V^\Delta + V_L = 1, \tag{19}$$

where med$\{\cdot, \cdot, \cdot\}$ gives the median of the input.

To prove equilibrium existence, it suffices to show that there exists $(\boldsymbol{p}, \boldsymbol{\mu}, Y, V^\Delta, V_L)$ to satisfy conditions (15)-(19). The proof consists of four steps. First, fixing (Y, V^Δ, V_L), we show that there exists a unique $(\boldsymbol{p}, \boldsymbol{\mu})$ to satisfy (16) and (17). Second, fixing (V^Δ, V_L), there exists $Y \geq \sum_{i \in \mathcal{N}} f_i(0)$ to satisfy (15). Third, fixing V_L, there exists V^Δ to satisfy (18). Last, we show that there exists V_L to satisfy (19).

Step I Substituting (17) into (16) yields

$$\frac{Y c_i'\left(f_i^{-1}(Y p_i)\right)}{f_i'\left(f_i^{-1}(Y p_i)\right)} \geq \text{med} \left\{ (1 - p_i)(V_L + V^\Delta), (1 - p_i)V_L, V_L + V^\Delta - \frac{V^\Delta}{\delta_i} - c_i(f_i^{-1}(Y p_i)) \right\}, \tag{20}$$

with equality holding if $p_i > \frac{f_i(0)}{Y}$.

Let

$$\phi(p_i) := \frac{Y c_i'\big(f_i^{-1}(Y p_i)\big)}{f_i'(f_i^{-1}(Y p_i))} - \operatorname{med}\left\{(1-p_i)(V_L+V^\Delta), (1-p_i)V_L, V_L+V^\Delta-\frac{V^\Delta}{\delta_i}-c_i\big(f_i^{-1}(Y p_i)\big)\right\}.$$

Note that $f_i(\cdot)$ is increasing and concave by assumption. This implies that $\phi(\cdot)$ strictly increases with p_i. Therefore, if $\phi\big(\frac{f_i(0)}{Y}\big) \geq 0$, or equivalently, if

$$\frac{Y c_i'(0)}{f_i'(0)} \geq \operatorname{med}\left\{\left(1-\frac{f_i(0)}{Y}\right)V_L, \left(1-\frac{f_i(0)}{Y}\right)(V_L+V^\Delta), V_L+V^\Delta-\frac{V^\Delta}{\delta_i}\right\}, \tag{21}$$

then $p_i = \frac{f_i(0)}{Y}$. Otherwise, if $\phi\big(\frac{f_i(0)}{Y}\big) < 0$, or equivalently, if

$$\frac{Y c_i'(0)}{f_i'(0)} < \operatorname{med}\left\{\left(1-\frac{f_i(0)}{Y}\right)V_L, \left(1-\frac{f_i(0)}{Y}\right)(V_L+V^\Delta), V_L+V^\Delta-\frac{V^\Delta}{\delta_i}\right\}, \tag{22}$$

then $p_i > \frac{f_i(0)}{Y}$; moreover, p_i is uniquely pinned down by $\phi(p_i) = 0$, or equivalently,

$$\frac{Y c_i'\big(f_i^{-1}(Y p_i)\big)}{f_i'(f_i^{-1}(Y p_i))} = \operatorname{med}\left\{(1-p_i)(V_L+V^\Delta), (1-p_i)V_L, V_L+V^\Delta-\frac{V^\Delta}{\delta_i}-c_i\big(f_i^{-1}(Y p_i)\big)\right\}. \tag{23}$$

Further, μ_i can be uniquely solved from (17). Therefore, fixing (Y, V^Δ, V_L), there exists a unique pair (p_i, μ_i) to satisfy (16) and (17), which we denote by $\big(p_i(Y, V^\Delta, V_L), \mu_i(Y, V^\Delta, V_L)\big)$ with slight abuse of notation.

Step II We show that fixing (V^Δ, V_L) and $\{p_i(Y, V^\Delta, V_L), \mu_i(Y, V^\Delta, V_L))\}_{i\in\mathcal{N}}$, there exists $Y \geq \sum_{i\in\mathcal{N}} f_i(0)$ to satisfy (15). By definition of $p_i(Y, V^\Delta, V_L)$, we have that $p_i(Y, V^\Delta, V_L) \geq \frac{f_i(0)}{Y}$, which implies

$$\sum_{i\in\mathcal{N}} p_i\left(\sum\nolimits_{j\in\mathcal{N}} f_j(0), V^\Delta, V_L\right) \geq 1.$$

Next, we claim that

$$\lim_{Y\to+\infty} \sum_{i\in\mathcal{N}} p_i(Y, V^\Delta, V_L) = 0.$$

To see this, first consider the case of $f_i'(0) < +\infty$. Then (21) holds as Y approaches infinity, in which case $p_i = \frac{f_i(0)}{Y}$ and

$$\lim_{Y\to+\infty} p_i(Y, V^\Delta, V_L) = \lim_{Y\to+\infty} \frac{f_i(0)}{Y} = 0.$$

Next, consider the case of $f_i'(0) = +\infty$. Then (22) holds for all Y and $p_i(Y, V^\Delta, V_L)$ solves (23). As Y approaches infinity, the right-hand side of (23) approaches infinity; therefore, the left-hand side must be finite, which indicates that $p_i(Y, V^\Delta, V_L)$ approaches 0.

By the intermediate value theorem, there exists $Y \geq \sum_{i \in \mathcal{N}} f_i(0)$ such that

$$\sum_{i \in \mathcal{N}} p_i(Y, V^\Delta, V_L) = 1.$$

Fixing (V^Δ, V_L), we denote the largest Y that solves the above equation by $Y(V^\Delta, V_L)$ in the subsequent analysis.

Step III Fixing V_L, $Y(V^\Delta, V_L)$, and $\{p_i(Y, V^\Delta, V_L), \mu_i(Y, V^\Delta, V_L))\}_{i \in \mathcal{N}}$, we show that there exists V^Δ such that (18) holds, i.e.,

$$\sum_{i \in \mathcal{N}} \mu_i \left(Y(V^\Delta, V_L), V^\Delta, V_L \right) = k - 1. \tag{24}$$

First, consider the case in which V^Δ approaches 0. For each $i \in \mathcal{N}$, when $p_i = \frac{f_i(0)}{Y}$, we have that

$$\begin{aligned}
&\lim_{V^\Delta \searrow 0} \mu_i \left(Y(V^\Delta, V_L), V^\Delta, V_L \right) \\
=& \lim_{V^\Delta \searrow 0} \frac{1}{V^\Delta} \text{med} \left\{ 0, V^\Delta \left(1 - \frac{f_i(0)}{Y(V^\Delta, V_L)} \right), \frac{V^\Delta}{\delta_i} - \frac{f_i(0)}{Y(V^\Delta, V_L)} (V_L + V^\Delta) \right\} = 0,
\end{aligned}$$

where the second equality follows from the fact that $\frac{V^\Delta}{\delta_i} - \frac{f_i(0)}{Y(V^\Delta, V_L)}(V_L + V^\Delta) \leq 0 \leq V^\Delta \big(1 - \frac{f_i(0)}{Y(V^\Delta, V_L)}\big)$ as V^Δ approaches 0.

When $p_i > \frac{f_i(0)}{Y}$, by (17), $\mu_i \left(Y(V^\Delta, V_L), V^\Delta, V_L \right) = 0$ for sufficiently small V^Δ. Therefore, we have that

$$\lim_{V^\Delta \searrow 0} \sum_{i \in \mathcal{N}} \mu_i \left(Y(V^\Delta, V_L), V^\Delta, V_L \right) = 0.$$

Next, consider the case in which V^Δ approaches infinity. For each $i \in \mathcal{N}$, we have that

$$\begin{aligned}
0 \leq& V^\Delta \left[1 - p_i \left(Y(V^\Delta, V_L), V^\Delta, V_L \right) \right] \\
\leq& \frac{V^\Delta}{\delta_i} - p_i \left(Y(V^\Delta, V_L), V^\Delta, V_L \right) (V_L + V^\Delta) + c_i \left(f_i^{-1} \left(Y(V^\Delta, V_L) p_i \left(Y(V^\Delta, V_L), V^\Delta, V_L \right) \right) \right);
\end{aligned}$$

together with (17), we can obtain that

$$\mu_i\left(Y(V^{\Delta}, V_L), V^{\Delta}, V_L\right) = 1 - p_i\left(Y(V^{\Delta}, V_L), V^{\Delta}, V_L\right), \text{ as } V^{\Delta} \to +\infty.$$

Therefore, we have that

$$\lim_{V^{\Delta}\to+\infty} \sum_{i\in\mathcal{N}} \mu_i\left(Y(V^{\Delta}, V_L), V^{\Delta}, V_L\right) = \lim_{V^{\Delta}\to+\infty} \sum_{i\in\mathcal{N}} \left[1 - p_i\left(Y(V^{\Delta}, V_L), V^{\Delta}, V_L\right)\right] = n - 1.$$

Note that $\mu_i(Y, V^{\Delta}, V_L)$ and $Y(V^{\Delta}, V_L)$ are continuous for all $i \in \mathcal{N}$, and $0 \le k - 1 \le n - 1$. It follows immediately that there exists $V^{\Delta} \ge 0$ to satisfy (24). In what follows, fixing V_L, let us denote the largest V^{Δ} that solves (24) by $V^{\Delta}(V_L)$.

Step IV We show that there exists $V_L \in [0, 1]$ to satisfy (19), i.e.,

$$\sum_{i\in\mathcal{N}_1} \frac{\delta_i}{1-\delta_i}\left[p_i V_L - c_i\left(f_i^{-1}(Yp_i)\right)\right] + \left(k - |\mathcal{N}_1|\right) V^{\Delta} + V_L = 1, \tag{25}$$

where $V^{\Delta} = V^{\Delta}(V_L)$, $Y = Y(V^{\Delta}, V_L)$, and $p_i = p_i(Y, V^{\Delta}, V_L)$ for $i \in \mathcal{N}$, as defined above.

Note that the left-hand side of (25) is always nonnegative; moreover, it is evident that the left-hand side is no less than 1 when $V_L = 1$. To conclude the proof, it suffices to show that $\lim_{V_L\searrow 0} V^{\Delta}(V_L) = 0$, from which we can conclude that the left-hand side of (25) approaches 0 as $V_L \searrow 0$.

Suppose, to the contrary, that $\limsup_{V_L\searrow 0} V^{\Delta}(V_L) > 0$. It can then be verified that the following strict inequality holds as $V_L \searrow 0$ and $V^{\Delta} \to \limsup_{V_L\searrow 0} V^{\Delta}(V_L)$:

$$V^{\Delta}(1 - p_i) < \frac{V^{\Delta}}{\delta_i} - p_i(V_L + V^{\Delta}) + c_i\left(f_i^{-1}(Yp_i)\right), \ \forall\, i \in \mathcal{N}.$$

Recall that $\mathcal{N}_2$ is nonempty by definition. That is, there exists some agent $j \in \mathcal{N}_2$. By (17), we have that

$$V^{\Delta}(1 - p_j) \ge \frac{V^{\Delta}}{\delta_j} - p_j(V_L + V^{\Delta}) + c_j\left(f_j^{-1}(Yp_j)\right).$$

A contradiction. ■

Proof of Theorems 2 and 3

Proof. We first prove Theorem 2. By (1), (2), and (9), we have

$$p_i = \frac{\alpha_i h_i(x_i) + \beta_i}{\sum_{j\in\mathcal{N}} \left[\alpha_j h_j(x_j) + \beta_j\right]},$$

and

$$c_i'(x_i)\frac{\alpha_i h_i(x_i) + \beta_i}{\alpha_i h_i'(x_i)} \geq p_i(1-p_i)\left(V_L + \frac{(1-p_i-\mu_i)V^\Delta}{1-p_i}\right), \tag{26}$$

with equality holding if $x_i > 0$.

We construct $(\widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}})$ as follows. For $x_i = 0$, we set $(\hat{\alpha}_i, \hat{\beta}_i) = (0, p_i)$. For $x_i > 0$, note by (13) that we have that

$$1 = \sum_{i\in\mathcal{N}_1}(\delta_i v_i) + (k - |\mathcal{N}_1|)V^\Delta + V_L \geq V^\Delta + V_L, \tag{27}$$

where the inequality follows from $v_i \geq 0$ and $|\mathcal{N}_1| \leq k-1$. Combining (26) and (27) yields

$$\frac{c_i'(x_i)h_i(x_i)}{h_i'(x_i)} \leq c_i'(x_i)\frac{\alpha_i h_i(x_i) + \beta_i}{\alpha_i h_i'(x_i)} = p_i(1-p_i)\left(V_L + \frac{(1-p_i-\mu_i)V^\Delta}{1-p_i}\right) \leq p_i(1-p_i).$$

Define $\hat{\theta}_i := p_i(1-p_i)h_i'(x_i)/c_i'(x_i) - h_i(x_i)$. The above inequality indicates $\hat{\theta}_i \geq 0$. Set

$$\left(\hat{\alpha}_i, \hat{\beta}_i\right) := \left(\frac{p_i}{h_i(x_i) + \hat{\theta}_i}, \hat{\alpha}_i\hat{\theta}_i\right). \tag{28}$$

It remains to verify that $(\boldsymbol{x}, \boldsymbol{p})$ constitutes the unique equilibrium effort profile and recognition probabilities under $(\widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}}, 1)$. When $k = 1$, the game degenerates to a standard static contest with prize value of 1. It suffices to show that the equilibrium recognition probability p_i satisfies

$$p_i = \frac{\hat{\alpha}_i h_i(x_i) + \hat{\beta}_i}{\sum_{j\in\mathcal{N}} \left[\hat{\alpha}_j h_j(x_j) + \hat{\beta}_j\right]}, \tag{29}$$

and x_i solves

$$\max_{x_i \geq 0} \frac{\hat{\alpha}_i h_i(x_i) + \hat{\beta}_i}{\sum_{j\in\mathcal{N}} \left[\hat{\alpha}_j h_j(x_j) + \hat{\beta}_j\right]} - c_i(x_i). \tag{30}$$

Note that $p_i = \hat{\alpha}_i h_i(x_i) + \hat{\beta}_i$ for all $i \in \mathcal{N}$ by construction (see, e.g., (28)). Therefore, $\sum_{j\in\mathcal{N}}(\hat{\alpha}_j h_j(x_j) + \hat{\beta}_j) = \sum_{j\in\mathcal{N}} p_j = 1$, which implies (29).

Next, we verify that x_i solves the maximization problem (30). For agent $i \in \mathcal{N}$ with $x_i = 0$, it is evident that choosing $x_i = 0$ dominates $x_i > 0$ under $(\widehat{\boldsymbol{\alpha}}, \widehat{\boldsymbol{\beta}}, 1)$ because $\hat{\alpha}_i = 0$.

For agent $i \in \mathcal{N}$ with $x_i > 0$, by (28), we have that

$$c_i'(x_i)\frac{\hat{\alpha}_i h_i(x_i) + \hat{\beta}_i}{\hat{\alpha}_i h_i'(x_i)} = c_i'(x_i)\frac{h_i(x_i) + \hat{\theta}_i}{h_i'(x_i)} = p_i(1 - p_i),$$

which is exactly the first-order condition for the maximization problem (30).

The above analysis shows that the optimum can be achieved by $k = 1$, in which case the game reduces to a standard static contest. By Theorem 2 in Fu and Wu (2020), the optimum can be achieved by choosing multiplicative biases $\boldsymbol{\alpha}$ only and setting headstart $\boldsymbol{\beta}$ to zero if fixing $\boldsymbol{p}$, $\Lambda(\boldsymbol{x}, \boldsymbol{p})$ weakly increases with x_i for all $i \in \mathcal{N}$. ■

Derivation for Equilibria in Example 1

Proof. First, consider the case of $k = 1$. The game reduces to a static Tullock contest. Let $Y := \sum_{i \in \mathcal{N}} \eta_i x_i$. The equilibrium conditions can be derived as

$$Y = 1 - p_i,$$

from which we can solve for the equilibrium aggregate effort Y, the equilibrium recognition probabilities $\boldsymbol{p} = (p_1, p_2, p_3, p_4)$, and the equilibrium efforts $\boldsymbol{x} = (x_1, x_2, x_3, x_4)$ as follows:

$$Y = \frac{3}{4},$$

$$p_i = 1 - Y = \frac{1}{4}, \ \forall i \in \mathcal{N},$$

and

$$\boldsymbol{x} = Y\boldsymbol{p} \oslash \boldsymbol{\eta} = \left(\frac{3}{16}, \frac{15}{16}, \frac{15}{16}, \frac{15}{16}\right).$$

Next, consider the case of $k = 2$. The equilibrium conditions in the proof of Theorem 1—i.e., conditions (14), (15), (16), (17), (18), and (19)—for this example can be expressed as follows:

$$Yp_i = \eta_i x_i,$$

$$\sum_{i \in \mathcal{N}} p_i = 1,$$

$$Y = (1 - p_i)(V_L + V^{\Delta}) - \mu_i V^{\Delta},$$

$$\mu_i = \frac{1}{\delta_i} - p_i - \frac{p_i V_L - Y p_i}{V^{\Delta}},$$

$$\sum_{i \in \mathcal{N}} \mu_i = 1,$$

$$\frac{1}{9}\left[p_1 V_L - \Upsilon p_1\right] + V_L + V^{\Delta} = 1.$$

It can be verified that $\boldsymbol{p} = (0.2322, 0.2559, 0.2559, 0.2559)$, $\boldsymbol{x} = (0.1711, 0.9433, 0.9433, 0.9433)$, $\boldsymbol{\mu} = (0.7678, 0.0774, 0.0774, 0.0774)$, $V_L = 0.9600$, and $V^{\Delta} = 0.0342$ constitute an SSPE of the game. The equilibria for the cases of $k = 3$ and $k = 4$ can be similarly verified. Numerical verification further confirms that, for this parameterization, the equilibrium outcome reported in Table 1 is unique for $k \in \{2, 3, 4\}$. ■

Derivation for the Optimal Recognition Mechanism in Example 2

Proof. We demonstrate the optimality of $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ in Example 2. When the designer sufficiently cares about the profile of agents' recognition probabilities—i.e., when $\lambda \gg 1/c$—the optimal equilibrium winning probability profile must be $\boldsymbol{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and the designer's payoff at $\boldsymbol{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ reduces to $\Lambda = x_1 + x_2 + x_3$. When c is sufficiently small, agent 3 is excessively strong and the designer's payoff is mainly determined by x_3. Therefore, it suffices to show that $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ maximizes x_3 among all rules $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ that induce $\boldsymbol{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Fix $\boldsymbol{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. We first rewrite the equilibrium conditions in the proof of Theorem 1—i.e., conditions (14)-(19). Evidently, condition (15) is satisfied and condition (14) becomes

$$\alpha_i^* x_i + \beta_i^* = \frac{\Upsilon}{3}, \ \forall\, i \in \{1, 2, 3\}. \tag{31}$$

Next, consider condition (16). The condition holds with equality for $x_i > 0$. Further, if $x_i = 0$ for some $i \in \mathcal{N}$ and the strict inequality holds, we can increase α_i until the equality holds and at the same time keep unchanged the equilibrium effort profile $\boldsymbol{x}$ and recognition probabilities $\boldsymbol{p}$. Therefore, we can assume that equality holds for all agents and the condition becomes

$$\frac{\Upsilon c_i}{\alpha_i^*} = \frac{2(V_L + V^{\Delta})}{3} - \mu_i V^{\Delta}, \ \forall\, i \in \{1, 2, 3\}. \tag{32}$$

Substituting (32) into (31) yields

$$3 c_i x_i \leq \frac{2(V_L + V^{\Delta})}{3} - \mu_i V^{\Delta}, \ \forall\, i \in \{1, 2, 3\}, \tag{33}$$

with equality holding if $\beta_i^* = 0$. To establish the optimality of headstarts, it suffices to show that the inequality is strict for at least one agent.

Conditions (17), (18), and (19) are

$$\mu_i = \begin{cases} \frac{2}{3} \le \frac{1}{\delta_i} - \frac{1}{3} - \frac{V_L}{3V^\Delta} + \frac{c_i x_i}{V^\Delta}, & i \in \mathcal{N}_1, \\ \frac{1}{\delta_i} - \frac{1}{3} - \frac{V_L}{3V^\Delta} + \frac{c_i x_i}{V^\Delta} \in \left[0, \frac{2}{3}\right], & i \in \mathcal{N}_2, \\ 0 \ge \frac{1}{\delta_i} - \frac{1}{3} - \frac{V_L}{3V^\Delta} + \frac{c_i x_i}{V^\Delta}, & i \in \mathcal{N}_3, \end{cases} \tag{34}$$

$$\mu_1 + \mu_2 + \mu_3 = 1, \tag{35}$$

and

$$\sum_{i \in \mathcal{N}_1} \frac{\delta_i}{1-\delta_i} \left(\frac{V_L}{3} - c_i x_i \right) + \left(2 - |\mathcal{N}_1|\right) V^\Delta + V_L = 1. \tag{36}$$

Substituting (34) in (33) yields that

$$c_i x_i \le \begin{cases} \frac{2}{9} V_L, & i \in \mathcal{N}_1, \\ \frac{1}{4}\left[V_L - (\frac{1}{\delta_i} - 1) V^\Delta \right], & i \in \mathcal{N}_2, \\ \frac{2}{9}(V_L + V^\Delta), & i \in \mathcal{N}_3, \end{cases} \tag{37}$$

from which we can conclude $c_i x_i \le \frac{2V_L}{9}$ for $i \in \mathcal{N}_1$; together with (36), we have that

$$\sum_{i \in \mathcal{N}_1} \frac{\delta_i}{1-\delta_i} \times \frac{V_L}{9} + \left(2 - |\mathcal{N}_1|\right) V^\Delta + V_L \le 1. \tag{38}$$

In what follows, we will show that $c_3 x_3 \le \frac{30}{144-\delta_3}$, and the equality holds if and only if $\boldsymbol{\alpha}^* = (\frac{62Y}{35}, \frac{62Y}{37}, \frac{62Yc}{39})$ and $\boldsymbol{\beta}^* = (0, \frac{17Y}{222}, 0)$. Consider the following three cases.

Case I: $3 \in \mathcal{N}_1$. Note that $|\mathcal{N}_1| \le k - 1 = 1$, we have that $\mathcal{N}_1 = \{3\}$. By (38), we can obtain that

$$\left[1 + \frac{\delta_3}{9(1-\delta_3)} \right] V_L + V^\Delta \le 1;$$

together with (36), we can obtain that

$$c_3 x_3 \le \frac{2V_L}{9} \le \frac{2(1-\delta_3)}{9 - 8\delta_3} < \frac{30}{144 - \delta_3}.$$

Case II: $3 \in \mathcal{N}_2$. By (34) and (37), we have that

$$0 \le \frac{1}{\delta_3} - \frac{1}{3} - \frac{V_L}{3V^\Delta} + \frac{c_3 x_3}{V^\Delta} \le \frac{1}{\delta_3} - \frac{1}{3} - \frac{V_L}{3V^\Delta} + \frac{V_L - (\frac{1}{\delta_3} - 1)V^\Delta}{4V^\Delta}.$$

Carrying out the algebra, we can obtain that

$$V_L \le \left(\frac{9}{\delta_3} - 1\right) V^\Delta = \frac{35}{4} V^\Delta. \tag{39}$$

Further, $3 \notin \mathcal{N}_1$ implies that $\mathcal{N}_1 \in \left\{\{1\},\{2\},\emptyset\right\}$, and thus (38) becomes

$$1 \ge \left\{ \begin{array}{ll} \frac{16}{15} V_L + V^\Delta, & \text{if } \mathcal{N}_1 = \{1\} \\ \frac{10}{9} V_L + V^\Delta, & \text{if } \mathcal{N}_1 = \{2\} \\ V_L + 2V^\Delta, & \text{if } \mathcal{N}_1 = \emptyset \end{array} \right\} \ge \frac{16}{15} V_L + V^\Delta, \tag{40}$$

where the last inequality follows from (39).

Combining (37), (39), and (40), we have that

$$c_3 x_3 \le \frac{1}{4}\left[V_L - (\frac{1}{\delta_3} - 1)V^\Delta\right] \le \frac{30}{144 - \delta_3} = \frac{13}{62}.$$

Note that equality holds in condition (37) if and only if $\beta_3^* = 0$. Further, equality holds in condition (39) only if $\mu_3 = 0$. Last, equality holds in condition (40) if and only if $\mathcal{N}_1 = \{1\}$ and $\beta_1^* = 0$.

Because $\mathcal{N}_1 = \{1\}$ and $\mu_3 = 0$, we have that $\mu_1 = \frac{2}{3}$ from (34); together with (35), we have $\mu_2 = \frac{1}{3}$. Moreover, by (34), we can conclude $2 \in \mathcal{N}_2$, which implies that $\mathcal{N}_2 = \{2,3\}$ and $\mathcal{N}_3 = \emptyset$.

Combining (39) and (40) (recall that equality holds in these two conditions), we can obtain $V_L = \frac{105}{124}$ and $V^\Delta = \frac{3}{31}$; together with (37), we have $x_1 = \frac{2V_L}{9} = \frac{35}{186}$. Substituting $\mu_2 = \frac{1}{3}$, $V_L = \frac{105}{124}$ and $V^\Delta = \frac{3}{31}$ in (34), we can obtain that $x_2 = \frac{V_L - 4V^\Delta}{3} = \frac{19}{124}$.

Last, we solve for $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$. Recall that $\beta_i^* = 0$ for $i \in \{1,3\}$. Therefore, $\alpha_i^* = \frac{Y}{3x_i}$ from (31). For $i = 2$, we have $x_2 = \frac{19}{124}$. Further, by (32), we have $\frac{Y}{\alpha_2^*} = \frac{2V_L + V^\Delta}{3} = \frac{37}{62}$, which implies that $\alpha_2^* = \frac{62Y}{37}$; together with (31), we can conclude that $\beta_2^* = \frac{Y}{3} - \alpha_2^* x_2 = \frac{17Y}{222}$.

In summary, the equality holds in $c_3 x_3 \le \frac{30}{144 - \delta_3}$ if and only if $\boldsymbol{\alpha}^* = (\frac{62Y}{35}, \frac{62Y}{37}, \frac{62Yc}{39})$ and $\boldsymbol{\beta}^* = (0, \frac{17Y}{222}, 0)$, under which the equilibrium is $\boldsymbol{x} = (\frac{35}{186}, \frac{19}{124}, \frac{39}{186c})$, $\boldsymbol{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $\boldsymbol{\mu} = (\frac{2}{3}, \frac{1}{3}, 0)$, $V_L = \frac{105}{124}$, and $V^\Delta = \frac{3}{31}$.

Case III: $\mathbf{3 \in \mathcal{N}_3}$. Condition (34), together with the postulated $3 \in \mathcal{N}_3$, implies that $\mu_3 = 0$. Analogous to the derivation of (39), we can obtain that

$$V_L > \left(\frac{9}{\delta_3} - 1\right) V^\Delta = \frac{35}{4} V^\Delta. \tag{41}$$

First, suppose $\mathcal{N}_1 \neq \emptyset$. By (40), we have that

$$1 \geq \frac{16V_L}{15} + V^{\Delta}. \tag{42}$$

Combining (33), (41), and (42) yields that

$$c_3 x_3 \leq \frac{2(V_L + V^{\Delta})}{9} < \frac{30}{144 - \delta_3}.$$

Next, suppose $\mathcal{N}_1 = \emptyset$; together with $3 \in \mathcal{N}_3$ and $k = 2$, we can conclude that $\mathcal{N}_2 = \{1, 2\}$. It follows from (36) that

$$V_L + 2V^{\Delta} = 1. \tag{43}$$

Recall $\mu_3 = 0$. Combining (34), (35), and (37), we can obtain that

$$1 = \mu_1 + \mu_2 = \frac{1}{\delta_1} + \frac{1}{\delta_2} - \frac{2}{3} - \frac{2V_L}{3V^{\Delta}} + \frac{x_1 + x_2}{V^{\Delta}} \leq 4 - \frac{2V_L}{3V^{\Delta}} + \frac{V_L}{2V^{\Delta}} - \frac{2}{3},$$

which in turn implies that

$$V_L \leq 14V^{\Delta}. \tag{44}$$

Therefore,

$$c_3 x_3 \leq \frac{2(V_L + V^{\Delta})}{9} \leq \frac{5}{24} < \frac{30}{144 - \delta_3},$$

where the first inequality follows from (33) and the second inequality from (43) and (44).
■

Proof of Theorem 4

Proof. Plugging (11) into (12), we have that

$$\sum_{i \in \mathcal{N}_2} \left[\left(\frac{1}{\delta_i} - p_i^* \right) V^{\Delta} - p_i^* V_L + c_i(x_i^*) \right] + \sum_{i \in \mathcal{N}_1} (1 - p_i^*) = (k-1)V^{\Delta}. \tag{45}$$

Recall by (19), we have that

$$\sum_{i \in \mathcal{N}_1} \frac{\delta_i}{1 - \delta_i} \left[p_i^* V_L - c_i(x_i^*) \right] + \left(k - |\mathcal{N}_1| \right) V^{\Delta} + V_L = 1. \tag{46}$$

Note that holding fixed $(\boldsymbol{x}^*, \boldsymbol{p}^*)$, we can adjust the contest rule to satisfy the above two equilibrium conditions as the voting rule k varies, which yields a new pair (V^{Δ}, V_L). To prove the theorem, it remains to verify the following first-order condition under the less inclusive voting rule $k-1$ and the new pair (V^Δ, V_L):

$$\frac{c_i'(x_i^*)h_i(x_i^*)}{h_i'(x_i^*)} \le p_i^*(1-p_i^*)\left[V_L + V^\Delta - \frac{\mu_i}{1-p_i^*}V^\Delta\right], \forall\, i \in \mathcal{N}.$$

Evidently, it suffices to show that the effective prize spread, $V_L + V^\Delta - \frac{\mu_i}{1-p_i^*}V^\Delta$, is nonincreasing in k.

We treat k as a continuous variable. Clearly, μ_i, V^Δ, and V_L are all continuous in k. Moreover, for all but finitely many values of k, the sets $\mathcal{N}_1$, $\mathcal{N}_2$, and $\mathcal{N}_3$ remain unchanged in a neighborhood of k, which indicates that μ_i, V^Δ, and V_L are differentiable with respect to k. Therefore, it suffices to show that the derivative of the effective prize spread, $V_L + V^\Delta - \frac{\mu_i}{1-p_i^*}V^\Delta$, with respect to k is nonpositive whenever it is differentiable. Taking the derivatives of (45) and (46) with respect to k yields that

$$\frac{dV_L}{dk} = -\frac{(\mathcal{B}+\mathcal{D})V^\Delta}{\mathcal{A}\mathcal{D}+\mathcal{B}\mathcal{C}} \text{ and } \frac{dV^\Delta}{dk} = -\frac{(\mathcal{C}-\mathcal{A})V^\Delta}{\mathcal{A}\mathcal{D}+\mathcal{B}\mathcal{C}},$$

where

$$\begin{aligned}
\mathcal{A} :=& 1 + \sum_{i\in\mathcal{N}_1}\frac{p_i^*\delta_i}{1-\delta_i} > 0,\\
\mathcal{B} :=& k - |\mathcal{N}_1| > 0,\\
\mathcal{C} :=& \sum_{i\in\mathcal{N}_2} p_i^* > 0,\\
\mathcal{D} :=& \sum_{i\in\mathcal{N}_1}(1-p_i^*) + \sum_{i\in\mathcal{N}_2}\left(\frac{1}{\delta_i} - p_i^*\right) - (k-1) > 0.
\end{aligned}$$

Evidently, $\mathcal{A}\mathcal{D}+\mathcal{B}\mathcal{C} > 0$ and $\mathcal{B}+\mathcal{D} > 0$. Moreover, we have that

$$\mathcal{C}-\mathcal{A} = \sum_{i\in\mathcal{N}_2} p_i^* - 1 - \sum_{i\in\mathcal{N}_1}\frac{p_i^*\delta_i}{1-\delta_i} \le -\sum_{i\in\mathcal{N}_1}\frac{p_i^*\delta_i}{1-\delta_i} \le 0.$$

Therefore, we have that

$$\frac{dV_L}{dk} = -\frac{(\mathcal{B}+\mathcal{D})V^\Delta}{\mathcal{A}\mathcal{D}+\mathcal{B}\mathcal{C}} \le 0 \text{ and } \frac{dV^\Delta}{dk} = -\frac{(\mathcal{C}-\mathcal{A})V^\Delta}{\mathcal{A}\mathcal{D}+\mathcal{B}\mathcal{C}} \ge 0. \tag{47}$$

We consider the following three cases.

Case I: $i \in \mathcal{N}_1$. By (11), $\mu_i = 1 - p_i^*$ and the effective prize spread is V_L. We can then conclude from (47) that V_L is non-increasing in k.

Case II: $i \in \mathcal{N}_2$. By (11), the effective prize spread is

$$\begin{aligned} V_L + V^\Delta - \frac{\mu_i}{1-p_i^*}V^\Delta =& V_L + V^\Delta - \frac{V^\Delta}{\delta_i(1-p_i^*)} + \frac{p_i^*}{1-p_i^*}(V_L + V^\Delta) - c_i(x_i^*) \\ =& \frac{V_L}{1-p_i^*} - \frac{(1-\delta_i)V^\Delta}{\delta_i(1-p_i^*)} - c_i(x_i^*). \end{aligned}$$

Carrying out the algebra, we can obtain that

$$\frac{d}{dk}\left(V_L + V^\Delta - \frac{\mu_i}{1-p_i^*}V^\Delta\right) = \frac{1}{1-p_i^*} \times \frac{dV_L}{dk} - \frac{1-\delta_i}{\delta_i(1-p_i^*)} \times \frac{dV^\Delta}{dk} \leq 0.$$

Case III: $i \in \mathcal{N}_3$. By (11), $\mu_i = 0$ and the effective prize spread is $V_L + V^\Delta$. By (47), we have that

$$\frac{d(V_L + V^\Delta)}{dk} = -\frac{(\mathcal{B} + \mathcal{C} + \mathcal{D} - \mathcal{A})V^\Delta}{\mathcal{A}\mathcal{D} + \mathcal{B}\mathcal{C}}.$$

It remains to prove

$$\mathcal{B} + \mathcal{C} + \mathcal{D} - \mathcal{A} \geq 0.$$

Carrying out the algebra, we have that

$$\begin{aligned} \mathcal{B} + \mathcal{C} + \mathcal{D} - \mathcal{A} =& k - |\mathcal{N}_1| + \sum_{i\in\mathcal{N}_2} p_i^* + \sum_{i\in\mathcal{N}_1}(1-p_i^*) + \sum_{i\in\mathcal{N}_2}\left(\frac{1}{\delta_i} - p_i^*\right) - (k-1) - 1 - \sum_{i\in\mathcal{N}_1}\frac{p_i^*\delta_i}{1-\delta_i} \\ =& \sum_{i\in\mathcal{N}_2}\frac{1}{\delta_i} - \sum_{i\in\mathcal{N}_1}\frac{p_i^*}{1-\delta_i} \\ \geq& 2|\mathcal{N}_2| - 2\sum_{i\in\mathcal{N}_1} p_i^* \geq 0, \end{aligned}$$

where the first inequality follows from $\delta_i \leq \frac{1}{2}$. This concludes the proof. ■