Research Contest Design with Resource Allocation and Entry Fees*

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Abstract

This paper explores the design of an R&D contest by a sponsor who can charge entry fees and allocate a fixed amount of productive resources across firms. We characterize the respective optimal contests for two objectives: (i) maximizing total effort in the contest and (ii) maximizing the expected quality of the product provided by the winning firm. The baseline model assumes a uniform entry fee, while an extension allows the sponsor to set firm-dependent entry fees. We show that the optimal contest induces the entry of only the two most efficient firms. The resource allocation plan in the optimum may favor the initially more competent firm—and thus promote a national champion—instead of leveling the playing field, and the optimum depends on the nature of the R&D task. Our analysis sheds light on the role played by these instruments in shaping optimal research contests.

Keywords: Research Contest; Contest Design; Resource Allocation; Entry Fee.

JEL Classification Codes: C72, D72.

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1 Introduction

A wealth of significant scientific and technological advances have been motivated by prizes (see, e.g., Terwiesch and Ulrich, 2009). For instance, the Rainhill Trials and George Stephenson's success in 1819 launched the age of railway transportation, the Longitude Prize awarded by the British government in 1714 enabled safe and precise ship navigation, and the first solo transatlantic flight was the result of winning the Orteig Prize in 1919. Inducement prize contests—which offer prizes to elicit efforts to achieve defined goals—are increasingly being recognized in the modern economic landscape as a cost-effective and efficient mechanism to procure technological solutions for specific needs, promote innovative research of scientific significance, or encourage entrepreneurial efforts toward socially valuable goals. The X Prize Foundation (XPRIZE), for instance, has sponsored numerous high-profile public innovation challenges "to encourage technological development to benefit humanity," and the U.S. government created an online platform, Challenge.gov, to facilitate the use of contest protocols and match federal agencies' needs with public innovators. The Department of Defense (DoD) engages in private R&D for defense technologies using competitive procurement exercises and awards contracts to private firms that develop prototypes of superior quality.

Due to their popularity and success, innovation contests have invigorated scholarly efforts to explore efficient ways to administer such competitions based on a wide range of perspectives and disciplines, from economics and operations management to information systems (see, e.g., Taylor, 1995; Fullerton and McAfee, 1999; Che and Gale, 2003; Terwiesch and Xu, 2008; Bimpikis, Ehsani, and Mostagir, 2019; Letina and Schmutzler, 2019; Benkert and Letina, 2020). This paper contributes to this research stream by analyzing the optimal design of an innovation contest. In particular, we arm the sponsor of a contest—who aims to secure an innovative product—with two instruments in addition to the posted prize: (i) entry fees and (ii) allocation of productive resources. To the best of our knowledge, we are the first to study the optimal combination of these two instruments in designing R&D contests. Our analysis addresses two fundamental questions in the literature on contest design: (i) How many and which firms should be included in the competitions? and (ii) How should productive resources be allocated among firms?

The instruments studied in the paper are prevalent in practice. Entry fees, for instance, are required by a number of XPRIZE challenges—e.g., the Google Lunar XPRIZE and XPRIZE Carbon Removal—and the prizes offered by data science competitions organized by Kaggle are funded by entry fees. These contests not only reward winners with prizes, but also often provide participants with various resources that bolster their productivity. Entrants in Mozilla's Open Innovation Challenge, for instance, receive mentorship and are provided with Mozilla's development tools, and the IBM Watson AI XPRIZE enables participants to access IBM Watson's APIs. Furthermore, the DoD's Small Business Innovation Research Program not only rewards winners with procurement contracts, but also provides an "implicit subsidy" to selected private contractors to support their development efforts (Lichtenberg, 1990). The DARPA Robotics Challenge charges an entry fee, but also provides various resources that facilitate participating teams' development, such as access to DARPA's robotics lab and software.

These examples inspire us to explore and analyze the design of innovation contests to achieve the optimal mix of entry fees and allocation of productive resources.

Snapshot of the Model A sponsor seeks to procure an innovative product and organizes an R&D contest that attracts a pool of firms as potential entrants. Each firm pays a fee for entry in the contest, and the winner—i.e., the firm that offers the product of highest quality—is awarded a prize; e.g., a procurement contract. A firm, on entry, commits to its R&D effort to develop the product (i.e., a prototype); the quality of its product is a random variable, and higher effort more likely yields higher quality. Firms differ in their initial competence, which is measured by their marginal R&D effort costs.

The sponsor is endowed with an initial prize purse (monetary resources) and a fixed amount of (nonmonetary) productive resources. The resources—e.g., the mentorship Mozilla provides development teams or access to DAPRA's robotics lab—improve the productivity of the recipient, such that a firm is more likely to achieve a higher-quality product when receiving a larger amount of resources. The sponsor distributes the resources among the firms to meet her goal. The game thus proceeds in two stages. The sponsor sets and announces the contest rule to firms in the first stage, and firms commit to their entry and effort choice in the second stage. In particular, the contest rule consists of three elements: (i) a prize for the winner, (ii) a fee required for entry, and (iii) an allocation profile of the productive resources. The sponsor sets the rule optimally in anticipation of firms' responses—i.e., their entry and effort decisions. The prize must be fully funded by her initial prize purse plus entry fees.

We consider two scenarios for contest design that accommodate the sponsor's possibly diverse preferences. Specifically, she sets the contest rule to maximize either (i) the total effort of the contest (effort-maximizing contest) or (ii) the expected quality of the winning product (quality-maximizing contest). The first objective is commonly assumed in the literature on contest design. Imagine a nonprofit organization—e.g., the XPrize Foundation—that aims to rally social effort toward or stimulate ideas about a fundamental challenge, such as rainforest conservation or decarbonization, in which case the first objective tends to be more relevant. Alternatively, imagine a pharmaceutical company that seeks a cost-efficient solution to synthesize an ingredient in its drugs, in which case the second objective would presumably apply.

Entry fee and resource allocation, as design instruments, can play multiple roles that intertwine and trigger subtle trade-offs. For instance, an entry fee may deter firms from entering the competition. This allows the sponsor to select proper candidates and, at the same time, limit competition. However, the sponsor raises revenues through entry fees, which enables her to increase the prize for the winner and boost the incentives for participants. When allocating productive resources among participants, the sponsor not only improves their productivity, but also varies firms' relative competitiveness in the contest. This manipulates the competitive balance of the competition and affects each firm's payoff in the contest—which, in turn, determines its willingness to participate.

Summary of the Results We begin with a baseline model in which the sponsor imposes a uniform entry fee on all participating firms. We show that the optimal contest always involves only two active firms regardless of the sponsor's objective, with the two most competent firms entering the contest. When the sponsor aims to maximize the total effort of the contest, the resource allocation plan fully levels the playing field, such that the two firms win with equal probability (Proposition 1). That is, the initially weaker firm is prioritized for resource allocation, which closes the gap between firms' levels of competence and creates an ex post even race. The result thus reflects the conventional wisdom in the contest literature of leveling the playing field. In contrast, when the sponsor is concerned about the expected quality of the winning product, she may promote a "national champion": The initially more competent firm receives more resources, which enlarges the gap in competitiveness and results in a more lopsided competition (Proposition 2). We characterize the conditions under which the optimal contest would embrace a national champion instead of an even race (Corollary 1). The optimum depends on the nature of the R&D task and the degree of heterogeneity of the two most competent firms.

The two objectives are not aligned and entail different trade-offs. The distribution of a firm's product quality depends not only on its effort, but also the resources it receives. More specifically, a firm's effort and the resources available to it are complementary to each other in producing a high-quality submission. As a result, the sponsor is compelled to spend more resources on the more competitive firm when she is concerned about maximizing the winning product's quality: The more competitive firm bears a lower marginal effort cost and presumably expends a higher effort, so an allocation plan that prioritizes the initial favorite improves the allocative efficiency of the contest. However, this further upsets the competitive balance of the contest and tends to soften the competition, as the conventional wisdom of the contest literature would predict. The former concern—i.e., allocative efficiency—does not arise when the sponsor maximizes total effort, so the latter prevails and leads to the usual prediction of leveling the playing field. The quality-maximizing contest must reconcile the trade-off, and thus a level playing field can be suboptimal.

Notably, our result of minimum entry—i.e., two firms—echoes that of the seminal work of Fullerton and McAfee (1999). They demonstrate that a contest of minimum entry can be optimal when the firms' cost profile meet certain conditions. We establish the optimality of minimum entry without restrictions on firms' cost structures.[1]

[1] Fullerton and McAfee (1999) allow the sponsor to use entry fees, but the model does not involve resource allocation that varies firms' productivity and the competitive balance of the contest. In their model, quality maximization and effort maximization perfectly coincide, as the total effort determines the distribution of the quality of the winning

Our analysis sheds light on the role played by entry fees. Suppose, for instance, that the sponsor is unable to collect entry fees and can only allocate productive resources. The effort-maximizing contest should always involve at least three active firms whenever possible, as shown in previous literature—e.g., Franke, Kanzow, Leininger, and Schwartz (2013) and Fu and Wu (2020). We also demonstrate that without entry fees, a national champion is more likely to result when the sponsor is concerned about the expected quality of the winning product. Put differently, the ability to collect entry fees and use the revenue to supplement the prize purse enables the sponsor to create more balanced competition through resource allocation. This subtlety inspires us to further explore the role played by entry fees. We extend our model to allow for discriminatory entry fees, such that the sponsor may condition entry fees on firms' identities. The results and economic logic are presented and discussed in Section 4.

Related Literature Our paper contributes to the strand of literature that espouses the merit of limiting contestants' participation in a contest, such as Baye, Kovenock, and De Vries (1993); Fullerton and McAfee (1999); and Che and Gale (2003). Fullerton and McAfee (1999) suggest that a contest organizer strategically sets entry to filter entrants and show that a contest of two active contenders can be optimal when the cost profile of eligible firms meets certain conditions. As stated above, our paper reinstates their key result without requiring a particular cost structure.[2]

Our paper is naturally linked to the growing literature on contests with endogenous entry. A handful of papers assume that participation requires a fixed and exogenous entry cost—e.g., Boosey, Brookins, and Ryvkin (2020), Shelegia and Wilson (2022), Fu, Jiao, and Lu (2015), and Stouras, Hutchison-Krupat, and Chao (2022). We instead assume an endogenously set entry fee and that the revenues from fees are added to the prize purse, which puts our paper in the company of Fullerton and McAfee (1999), Taylor (1995), Moldovanu, Sela, and Shi (2012), Hammond, Liu, Lu, and Riyanto (2019), Liu, Lu, Wang, and Zhang (2018), Liu and Lu (2019, 2022), Fu and Lu (2010) and Jia and Sun (2021). None of these studies involve the allocation of productive resources among firms.[3,4]

In our setting, the resource allocation profile affects firms' relative competitiveness, which determines their payoffs in the contest and, in turn, contributes to their entry decisions. Our paper is thus closely related to the immense literature on contests with identity-dependent preferential treatment, such as Franke, Kanzow, Leininger, and Schwartz (2013, 2014); Drugov and Ryvkin

product. In contrast, our setting, by allowing for resource allocation, causes the two objectives to diverge.

[2]Terwiesch and Xu (2008), in contrast, contend that broader participation allows the contest organizer to secure more diverse solutions for the problem she aims to tackle. Boudreau, Lacetera, and Lakhani (2011) demonstrate empirically that the proper number of participants depends on the nature of the underlying research problem, as well as the uncertainty it entails.

[3]Azmat and Möller (2009, 2018) allow each contestant to choose which contest to enter when multiple contests are available.

[4]Instead of the explicit decision of entry, Lemus and Marshall (2021) examine empirically a dynamic contest setting in which contestants decide whether to continue their participation.

(2017); Franke, Leininger, and Wasser (2018); and Fu and Wu (2020). These studies typically view the identity-dependent biases imposed on contestants' effort entry as a nominal scoring rule. In contrast, the resources allocated to a firm not only increase its relative competitiveness but also its actual output. A handful of studies—e.g., Fu, Lu, and Lu (2012), Deng, Fu, and Wu (2021), and Gao, Fan, Huang, and Chen (2022)—address a similar productive resource allocation problem.[5] However, the contest design in their settings does not involve entry fees. The resource allocation profile set by the designer determines the mapping of firms' efforts to the probabilities of their winning the contest. This links our paper to the studies that endogenize the winning probability specification of a contest (i.e., contest success function) through optimal contest design. Letina, Liu, and Netzer (2023), for instance, let a designer decide how to allocate prizes based on the noisy signals of contestants' efforts and find that the optimum boils down to a nested Tullock contest.

In our setting, the entry fee is a source of revenue to fund the prize purse, which also links our study to the extensive literature on the optimal prize structure in contests, such as Moldovanu and Sela (2001); Kalra and Shi (2001); Terwiesch and Xu (2008); Ales, Cho, and Körpeoğlu (2017); and Fu, Wang, and Wu (2021). Fu, Lu, and Lu (2012) also examine contest design that involves both prize setting and resource allocation. However, they assume a two-firm contest without endogenous entry. Further, they assume that the sponsor is subject to a single budget constraint on monetary resources; the sponsor splits the budget between a prize for a winner and subsidies provided to respective firms.

2 The Model

A sponsor organizes an R&D contest to acquire an innovative product. The contest posts a prize of a value $V > 0$—e.g., a procurement contract—for the winner. A pool of $n \geq 2$ firms are interested in the competition. A firm's participation incurs an exogenous fixed cost $\gamma > 0$—e.g., the costs of preparation for the project and forgone revenues from alternative engagement. Meanwhile, the contest charges a fee $\phi \geq 0$ to each entrant. Each firm $i \in \mathcal{N} := \{1, \ldots, n\}$, on entry, commits to its effort $x_i > 0$ to develop the product sought by the sponsor. In the case in which a firm i chooses to opt out of the research contest, we set $x_i = 0$. The firm that submits the highest-quality product wins. A firm i bears a constant marginal effort cost $c_i > 0$. Assume without loss of generality that firms are ordered such that $c_1 \leq \cdots \leq c_n$, with a lower marginal cost to imply a greater level of innate ability.

The sponsor is endowed with an initial (monetary) prize purse $b > 0$ and a fixed sum of (nonmonetary) productive resources, which we normalize to unity. The sponsor may fund the prize with the proceeds from entry fees. Further, she splits her endowed productive resources—

[5] Relatedly, He, Ke, and Zhang (2022) consider a horizontally differentiated market and investigate how firms should allocate their marketing effort across different products to maximize profits.

e.g., access to equipment, laboratory facilities, and computing infrastructure—and allocates the resources among participating firms to improve their productivity.

R&D Contest The winner is selected through a standard "best of simultaneous submissions" R&D contest, modeled as an extension of the classic research contest framework proposed by Fullerton and McAfee (1999). The quality q_i of a firm i's product is randomly drawn from a distribution with cumulative distribution function (CDF) $[F(q_i)]^{\alpha_i x_i^r}$, with $r \in (0,1]$, where $\alpha_i \geq 0$ is the amount of productive resource firm i receives from the sponsor and $F(\cdot)$ is a continuous CDF on a support $\left[\underline{q}, \overline{q}\right]$. The term $\alpha_i x_i^r$ can intuitively be interpreted as the number of effective trials or draws of ideas, with the quality of each trial or draw following the distribution $F(\cdot)$. The firm simply presents the output of the most successful trial or draw—with a quality q_i—as its submission to the contest.

A larger $\alpha_i x_i^r$ implies that a higher q_i is more likely to be realized and firm i is more likely to leapfrog its opponents. The resource α_i can presumably be viewed as a capital input that improves the firm's efficiency—e.g., access to computing facilities. The effort x_i can conveniently be interpreted as a labor input sunk by the firm—e.g., the time, energy, and intellectual resources dedicated to the project. We assume the term $\alpha_i x_i^r$ to be concave—i.e., $r \leq 1$—which describes a development process with diminishing marginal returns: To put this intuitively, doubling input cannot more than double the likelihood of a breakthrough.

If no firm enters, the contest is cancelled. If only one firm enters, the entrant automatically wins the prize. Otherwise, by Fullerton and McAfee (1999) and Baye and Hoppe (2003), fixing an effort profile $\boldsymbol{x} := (x_1, \ldots, x_n)$, with $\sum_{j=1}^n \alpha_j \cdot x_j^r > 0$, each firm $i \in \mathcal{N}$ wins the contest with a probability[6,7]

$$p_i(\boldsymbol{x}) := \Pr\left(q_i > \max_{j \neq i} q_j\right) = \frac{\alpha_i \cdot x_i^r}{\sum_{j=1}^n \alpha_j \cdot x_j^r}. \tag{1}$$

Contest Design Prior to the contest, the sponsor sets and publicly announces the contest rule, which is described by a triple $(V, \phi, \boldsymbol{\alpha})$: (i) the posted prize value $V > 0$; (ii) the entry fee $\phi \geq 0$; and (iii) a resource allocation profile $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_n) \geq (0, \ldots, 0)$. We assume that the entry fee is *identical* for all firms and will relax this assumption in Section 4. The sponsor anticipates firms' responses to $(V, \phi, \boldsymbol{\alpha})$ and sets the contest rule accordingly.

The sponsor is subject to two prevailing budget constraints. First, the productive resources she can provide to the firms are limited, and we normalize the budget constraint to $\sum_{i \in \mathcal{N}} \alpha_i = 1$. Obviously, the sponsor is allowed to strategically choose the set of firms she intends to include in

[6] Another micro-foundation to obtain this success function is the following: Firm $i \in \mathcal{N}$ has a production technology in the form of $f_i(x_i) = \alpha_i \cdot x_i^r$, where α_i is the resource allocated to firm i. The sponsor receives a noisy signal s_i of firm i's performance or output, with $\log s_i = \log f_i(x_i) + \epsilon_i$, where ϵ_i follows a type I extreme-value distribution (i.e., Gumbel distribution). The prize is awarded to the firm with the highest signal. See Fu and Lu (2012) for more details.

[7] In the case of $\sum_{j=1}^n \alpha_j \cdot x_j^r = 0$, we let $p_i(\boldsymbol{x}) = 1/|\{j \in \mathcal{N} | x_j > 0\}|$ if $x_i > 0$.

the competition: She can effectively exclude a firm by providing it with zero or a sufficiently small amount of resources. Second, the value of the posted prize, V, is bounded by her initial endowment b and the proceeds collected through entry fees. Denote by $k(V, \phi, \boldsymbol{\alpha})$ the number of entrants in the equilibrium for a given contest rule $(V, \phi, \boldsymbol{\alpha})$. The budget constraint for the prize purse is thus $V \leq b + k(V, \phi, \boldsymbol{\alpha})\phi$.

The sponsor can have two design objectives. She may intend to promote technological efforts for socially valuable missions—e.g., an XPRIZE challenge to discover clean and renewable energy in response to climate change. The sponsor, under such a circumstance, aims to maximize firms' total effort, which is given by

$$Z^* := \sum\nolimits_{i\in\mathcal{N}} x_i. \tag{2}$$

Alternatively, the sponsor can be concerned about the quality of the winning product—e.g., when the DoD procures military equipment from private contractors. Denote by $q_{max} = \max\{q_1, \ldots, q_n\}$ the quality of the winning product. For a given effort profile $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$, q_{max} is the first-order statistic of the quality of firms' submissions q_i, which follows a distribution with CDF $\big[F(q_{max})\big]^{\sum_{i=1}^n \alpha_i \cdot x_i^r}$. The sponsor thus sets $(V, \phi, \boldsymbol{\alpha})$ to maximize

$$Z^\star := \sum\nolimits_{i\in\mathcal{N}} \alpha_i \cdot x_i^r. \tag{3}$$

Timeline and Payoff The game proceeds in two stages. In the first, the sponsor commits to and announces the contest rule $(V, \phi, \boldsymbol{\alpha})$. The contest takes place in the second stage. Firms observe $(V, \phi, \boldsymbol{\alpha})$ and simultaneously make their entry and effort decisions.

For a given effort profile $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$, a firm i's expected payoff in a contest $(V, \phi, \boldsymbol{\alpha})$ is given by

$$\pi_i(\boldsymbol{x}; V, \phi, \boldsymbol{\alpha}) = \begin{cases} p_i(\boldsymbol{x}) \cdot V - c_i x_i - \phi - \gamma, & \text{if } x_i > 0, \\ 0, & \text{if } x_i = 0, \end{cases}$$

where $p_i(\boldsymbol{x})$ is defined in (1).

3 Analysis

We now characterize the optimal contest. We focus on pure-strategy equilibrium in the second-stage contest game. For simplicity, we assume that (i) the sponsor is restricted to choosing from the set of contest rules $(V, \phi, \boldsymbol{\alpha})$ under which an equilibrium exists and (ii) the equilibrium most favorable to the sponsor is selected when multiple equilibria exist.[8] It is noteworthy, though, that the prevailing equilibrium selection criterion and restrictions on choice set do not affect our

[8] In Appendix B, we provide two examples in which an entry fee may cause the nonexistence or multiplicity of pure-strategy equilibria.

prediction: We will verify later that the optimum can always be achieved by a contest rule that induces a unique pure-strategy equilibrium (see Footnote 13).

The second-stage contest game, in general, does not yield a closed-form equilibrium solution. This nullifies the traditional implicit programming approach to optimal contest design, which requires a closed-form equilibrium solution for every possible contest rule (see, e.g., Franke, Kanzow, Leininger, and Schwartz, 2013, 2014; Franke, Leininger, and Wasser, 2018). We adopt the technique developed by Fu and Wu (2020) and Deng, Fu, and Wu (2021) to circumvent the challenge and solve for the optimum.

Specifically, we do not explicitly solve for the equilibrium effort profile $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$ of the contest game. Instead, we reformulate the optimization problem by treating the equilibrium winning probabilities $\boldsymbol{p} \equiv (p_1, \ldots, p_n)$ as the design variables, which are implicitly determined by the prevailing contest rule $(V, \phi, \boldsymbol{\alpha})$ and resultant equilibrium effort profile. We search for the equilibrium winning probabilities $\boldsymbol{p}$ associated with the optimal contest rule $(V, \phi, \boldsymbol{\alpha})$ and recover the optimum from equilibrium conditions.

With simple algebraic transformation, the first-order condition $\partial \pi_i (\boldsymbol{x}; V, \phi, \boldsymbol{\alpha})/\partial x_i = 0$ for a firm $i \in \mathcal{N}$ that chooses to exert a strictly positive effort can be expressed as follows:

$$x_i = r p_i(\boldsymbol{x}) \left[1 - p_i(\boldsymbol{x})\right] \times \frac{V}{c_i}. \tag{4}$$

Note that a firm stands zero chance of winning the contest if it exerts zero effort. As a result, the above condition continues to hold for a firm that opts out of the contest.

Denote by $\mathcal{N}_+(\boldsymbol{p})$ and $k(\boldsymbol{p})$, respectively, the set and number of firms with strictly positive equilibrium winning probabilities:

$$\mathcal{N}_+(\boldsymbol{p}) := \left\{ i = 1, \ldots, n \,|\, p_i > 0 \right\} \tag{5}$$

and

$$k(\boldsymbol{p}) := \left| \mathcal{N}_+(\boldsymbol{p}) \right|. \tag{6}$$

The following lemma establishes a correspondence between firms' equilibrium winning probabilities $\boldsymbol{p} \equiv (p_1, \ldots, p_n)$ and the resource allocation profile $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$.

Lemma 1 *Consider a second-stage contest and ignore for now firms' participation constraints.*[9] *Any profile of the equilibrium winning probabilities* $\boldsymbol{p} \equiv (p_1, \ldots, p_n) \in \Delta^{n-1}$*, with* $p_i \neq 1$ *for all*

[9] Or equivalently, consider a second-stage research contest, with $\phi = \gamma = 0$.

$i \in \mathcal{N}$, can be induced by the following resource allocation profile $\boldsymbol{\alpha}(\boldsymbol{p}) \equiv \big(\alpha_1(\boldsymbol{p}), \ldots, \alpha_n(\boldsymbol{p})\big)$:[10]

$$\alpha_i(\boldsymbol{p}) = \begin{cases} \frac{c_i^r p_i^{1-r}}{(1-p_i)^r} \times \frac{1}{\eta(\boldsymbol{p})}, & \text{if } p_i > 0, \\ 0, & \text{if } p_i = 0, \end{cases}$$

where $\eta(\boldsymbol{p}) := \sum_{j \in \mathcal{N}_+(\boldsymbol{p})} \frac{c_j^r p_j^{1-r}}{(1-p_j)^r}$.

Before we proceed, we impose the following regularity condition throughout the paper.

Assumption 1 ***(Sufficient Budget for Sponsor)*** *$b > \max\{\underline{b}^*, \underline{b}^\star\}$, where $\underline{b}^*$ and $\underline{b}^\star$ are to be defined later in (13) and (15), respectively.*

Intuitively, this condition ensures that the sponsor has adequate budget b to attract the participation of the set of firms she desires for all possible scenarios we will consider in the subsequent analysis.[11]

We are now ready to characterize the optimal contests for total effort maximization and quality maximization, respectively.

3.1 Effort-maximizing Contests

Suppose that the sponsor aims to maximize the total effort of the contest. The correspondence established by Lemma 1 allows us to reformulate the optimization problem and treat the distribution of *equilibrium* winning probabilities $\boldsymbol{p}$ as the design variable. Instead of searching for the optimal $(V, \phi, \boldsymbol{\alpha})$, the sponsor literally chooses $(V, \phi, \boldsymbol{p})$ to maximize Z^* as specified in (2), i.e.,

$$Z^* \equiv \sum_{i \in \mathcal{N}} x_i = \sum_{i \in \mathcal{N}} \left[r p_i (1 - p_i) \frac{V}{c_i} \right],$$

subject to the following constraints:

$$\sum_{i \in \mathcal{N}} p_i = 1, \text{ and } p_i \geq 0, \text{ for all } i \in \mathcal{N}, \tag{7}$$

$$\min_{i \in \mathcal{N}_+(\boldsymbol{p})} \Big\{ p_i \big[1 - r(1 - p_i)\big] \times V \Big\} \geq \phi + \gamma, \tag{8}$$

and

$$V - k(\boldsymbol{p})\phi = b, \tag{9}$$

where $\mathcal{N}_+(\boldsymbol{p})$ and $k(\boldsymbol{p})$ are defined in (5) and (6), respectively. Constraint (7) simply requires that firms' winning probabilities be nonnegative and sum to one; (8) is the participation constraint

[10] It is straightforward to verify that $p_i = 1$ cannot arise in the equilibrium.

[11] Alternatively, we can assume that a firm's exogenous fixed cost of entry $\gamma > 0$ is sufficiently small.

for an active firm;[12] and (9) ensures budget balance, which requires that the prize be sufficiently funded by the sponsor's initial prize purse b and the revenues of entry fees, $k(\boldsymbol{p})\phi$, collected from the $k(\boldsymbol{p})$ entrants.

Note that constraint (8) must bind in the optimal R&D contest. The sponsor can otherwise increase V and ϕ simultaneously—while holding fixed $\boldsymbol{p} \equiv (p_1, \ldots, p_n)$—which improves her payoff without violating constraints (8) and (9). As a result, for a given profile of equilibrium winning probabilities $\boldsymbol{p} \equiv (p_1, \ldots, p_n)$ the sponsor intends to induce, she sets the entry fee such that

$$\phi = \min_{i \in \mathcal{N}_+(\boldsymbol{p})} \Big\{ p_i \left[1 - r(1 - p_i) \right] \Big\} \times V - \gamma. \tag{10}$$

Combining (9) and (10) yields

$$V = \frac{b - k(\boldsymbol{p})\gamma}{1 - k(\boldsymbol{p}) \min_{i \in \mathcal{N}_+(\boldsymbol{p})} \Big\{ p_i \left[1 - r(1 - p_i) \right] \Big\}}, \tag{11}$$

and

$$\phi = \frac{b \min_{i \in \mathcal{N}_+(\boldsymbol{p})} \Big\{ p_i \left[1 - r(1 - p_i) \right] \Big\} - \gamma}{1 - k(\boldsymbol{p}) \min_{i \in \mathcal{N}_+(\boldsymbol{p})} \Big\{ p_i \left[1 - r(1 - p_i) \right] \Big\}}. \tag{12}$$

For a given cost profile $\boldsymbol{c} \equiv (c_1, \ldots, c_n)$, the sponsor's optimization problem can be simplified as the following:

$$\max_{\boldsymbol{p} \in \Delta^{n-1},\, k(\boldsymbol{p}) \geq 2} \mathcal{W}(\boldsymbol{p}, \boldsymbol{c}) := \left(\sum_{i \in \mathcal{N}} \frac{r p_i (1 - p_i)}{c_i} \right) \times \frac{b - k(\boldsymbol{p})\gamma}{1 - k(\boldsymbol{p}) \min_{i \in \mathcal{N}_+(\boldsymbol{p})} \Big\{ p_i \left[1 - r(1 - p_i) \right] \Big\}},$$

where Δ^{n-1} is an $(n-1)$-dimensional simplex as defined by (7).

Denote by $\boldsymbol{p}^* \equiv (p_1^*, \ldots, p_n^*)$ the effort-maximizing equilibrium winning probabilities and let $k^* := \big| \mathcal{N}_+(\boldsymbol{p}^*) \} \big|$. Further define

$$\underline{b}^* := \frac{4\gamma}{2 - r} > 0. \tag{13}$$

The following result can be obtained.

Proposition 1 ***(Effort-maximizing Research Contest with Uniform Entry Fees)*** *Suppose that the sponsor aims to maximize the total effort of the R&D contest and Assumption 1 is satisfied, i.e.,* $b > \underline{b}^*$. *The optimal contest* $(V^*, \phi^*, \boldsymbol{\alpha}^*)$ *is given by*

$$V^* = \frac{2b - 4\gamma}{r}, \phi^* = \frac{(2 - r)b}{2r} - \frac{2\gamma}{r},$$

[12] This can be implied by active firms' first-order conditions (4) in equilibrium.

$$\textit{and } \boldsymbol{\alpha}^* \equiv (\alpha_1^*, \alpha_2^*, \alpha_3^* \dots, \alpha_n^*) = \left(\frac{c_1^r}{c_1^r + c_2^r}, \frac{c_2^r}{c_1^r + c_2^r}, 0, \dots, 0 \right).$$

The contest induces a profile of equilibrium winning probabilities $\boldsymbol{p}^* \equiv (p_1^*, \dots, p_n^*) = (1/2, 1/2, 0, \dots, 0)$*: The two most competent firms enter the contest, with* $k^* = 2$ *and* $\mathcal{N}_+(\boldsymbol{p}^*) = \{1, 2\}$*.*[13]

By Proposition 1, it takes (exactly) two to tango: The optimal R&D contest involves the two most efficient firms. Furthermore, the optimal contest fully levels the playing field, such that firms 1 and 2 win with equal probability. To achieve this, the sponsor arms the less efficient firm—i.e., firm 2—with a larger amount of resources, with $\alpha_2^* = c_2^r/(c_1^r + c_2^r) \geq c_1^r/(c_1^r + c_2^r) = \alpha_1^*$.

Proposition 1 echoes Theorem 3 of Fullerton and McAfee (1999), which states that restricting entry to two competitors optimizes a research contest absent resource allocation. It is noteworthy that the result of Fullerton and McAfee (1999) imposes a restriction on the profile of firms' marginal costs $(c_1, \dots, c_n)$: They require that $\Delta_i = (i \cdot c_i)/\sum_{j=1}^n c_j$ be nondecreasing in $i \in \mathcal{N}$. This condition is violated whenever one firm is a (sufficiently) close competitor with another firm.[14] Proposition 1 demonstrates that the condition is no longer binding when the sponsor is able to allocate productive resources across participants. Our result thus complements Fullerton and McAfee (1999) and revives the optimality of bilateral competitions without restrictions on firms' cost profile.

Entry fee plays a critical role in the contest mechanism. Suppose otherwise that we exclude the entry fee from the sponsor's toolkit and let her optimize the contest with only the choice of $\boldsymbol{\alpha}$. The sponsor is unable to enlarge her prize purse, so she posts a prize $V = b$. The contest design problem is no different from the exercises of Franke, Kanzow, Leininger, and Schwartz (2013) and Fu and Wu (2020), who set identity-dependent multiplicative biases on contestants' output to maximize total effort. However, in our setting $\boldsymbol{\alpha}$ is interpreted as a profile of productive resource allocation instead of a nominal scoring rule. Their results show that at least *three firms* will be kept active in the optimum.

Two competing forces come into play when an entry fee is in place. On the one hand, an entry fee discourages participation, which weakens the competition and limits effort contribution. On the other hand, the revenue collected through entry fees can increase the prize purse and bolster the incentive provided to participating firms, which motivates their input. Proposition 1 demonstrates that the latter positive effect outweighs the former negative one: The contest provides a larger prize to incentivize entrants at the cost of limited participation, which leads to the result of two

[13]Note that there exists a unique pure-strategy equilibrium under the optimal contest rule $(V^*, \phi^*, \boldsymbol{\alpha}^*)$. To see this, note that firms 3 to n each receive zero resources and would choose to opt out in the equilibrium. Further, firms 1 and 2 must each exert a strictly positive effort in the equilibrium. To see this, suppose, to the contrary, that one firm chooses zero effort. Then the opponent's best response is not well defined: By exerting an infinitesimal effort, the firm is able to secure a sure win and a positive payoff close to $V^* - \phi^* - \gamma$, while exerting zero effort—i.e., opting out—yields zero payoff to the firm.

[14]See Dizdar (2021) for a detailed discussion of the monotonicity condition on $\{\Delta_i\}_{i=1}^n$ and the performance of a two-firm research contest when the condition is not satisfied.

active firms. The trade-off fades away when entry fees are disabled, which mutes both effects. The optimal contest thus involves broader participation, which requires at least three active firms whenever feasible.

3.2 Quality-maximizing Contests

Now suppose that the sponsor is concerned about the quality of the winning product. By Lemma 1 and Equation (4), we can rewrite $Z^\star$ defined in (3) as the following:

$$Z^\star \equiv \sum_{i \in \mathcal{N}} \alpha_i \cdot x_i^r = \sum_{i \in \mathcal{N}} \left[\alpha_i p_i^r (1-p_i)^r \frac{(rV)^r}{c_i^r} \right] = \frac{(rV)^r}{\eta(\boldsymbol{p})}, \tag{14}$$

where $\eta(\boldsymbol{p})$ is defined in Lemma 1.

The sponsor chooses $(V, \phi, \boldsymbol{p})$ to maximize (14), subject to constraints (7), (8), and (9). Denote by $\boldsymbol{p}^\star \equiv (p_1^\star, \ldots, p_n^\star)$ the equilibrium winning probabilities in the quality-maximizing contest. Let $k^\star := \left| \mathcal{N}_+(\boldsymbol{p}^\star) \} \right|$ denote the number of active firms in the optimum. Further define

$$\underline{b}^\star := \frac{\gamma}{p_2^\star \left(1 - r p_1^\star\right)}, \tag{15}$$

where $(p_1^\star, p_2^\star)$ gives the solution to the optimization problem (16) laid out below. The following result ensues.

Proposition 2 ***(Quality-maximizing R&D Contest)*** *Suppose that the sponsor aims to maximize the expected quality of the winning product of the R&D contest and Assumption 1 is satisfied, i.e., $b > \underline{b}^\star$. The optimal contest induces $k^\star = 2$ entrants. The maximum expected quality of the winning product can be obtained by inducing $\boldsymbol{p}^\star \equiv (p_1^\star, \ldots, p_n^\star) = (p_1^\star, p_2^\star, 0, \ldots, 0)$, where $(p_1^\star, p_2^\star)$, with $p_1^\star \geq 1/2 \geq p_2^\star$, solves*

$$\min_{p_1+p_2=1,\, p_1 \geq p_2 > 0} \left(\frac{c_1^r p_1^{1-r}}{p_2^r} + \frac{c_2^r p_2^{1-r}}{p_1^r} \right) \times \left[1 - 2p_2\left(1 - r p_1\right)\right]^r. \tag{16}$$

The corresponding contest rule $(V^\star, \phi^\star, \boldsymbol{\alpha}^\star)$ is given by

$$V^\star = \frac{b - 2\gamma}{1 - 2p_2^\star(1 - r p_1^\star)},\ \phi^\star = \frac{b p_2^\star \left(1 - r p_1^\star\right) - \gamma}{1 - 2p_2^\star \left(1 - r p_1^\star\right)},$$

and

$$\boldsymbol{\alpha}^\star \equiv \left(\alpha_1^\star, \alpha_2^\star, \alpha_3^\star \ldots, \alpha_n^\star\right) = \left(\frac{\frac{c_1^r (p_1^\star)^{1-r}}{(p_2^\star)^r}}{\frac{c_1^r (p_1^\star)^{1-r}}{(p_2^\star)^r} + \frac{c_2^r (p_2^\star)^{1-r}}{(p_1^\star)^r}}, \frac{\frac{c_2^r (p_2^\star)^{1-r}}{(p_1^\star)^r}}{\frac{c_1^r (p_1^\star)^{1-r}}{(p_2^\star)^r} + \frac{c_2^r (p_2^\star)^{1-r}}{(p_1^\star)^r}}, 0, \ldots, 0 \right).$$

Similar to Proposition 1, Proposition 2 also predicts that the optimal contest involves only two firms and concentrates its resources on the two most efficient ones. Despite the similarity, quality maximization stands in sharp contrast to effort maximization in terms of the underlying trade-offs. Note that the productive resources $\boldsymbol{\alpha} \equiv (\alpha_1, \ldots, \alpha_n)$ do not directly affect the sponsor's payoff when she maximizes total effort. In contrast, $\boldsymbol{\alpha}$ directly enters the objective function (3) and generates intrinsic value to the sponsor in a quality-maximizing contest. Because of the *complementarity* between the resource α_i and a firm's input x_i, the sponsor must avoid spreading costly and scarce resources across less productive firms. This effect leads to a limited competition that involves only the two most efficient firms.

The sponsor, when allocating resources between heterogeneous firms, must strike a balance between two competing effects. Prioritizing the weaker firm fuels competition, which we call the *competition effect*. However, this undermines *allocative efficiency*, which requires that the resources be concentrated on the more competent firm, since resources and effort are complementary. Tension between the two concerns tilts the optimum away from its counterpart, which maximizes total effort and may even overturn the conventional wisdom by further upsetting the balance of the competition. A closer look at $(\alpha_1^\star, \alpha_2^\star)$ yields the following.

Corollary 1 ***(National Champion vs. Handicapping)*** *Suppose that $c_1 < c_2$. The following statements hold:*

(i) If $r \geq 1/2$, then $\alpha_1^\star < \alpha_2^\star$.

(ii) If $r < 1/2$, then there exists a threshold ℓ—which depends on r—such that $\alpha_1^\star \gtreqless \alpha_2^\star$ if and only if $c_2/c_1 \gtreqless \ell$.

The sponsor may create a national champion by prioritizing the more competent firm in resource allocation, which further upsets the competitive balance of the playing field. Figure 1 depicts the comparison between $\alpha_1^\star$ and $\alpha_2^\star$ in the optimum under various parameterizations. The horizontal axis measures the degree of heterogeneity between the two most efficient firms, $\log(c_2/c_1)$, and the vertical axis traces the value of the exponential term r.

Recall that $\alpha_i x_i^r$ is interpreted as the number of trials. The term r thus measures how effectively effort x_i can be converted into output and provides an intuitive account of the R&D task's technological nature. A smaller r intuitively alludes to a more challenging R&D task or a more strenuous R&D process, since a given input is less likely to deliver high-quality trials. For instance, a research project that aims for major scientific discovery—e.g., a universal flu vaccine—can be described by a very small r; in contrast, an effort to incrementally improve an engineering process involves less uncertainty and presumably alludes to a larger r. By Corollary 1 and Figure 1, $\alpha_1^\star > \alpha_2^\star$ when $r < 1/2$ and $c_2/c_1 > \ell$. That is, the optimal contest favors the more competent firm if and only if (i) the R&D process is sufficiently difficult and (ii) firms are substantially heterogeneous.

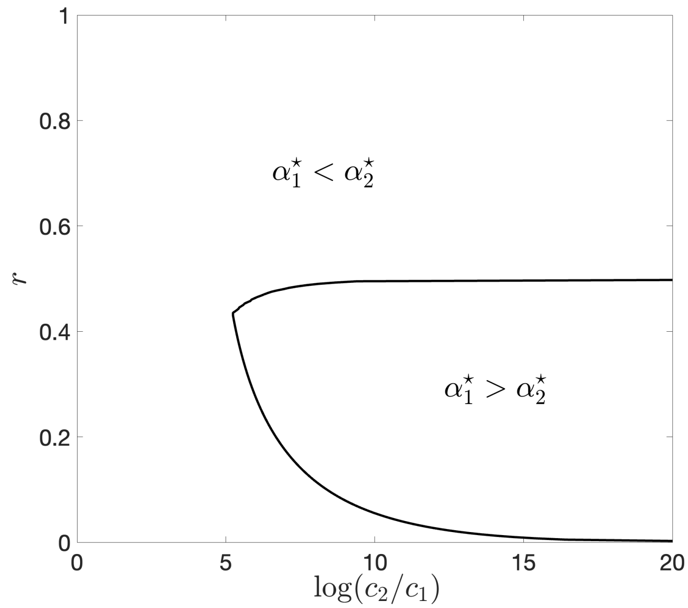

Figure 1: Quality-maximizing Resource Allocation Scheme.

First, the competition effect wanes when the difficulty of the task increases—i.e., with a smaller r: The additional incentive provided by a level playing field is diminished by the lower marginal return to effort, so a more even race incentivizes firms less effectively. As a result, $\alpha_1^\star < \alpha_2^\star$ may not hold when r falls below $1/2$, in which case the optimum also depends on the degree of heterogeneity between firms. Second, an increase in the degree of heterogeneity between firms—i.e., a larger c_2/c_1—magnifies the loss of allocative efficiency when assigning resources to the weaker firm, which further diminishes the appeal of a level playing field. As a result, the sponsor is compelled to cultivate a national champion—i.e., $\alpha_1^\star > \alpha_2^\star$—when the R&D process is sufficiently difficult and the degree of heterogeneity between firms is significant, i.e., $c_2/c_1 > \ell$.

Deng, Fu, and Wu (2021) consider a similar resource allocation problem in an R&D contest, but their setting does not involve the use of entry fees. They show that a national champion arises in the optimum whenever r falls below $1/2$. Comparing their results with Corollary 1 indicates that the presence of entry fees renders a national champion less likely: By Corollary 1, a national champion requires not only $r < 1/2$ but also $c_2/c_1 > \ell$. The sponsor's ability to charge an entry fee tilts the aforementioned trade-off in favor of the competition effect. Recall that the revenue collected through entry fees enlarges the prize purse. Firms can be motivated more effectively when attracted by a more generous prize, which amplifies the gain from a level playing field. Meanwhile, concern about allocative efficiency can be ameliorated because less efficient firm 2 contributes more effort when a larger prize is in place. We then observe that the entry fee catalyzes more even races in the optimum.

4 Discriminatory Entry Fees

We now relax the assumption of uniform entry fees and allow them to depend on firms' identities. Denote by $\phi_i \geq 0$ the entry fee imposed for a firm $i \in \mathcal{N}$ and let $\boldsymbol{\phi} := (\phi_1, \ldots, \phi_n)$. We first consider the optimal contest that maximizes total effort, then proceed to the case of quality maximization.

4.1 Effort-maximizing Contests

Similar to the analysis in Section 3.1, the optimal contest design problem can be reformulated as follows: The sponsor chooses $(V, \boldsymbol{\phi}, \boldsymbol{p})$ to maximize (2), subject to constraints (7),

$$p_i \left[1 - r(1 - p_i)\right] \times V \geq \phi_i + \gamma, \text{ for all } i \in \mathcal{N}_+(\boldsymbol{p}), \tag{17}$$

and

$$V - \sum\nolimits_{i \in \mathcal{N}_+(\boldsymbol{p})} \phi_i = b. \tag{18}$$

Similar to (8), (17) provides participation constraints for firms. Constraint (18) requires a binding budget constraint. Denote by $\widehat{\boldsymbol{p}}^* \equiv (\widehat{p}_1^*, \ldots, \widehat{p}_n^*)$ the equilibrium winning probabilities in the optimum and let $\widehat{k}^* := \left|\mathcal{N}_+(\widehat{\boldsymbol{p}}^*)\}\right|$. The following result can be obtained.

Proposition 3 ***(Effort-maximizing Contest with Discriminatory Entry Fees)*** *Suppose that the sponsor aims to maximize the total effort of the R&D contest and is allowed to impose discriminatory entry fees. Moreover, Assumption 1 is satisfied, i.e.,* $b > \underline{b}^*$*. The optimal contest involves* $\widehat{k}^* = 2$ *active firms and induces* $\widehat{\boldsymbol{p}}^* \equiv (\widehat{p}_1^*, \ldots, \widehat{p}_n^*) = (\widehat{p}_1^*, \widehat{p}_2^*, 0, \ldots, 0)$*, where* $(\widehat{p}_1^*, \widehat{p}_2^*) > (0, 0)$ *satisfies*

$$\widehat{p}_1^* + \widehat{p}_2^* = 1, \text{ and } b \times \min\nolimits_{i \in \{1,2\}} \left\{1 - r\widehat{p}_i^*\right\} - 2\gamma \geq 0. \tag{19}$$

The corresponding contest—which we denote by $\left(\widehat{V}^*, \widehat{\boldsymbol{\phi}}^*, \boldsymbol{\alpha}^*\right)$*—involves*

$$\widehat{V}^* = \frac{b - 2\gamma}{2r\widehat{p}_1^*\widehat{p}_2^*}, \widehat{\boldsymbol{\phi}}^* = \left(\frac{b\left(1 - r\widehat{p}_2^*\right) - 2\gamma}{2r\widehat{p}_2^*}, \frac{b\left(1 - r\widehat{p}_1^*\right) - 2\gamma}{2r\widehat{p}_1^*}, 0, \ldots, 0\right)$$

and

$$\widehat{\boldsymbol{\alpha}}^* \equiv \left(\widehat{\alpha}_1^*, \widehat{\alpha}_2^*, \widehat{\alpha}_3^* \ldots, \widehat{\alpha}_n^*\right) = \left(\frac{\frac{c_1^r(\widehat{p}_1^*)^{1-r}}{(\widehat{p}_2^*)^r}}{\frac{c_1^r(\widehat{p}_1^*)^{1-r}}{(\widehat{p}_2^*)^r} + \frac{c_2^r(\widehat{p}_2^*)^{1-r}}{(\widehat{p}_1^*)^r}}, \frac{\frac{c_2^r(\widehat{p}_2^*)^{1-r}}{(\widehat{p}_1^*)^r}}{\frac{c_1^r(\widehat{p}_1^*)^{1-r}}{(\widehat{p}_2^*)^r} + \frac{c_2^r(\widehat{p}_2^*)^{1-r}}{(\widehat{p}_1^*)^r}}, 0, \ldots, 0\right).$$

By Proposition 3, the optimal R&D contest again involves two active firms when the sponsor can charge discriminatory entry fees. Two remarks are in order. First, the optimal contest is not unique, which stands in stark contrast to our previous findings. Multiple contests exist that generate maximum total effort while inducing different winning probability profiles in the equilibrium. The effort-maximizing contest in Section 3.1—which charges a uniform entry fee and induces an even contest with $(p_1, p_2) = (1/2, 1/2)$—remains one of the optima. To see this, note that if a tuple $(p_1, p_2) = (p_1^\natural, p_2^\natural)$ satisfies constraint (19), then $(p_1, p_2) = (p_2^\natural, p_1^\natural)$ also satisfies (19).

This observation leads to the second remark: The sponsor does not (strictly) benefit from the opportunity to charge discriminatory entry fees. Relaxing the constraint of uniform entry fees allows for multiple optima, but none of them strictly outperforms the original optimum in Proposition 1. The sponsor can charge a uniform entry fee and set $\widehat{\boldsymbol{\alpha}}^*$ to level the playing field, as she does in Section 3.1. She can also set $\widehat{\boldsymbol{\alpha}}^*$ to induce uneven winning odds and impose an entry fee equal to the surplus each active firm expects to earn in the contest. Regardless, firms end up with zero surplus in all of these candidate contests, and these competitions fully exhaust the rent.

4.2 Quality-maximizing Contest

Denote by $\widehat{\boldsymbol{p}}^\star \equiv (\widehat{p}_1^\star, \ldots, \widehat{p}_n^\star)$ the equilibrium winning probabilities in the quality-maximizing research contest and let $\widehat{k}^\star := \left|\mathcal{N}_+(\widehat{\boldsymbol{p}}^\star)\}\right|$. The following ensues.

Proposition 4 ***(Quality-maximizing R&D Contest with Discriminatory Entry Fees)*** *Suppose that the sponsor aims to maximize the expected quality of the winning product of the R&D contest and can charge discriminatory entry fees. Moreover, Assumption 1 is satisfied—i.e., $b > \underline{b}^*$—and $b < \frac{2\gamma}{1-r}$.*[15] *Then the optimal contest involves $\widehat{k}^\star = 2$ active firms.*

(i) If $c_1 < c_2$, then the optimal contest induces an equilibrium winning probability profile $\widehat{\boldsymbol{p}}^\star \equiv (\widehat{p}_1^\star, \ldots, \widehat{p}_n^\star) = (\widehat{p}_1^\star, \widehat{p}_2^\star, 0, \ldots, 0)$, with

$$\widehat{p}_1^\star = 1 - \widehat{p}_2^\star = \frac{1}{r} - \frac{2\gamma}{rb}. \tag{20}$$

The corresponding contest rule—which we denote by $(\widehat{V}^\star, \widehat{\boldsymbol{\phi}}^\star, \widehat{\boldsymbol{\alpha}}^\star)$—is

$$\widehat{V}^\star = \frac{b - 2\gamma}{2r\widehat{p}_1^\star\widehat{p}_2^\star}, \; \widehat{\boldsymbol{\phi}}^\star = \left(\frac{b(1 - r\widehat{p}_2^\star) - 2\gamma}{2r\widehat{p}_2^\star}, 0, 0, \ldots, 0\right),$$

and

$$\widehat{\boldsymbol{\alpha}}^\star \equiv \left(\widehat{\alpha}_1^\star, \widehat{\alpha}_2^\star, \widehat{\alpha}_3^\star \ldots, \widehat{\alpha}_n^\star\right) = \left(\frac{\frac{c_1^r(\widehat{p}_1^\star)^{1-r}}{(\widehat{p}_2^\star)^r}}{\frac{c_1^r(\widehat{p}_1^\star)^{1-r}}{(\widehat{p}_2^\star)^r} + \frac{c_2^r(\widehat{p}_2^\star)^{1-r}}{(\widehat{p}_1^\star)^r}}, \frac{\frac{c_2^r(\widehat{p}_2^\star)^{1-r}}{(\widehat{p}_1^\star)^r}}{\frac{c_1^r(\widehat{p}_1^\star)^{1-r}}{(\widehat{p}_2^\star)^r} + \frac{c_2^r(\widehat{p}_2^\star)^{1-r}}{(\widehat{p}_1^\star)^r}}, 0, \ldots, 0\right).$$

(ii) If $c_1 = c_2$, there exist multiple profiles of equilibrium winning probabilities and contest rules

[15] The assumption $b < \frac{2\gamma}{1-r}$ is imposed to guarantee that the equilibrium winning probability of the most efficient firm in (20) is smaller than one. Otherwise, a maximum does not exist when the two most efficient firms are heterogeneous; moreover, the supremum can be approached arbitrarily closely by giving the second most efficient firm an infinitesimal amount of winning probability and the most efficient firm complementary probability.

that generate the maximum expected quality of the winning product, and are the same as those provided in Proposition 3.

Analogous to Proposition 2, Proposition 4 states that the optimal R&D contest involves exactly two entrants. Further, it can be verified that the more competent firm stands a better chance to win the contest—i.e., $\widehat{p}_1^\star > 1/2 > \widehat{p}_2^\star$—whenever the two most competent firms are heterogeneous; i.e., $c_1 < c_2$. It is noteworthy, however, that the optimal contest collects the entry fee only from firm 1 in this case. Similar to Proposition 2, the sponsor may choose to cultivate a national champion or favor the underdog in the optimum, depending on the discriminatory power r and the degree of firm heterogeneity c_2/c_1. We obtain the following result, which paves the way for more detailed discussion of the underlying logic.

Corollary 2 ***(Discriminatory Entry Fees Render A National Champion More Likely)*** *Suppose that $c_1 < c_2$; then $\alpha_1^\star < \widehat{\alpha}_1^\star$.*

By Corollary 2, the optimal contest with discriminatory entry fees awards a larger share of productive resources to the ex ante stronger firm, i.e., $\alpha_1^\star < \widehat{\alpha}_1^\star$. Recall that the sponsor must factor in allocative efficiency when she maximizes the expected quality of the winning product. This compels her to prioritize the more competent firm when allocating resources to tap its superior productivity. Discriminatory entry fees afford the sponsor more flexibility in this respect. Uniform entry fees tempt the sponsor to level the playing field: The entry fee cannot exceed the surplus firm 2 is able to secure from the contest, so a more uneven race leaves more rent to firm 1 and limits the eventual prize purse. However, awarding more resources to less productive firm 2 wastes productivity and jeopardizes the allocative efficiency of the contest. Discriminatory entry fees offer a solution. The sponsor can privilege more competent firm 1 in resource allocation, while confiscating its rent by charging a larger entry fee ϕ_1. In the optimal contest, firm 2 ends up with zero surplus and pays zero entry fee, which leaves it indifferent between participating in or staying out of the contest. A lopsided contest can increase firm 1's surplus—which, however, is absorbed by a high entry fee; the revenue tops up the prize purse, which motivates the two firms to invest in their effort. As a result, Corollary 1 states that firm 1 tends to receive more resources when the entry fee is not forced to be uniform.

Two remarks are in order before we close this section. First, in contrast to Proposition 3, a quality-maximizing sponsor *strictly* benefits from the flexibility to charge discriminatory entry fees when the top two candidates are heterogeneous. Second, a closer look at (19) and (20) reveals that the optimal contest established in Proposition 4 not only maximizes the expected quality of the winning product but also the total effort of the R&D contest.[16] The flexibility to impose identity-

[16]Note that there does not always exist a contest that maximizes both total effort and the expected quality of the winning product under the constraint of a uniform entry fee. By Corollary 1, a quality-maximizing contest cultivates a national champion and gives the most efficient firm a higher equilibrium winning probability when $r < 1/2$ and $c_2/c_1 > \ell$, while according to Proposition 1, an effort-maximizing contest would completely level the playing field.

dependent entry fees allows the sponsor to fully extract surplus and maximizes the incentive the contest provides.

5 Concluding Remarks and Implications

In this paper, we explore the design of an R&D contest by a sponsor who can (i) charge entry fees and (ii) allocate a fixed amount of productive resources across firms. We revisit the classical optimal design problem of research contests pioneered by Fullerton and McAfee (1999). We show that the optimal contest induces the entry of only the two most efficient firms, which echoes the finding of Fullerton and McAfee (1999). However, they require that firms' cost profile meet certain requirement, while our results are immune to these conditions. We demonstrate that the optimal contest depends on the prevailing design objective—maximizing total effort or the expected quality of the winning product. The former requires a level playing field, as predicted by the conventional wisdom of the contest literature; in contrast, the latter never generates an even race unless the two ex ante most efficient firms are symmetric—and the optimum may even prioritize the initial favorite when allocating resources, which further upsets the balance of the competition and leads to a national champion. Our analysis sheds light on the role played by these instruments in shaping optimal contests.

Our results also generate useful implications for the practical design of contest mechanisms. For instance, we show that a level playing field may end up being suboptimal for quality maximization. By Proposition 2, the optimum depends on the nature of the R&D contest and the degree of heterogeneity of the firms, which provides a guideline for allocating productive resources to competing firms.

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Appendix A: Proofs

Proof of Lemma 1

Proof. The proof is similar to that of Theorem 1 in Deng, Fu, and Wu (2021) and is omitted for brevity. ■

Proof of Proposition 1

Proof. To prove the proposition, it suffices to show that

$$\mathcal{W}(\boldsymbol{p},\boldsymbol{c}) \leq \left(\frac{b}{2}-\gamma\right) \times \left(\frac{1}{c_1}+\frac{1}{c_2}\right), \tag{21}$$

with the equality holding if, and only if, $p_1 = p_2 = \frac{1}{2}$ and $p_3 = \cdots = p_n = 0$.

Note that

$$\begin{aligned}
\mathcal{W}(\boldsymbol{p},\boldsymbol{c}) &\equiv \left(\sum_{i\in\mathcal{N}} \frac{rp_i(1-p_i)}{c_i}\right) \times \frac{b-k(\boldsymbol{p})\gamma}{1-k(\boldsymbol{p})\times \min_{i\in\mathcal{N}_+(\boldsymbol{p})}\left\{p_i\left[1-r(1-p_i)\right]\right\}} \\
&\leq \left(\sum_{i\in\mathcal{N}} \frac{rp_i(1-p_i)}{c_i}\right) \times \frac{b-k(\boldsymbol{p})\gamma}{1-\sum_{i\in\mathcal{N}_+(\boldsymbol{p})}\left\{p_i\left[1-r(1-p_i)\right]\right\}} \\
&= \left(\sum_{i\in\mathcal{N}} \frac{p_i(1-p_i)}{c_i}\right) \times \frac{b-k(\boldsymbol{p})\gamma}{\sum_{i\in\mathcal{N}_+(\boldsymbol{p})}\left[p_i(1-p_i)\right]}.
\end{aligned} \tag{22}$$

Let $w_i := \frac{p_i(1-p_i)}{\sum_{j\in\mathcal{N}_+(\boldsymbol{p})}[p_j(1-p_j)]}$ for all $i \in \mathcal{N}$. It follows immediately that $\sum_{i\in\mathcal{N}} w_i = 1$ and

$$\begin{aligned}
w_1 = \frac{p_1(1-p_1)}{\sum_{j\in\mathcal{N}_+(\boldsymbol{p})}\left[p_j(1-p_j)\right]} &= \frac{p_1(1-p_1)}{p_1(1-p_1)+\sum_{j\in\mathcal{N}_+(\boldsymbol{p})\backslash\{1\}}\left[p_j(1-p_j)\right]} \\
&\leq \frac{p_1(1-p_1)}{p_1(1-p_1)+\sum_{j\in\mathcal{N}_+(\boldsymbol{p})\backslash\{1\}}\left(p_jp_1\right)} \\
&= \frac{p_1(1-p_1)}{2p_1(1-p_1)} = \frac{1}{2},
\end{aligned} \tag{23}$$

with the equality holding if, and only if, $k(\boldsymbol{p}) = 2$. Further, we have that

$$\begin{aligned}
\left(\sum_{i\in\mathcal{N}} \frac{p_i(1-p_i)}{c_i}\right) \times \frac{b-k(\boldsymbol{p})\gamma}{\sum_{i\in\mathcal{N}_+(\boldsymbol{p})} p_i(1-p_i)} &= \left[b-k(\boldsymbol{p})\gamma\right] \times \left(\frac{w_1}{c_1} + \sum_{i\in\mathcal{N}\backslash\{1\}} \frac{w_i}{c_i}\right) \\
&\leq \left[b-k(\boldsymbol{p})\gamma\right] \times \left(\frac{w_1}{c_1} + \sum_{i\in\mathcal{N}\backslash\{1\}} \frac{w_i}{c_2}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \left[b - k(\boldsymbol{p})\gamma\right] \times \left(\frac{w_1}{c_1} + \frac{1-w_1}{c_2}\right) \\
&\leq \left(\frac{b}{2} - \gamma\right) \times \left(\frac{1}{c_1} + \frac{1}{c_2}\right),
\end{aligned} \tag{24}
$$

where the first inequality follows from $c_2 \leq \cdots \leq c_n$; the second equality follows from $\sum_{i\in\mathcal{N}} w_i = 1$; and the second inequality follows from $k(\boldsymbol{p}) \geq 2$, $w_1 \leq \frac{1}{2}$, and $c_1 \leq c_2$.

Combining (22) and (24) yields (21), with the equality holding if, and only if, $p_1 = p_2 = \frac{1}{2}$ and $p_3 = \cdots = p_n = 0$. From $\boldsymbol{p}^* \equiv (p_1^*, \ldots, p_n^*) = (\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0)$ and (12), we can obtain the optimally designed entry fee as follows:

$$
\phi = \frac{b(2-r) - 4\gamma}{2r},
$$

which is positive if $b > \underline{b}^* \equiv \frac{4\gamma}{2-r}$. This concludes the proof. ■

Proof of Proposition 2

Proof. Combining (11) and (14), the optimization problem can be simplified as follows:

$$
\max_{\boldsymbol{p}\in\Delta^{n-1},\, k(\boldsymbol{p})\geq 2} \mathcal{M}(\boldsymbol{p}, \boldsymbol{c}) := \frac{\left[b - k(\boldsymbol{p})\gamma\right]^r}{\left(\sum_{i\in\mathcal{N}_+(\boldsymbol{p})} \frac{c_i^r p_i^{1-r}}{(1-p_i)^r}\right)\left(1 - k(\boldsymbol{p}) \min_{i\in\mathcal{N}_+(\boldsymbol{p})} \left\{p_i\left[1 - r(1-p_i)\right]\right\}\right)^r}. \tag{25}
$$

From the rearrangement inequality, we can show that $p_1 \geq \cdots \geq p_n$ in the optimal research contest. Next, we show that $\mathcal{N}_+(\boldsymbol{p}) = \{1, 2\}$. It is evident that $k(\boldsymbol{p}) \geq 2$ in the optimum, which in turn implies that $b - k\gamma \leq b - 2\gamma$.

Fixing an arbitrary equilibrium winning probability profile $\boldsymbol{p} = (p_1, \ldots, p_n)$, with $p_1 \geq \cdots \geq p_n$ and $p_3 > 0$, we construct $\boldsymbol{p}^\dagger := (p_1^\dagger, \ldots, p_n^\dagger)$ as follows:

$$
p_i^\dagger = \begin{cases} \max\left\{p_1, 1/2\right\}, & \text{for } i = 1, \\ \min\left\{1 - p_1, 1/2\right\}, & \text{for } i = 2, \\ 0, & \text{for } i \geq 3. \end{cases}
$$

It is straightforward to verify that $p_1 \leq p_1^\dagger$, $p_2 < p_2^\dagger$, and $p_i \leq p_2^\dagger \leq 1/2 \leq p_1^\dagger$ for all $i \in \{3, \ldots, n\}$, from which we can obtain that

$$
\frac{c_1^r p_1^{1-r}}{(1-p_1)^r} = p_1 \frac{c_1^r}{\left[p_1(1-p_1)\right]^r} \geq p_1 \frac{c_1^r}{\left[p_1^\dagger(1-p_1^\dagger)\right]^r} = p_1 \frac{c_1^r}{\left(p_1^\dagger p_2^\dagger\right)^r}, \tag{26}
$$

$$
\frac{c_2^r p_2^{1-r}}{(1-p_2)^r} = p_2 \frac{c_2^r}{\left[p_2(1-p_2)\right]^r} > p_2 \frac{c_2^r}{\left[p_2^\dagger(1-p_2^\dagger)\right]^r} = p_2 \frac{c_2^r}{\left(p_1^\dagger p_2^\dagger\right)^r}, \tag{27}
$$

and

$$\frac{c_i^r p_i^{1-r}}{(1-p_i)^r} = p_i \frac{c_i^r}{\left[p_i(1-p_i)\right]^r} \geq p_i \frac{c_2^r}{\left[p_2^\dagger(1-p_2^\dagger)\right]^r} = p_i \frac{c_2^r}{\left(p_1^\dagger p_2^\dagger\right)^r}, \text{ for } i \geq 3. \tag{28}$$

Therefore, we have that

$$\begin{aligned}
\sum_{i\in\mathcal{N}_+(\boldsymbol{p})} \frac{c_i^r p_i^{1-r}}{(1-p_i)^r} =& \frac{c_1^r p_1^{1-r}}{(1-p_1)^r} + \sum_{i\in\mathcal{N}_+(\boldsymbol{p})\setminus\{1\}} \frac{c_i^r p_i^{1-r}}{(1-p_i)^r} \\
>& p_1 \frac{c_1^r}{\left(p_1^\dagger p_2^\dagger\right)^r} + \sum_{i\in\mathcal{N}_+(\boldsymbol{p})\setminus\{1\}} p_i \frac{c_2^r}{\left(p_1^\dagger p_2^\dagger\right)^r} \\
=& p_1 \frac{c_1^r}{\left(p_1^\dagger p_2^\dagger\right)^r} + (1-p_1) \frac{c_2^r}{\left(p_1^\dagger p_2^\dagger\right)^r} \\
\geq& p_1^\dagger \frac{c_1^r}{\left(p_1^\dagger p_2^\dagger\right)^r} + p_2^\dagger \frac{c_2^r}{\left(p_1^\dagger p_2^\dagger\right)^r} \\
=& \frac{c_1^r \left(p_1^\dagger\right)^{1-r}}{\left(p_2^\dagger\right)^r} + \frac{c_2^r \left(p_2^\dagger\right)^{1-r}}{\left(p_1^\dagger\right)^r} = \sum_{i\in\mathcal{N}_+(\boldsymbol{p}^\dagger)} \frac{c_i^r \left(p_i^\dagger\right)^{1-r}}{\left(1-p_i^\dagger\right)^r},
\end{aligned} \tag{29}$$

where the first inequality follows from (26), (27), and (28) and the second inequality follows from $c_1 \leq c_2$ and $p_1 \leq p_1^\dagger$.

Next, note that $1 - p_{k(\boldsymbol{p})} \geq 1 - p_2^\dagger$ and

$$\begin{aligned}
k(\boldsymbol{p})p_{k(\boldsymbol{p})} \leq& \min\left\{1, \frac{k(\boldsymbol{p})}{k(\boldsymbol{p})-1} \times \sum_{i\in\mathcal{N}\setminus\{1\}} p_i\right\} \\
=& \min\left\{1, \frac{k(\boldsymbol{p})}{k(\boldsymbol{p})-1}(1-p_1)\right\} \leq \min\left\{1, 2(1-p_1)\right\} = 2p_2^\dagger,
\end{aligned}$$

from which we can conclude that

$$\begin{aligned}
1 - k(\boldsymbol{p}) \times \min_{i\in\mathcal{N}_+(\boldsymbol{p})} \left\{p_i\left[1 - r(1-p_i)\right]\right\} =& 1 - k(\boldsymbol{p})p_{k(\boldsymbol{p})}\left[1 - r\left(1 - p_{k(\boldsymbol{p})}\right)\right] \\
\geq& 1 - 2p_2^\dagger\left[1 - r\left(1 - p_2^\dagger\right)\right] \\
=& 1 - k(\boldsymbol{p}^\dagger) \times \min_{i\in\mathcal{N}_+(\boldsymbol{p}^\dagger)} \left\{p_i^\dagger\left[1 - r\left(1 - p_i^\dagger\right)\right]\right\}.
\end{aligned} \tag{30}$$

Combining (29) and (30) yields $\mathcal{M}(\boldsymbol{p}, \boldsymbol{c}) < \mathcal{M}(\boldsymbol{p}^\dagger, \boldsymbol{c})$, which implies that $\mathcal{N}_+(\boldsymbol{p}) = \{1, 2\}$ in the optimally designed contest. Therefore, the sponsor's optimization problem (25) boils down to

$$\min_{p_1+p_2=1,\, p_1\geq p_2>0} \left(\frac{c_1^r p_1^{1-r}}{p_2^r} + \frac{c_2^r p_2^{1-r}}{p_1^r}\right) \times \left[1 - 2p_2(1-rp_1)\right]^r.$$

Substituting the solution to the above optimization problem—which we denote by $(p_1^\star, p_2^\star)$—into (12), we can derive the corresponding entry fee as follows:

$$\phi^\star = \frac{bp_2^\star\left(1 - rp_1^\star\right) - \gamma}{1 - 2p_2^\star\left(1 - rp_1^\star\right)}.$$

The entry fee is positive if $b > \underline{b}^\star \equiv \frac{\gamma}{p_2^\star(1 - rp_1^\star)}$. This concludes the proof. ■

Proof of Corollary 1

Proof. It is useful to state an intermediate result.

Lemma 2 *Consider the following optimization problem:*

$$\min_{p_1+p_2=1,\, p_1 \geq p_2 > 0} \left(\frac{c_1^\dagger p_1^{1-r}}{p_2^r} + \frac{c_2^\dagger p_2^{1-r}}{p_1^r} \right),$$

where $c_i^\dagger := (c_i)^r$*, with* $i \in \{1, 2\}$*. Denote the solution by* $\widetilde{\boldsymbol{p}}^\star := (\widetilde{p}_1^\star, \widetilde{p}_2^\star)$ *and the corresponding resource allocation rule derived from Lemma 1 by* $\widetilde{\boldsymbol{\alpha}}^\star := (\widetilde{\alpha}_1^\star, \widetilde{\alpha}_2^\star)$*. Then* $\widetilde{\alpha}_1^\star \gtreqless \widetilde{\alpha}_2^\star$ *if and only if* $r \lesseqgtr \frac{1}{2}$*.*

Proof. Taking the logarithm of the objective function in the lemma yields

$$\psi(p_2, r) := \log\left(\frac{c_1^\dagger p_1^{1-r}}{p_2^r} + \frac{c_2^\dagger p_2^{1-r}}{p_1^r} \right) = \log\left(c_1^\dagger(1 - p_2) + c_2^\dagger p_2 \right) - r\log\left(p_2(1 - p_2)\right).$$

Carrying out the algebra, we can obtain that

$$\frac{\partial^2 \psi}{\partial p_2 \partial r} = -\frac{1 - 2p_2}{p_2(1 - p_2)} < 0.$$

Therefore, $\psi(p_2, r)$ is submodular in (p_2, r). By Topkis's theorem, $\widetilde{p}_2^\star$ is increasing in r, which in turn implies that

$$\frac{\widetilde{\alpha}_1^\star}{\widetilde{\alpha}_2^\star} = \frac{(c_1)^r \left(\widetilde{p}_1^\star\right)^{1-r} / \left(1 - \widetilde{p}_1^\star\right)^r}{(c_2)^r \left(\widetilde{p}_2^\star\right)^{1-r} / \left(1 - \widetilde{p}_2^\star\right)^r} = \frac{c_1^\dagger \widetilde{p}_1^\star}{c_2^\dagger \widetilde{p}_2^\star} = \frac{c_1^\dagger\left(1 - \widetilde{p}_2^\star\right)}{c_2^\dagger \widetilde{p}_2^\star}$$

is decreasing in r.

Therefore, to prove the lemma, it suffices to show that $\widetilde{\alpha}_1^\star = \widetilde{\alpha}_2^\star$ when $r = \frac{1}{2}$. In this case, the optimization problem can be written as

$$\min_{p_1+p_2=1,\, p_1 \geq p_2 > 0} c_1^\dagger \sqrt{\frac{p_1}{p_2}} + c_2^\dagger \sqrt{\frac{p_2}{p_1}}.$$

From the AM-GM inequality, we have that

$$c_1^\dagger\sqrt{\frac{p_1}{p_2}}+c_2^\dagger\sqrt{\frac{p_2}{p_1}}\geq 2\sqrt{c_1^\dagger c_2^\dagger},$$

with the equality holding if, and only if,

$$c_1^\dagger\sqrt{\frac{\widetilde{p}_1^\star}{\widetilde{p}_2^\star}}=c_2^\dagger\sqrt{\frac{\widetilde{p}_2^\star}{\widetilde{p}_1^\star}},$$

from which we can conclude that

$$\frac{\widetilde{\alpha}_1^\star}{\widetilde{\alpha}_2^\star}=\frac{c_1^\dagger\widetilde{p}_1^\star}{c_2^\dagger\widetilde{p}_2^\star}=1.$$

This concludes the proof. ■

Now we can prove part (i) of the corollary. Consider the following function:

$$\zeta(p_2;\theta):=\log\left(\frac{c_1^\dagger p_1^{1-r}}{p_2^r}+\frac{c_2^\dagger p_2^{1-r}}{p_1^r}\right)+\theta r\log\big(1-2p_2(1-rp_1)\big)\,,\text{ with }\theta\in[0,1].$$

It is evident that minimizing $\zeta(p_2;\theta)$ is equivalent to minimizing the objective function (16) when $\theta=1$. Similarly, minimizing $\zeta(p_2;\theta)$ is equivalent to minimizing the objective function stated in Lemma 2 when $\theta=0$.

Moreover, carrying out the algebra, we can obtain that

$$\frac{\partial^2\zeta}{\partial p_2\partial\theta}=\frac{2r^2\,(1-2p_2)-2r}{1-2p_2\,\big[1-r\,(1-p_2)\big]}<0.$$

Therefore, $\zeta(p_2;\theta)$ is submodular in (p_2,θ). Again, by Topkis's theorem, we have that $p_2^\star>\widetilde{p}_2^\star$; together with Lemma 2, we can conclude that $\alpha_1^\star<\alpha_2^\star$ for $r\geq 1/2$.

Next, we prove part (ii) of the corollary. Recall $\alpha_2/\alpha_1=(c_2^\dagger p_2)/(c_1^\dagger p_1)$. Define $c:=c_2^\dagger/c_1^\dagger$ and $\alpha:=\alpha_2/\alpha_1$. It follows immediately that $c>1$ and $c=\alpha\times(p_1/p_2)\geq\alpha$. The logarithm of the objective function (16)—or equivalently, $\zeta(p_2;1)$—can be viewed as a function of α and be expressed as

$$\eta(\alpha,c):=\zeta(p_2;1)=\log(1+\alpha)-\log(c+\alpha)+r\log\left(2r-\frac{\alpha}{c}+\frac{c}{\alpha}\right).$$

Fixing $c>1$, the optimization problem stated in Proposition 2 boils down to one in which the sponsor chooses $\alpha\in(0,c]$ to minimize $\eta(\alpha,c)$.

The proof consists of three steps. In the first step, we show that $\alpha^\star:=\alpha_2^\star/\alpha_1^\star$ is strictly decreasing in c. In the second step, we show that $\alpha^\star=c>1$ when c is sufficiently close to 1. Last, we show that $\alpha^\star\leq 1$ when c is sufficiently large. The three steps altogether imply that there exists a threshold $\bar{c}$ such that $\alpha^\star \equiv \alpha_2^\star/\alpha_1^\star > 1$ if $c < \bar{c}$ and $\alpha^\star \equiv \alpha_2^\star/\alpha_1^\star < 1$ if $c > \bar{c}$.

Step I For $\alpha > 0$, we can obtain that

$$\begin{aligned}\frac{\partial^2\eta}{\partial\alpha\partial c} =& \frac{c^4(1-2r^2)+4c^3\alpha r(2-r)-2c^2\alpha^2(1-4r-2r^2)+4c\alpha^3r^2+\alpha^4(1+2r^2)}{(c+\alpha)^2\left(c^2+2rc\alpha-\alpha^2\right)^2}\\ \geq& \frac{2c^2\alpha^2(\sqrt{1-4r^4}+4r+2r^2-1)}{(c+\alpha)^2\left(c^2+2rc\alpha-\alpha^2\right)^2} > 0,\end{aligned}$$

where the first inequality follows from the AM-GM inequality and the second inequality follows from $0 < r < 1/2$. Therefore, $\eta(\alpha, c)$ is supermodular in (α, c). By Topkis's theorem, $\alpha^\star$ is strictly decreasing in c.

Step II Note that

$$\eta(\alpha, c) - \eta(c, c) = r\log\left(1 + \frac{(c+\alpha)(c-\alpha)}{2rc\alpha}\right) - \log\left(1 + \frac{(c-\alpha)(c-1)}{2c(\alpha+1)}\right).$$

Next, we show that $\eta(\alpha, c) > \eta(c, c)$ for all $\alpha \in (0, c)$ when $c < 2^r$. Consider the following two cases depending on $\frac{(c+\alpha)(c-\alpha)}{2rc\alpha}$ relative to 1.

Case I: $\boldsymbol{\frac{(c+\alpha)(c-\alpha)}{2rc\alpha} > 1}$**.** Then we have that

$$\eta(\alpha, c) - \eta(c, c) > r\log 2 - \log\left(1 + \frac{(c-\alpha)(c-1)}{2c(\alpha+1)}\right) > r\log 2 - \log(c) > 0,$$

where the second inequality follows from the fact that $\frac{(c-\alpha)(c-1)}{2c(\alpha+1)} < \frac{c(c-1)}{2c} < c - 1$.

Case II: $\boldsymbol{\frac{(c+\alpha)(c-\alpha)}{2rc\alpha} \leq 1}$**.** Then we have that

$$\begin{aligned}\eta(\alpha, c) - \eta(c, c) \geq& r \times \frac{(c+\alpha)(c-\alpha)}{4rc\alpha} - \log\left(1 + \frac{(c-\alpha)(c-1)}{2c(\alpha+1)}\right)\\ \geq& \frac{(c+\alpha)(c-\alpha)}{4c\alpha} - \frac{(c-\alpha)(c-1)}{2c(\alpha+1)}\\ =& \frac{\left[c+(3-c)\alpha+\alpha^2\right](c-\alpha)}{4c\alpha(\alpha+1)} > 0,\end{aligned}$$

where the first inequality follows from the fact that $\log(1+x) \geq \frac{x}{2}$ for every $x \in [0, 1]$; the second inequality follows from the fact that $\log(1+x) \geq x$ for every $x > 0$; and the third inequality follows from $c < 2^r < 3$.

Step III Carrying out the algebra, we have that

$$\eta(\alpha, c) - \eta(1, c) = \log\left(1 + \frac{(c-1)(\alpha-1)}{2(c+\alpha)}\right) - r\log\left(1 + \frac{(\alpha-1)(\frac{c}{\alpha}+\frac{1}{c})}{2r + \frac{c}{\alpha} - \frac{\alpha}{c}}\right).$$

Next, we show that fixing $0 < r < 1/2$, $\eta(\alpha, c) - \eta(1, c) > 0$ for every $\alpha \in [1, c)$ when c is sufficiently large.

Note that fixing r, there exists a threshold δ such that $\log(1 + \frac{rx}{3}) > r\log(1+x)$ for every $x > \delta$. Consider the following two cases depending on α relative to 2δ.

Case I: $\boldsymbol{\alpha < 2\delta}$. Then we have that

$$\begin{aligned}\eta(\alpha, c) - \eta(1, c) &= \log\left(1 + \frac{(c-1)(\alpha-1)}{2(c+\alpha)}\right) - r\log\left(1 + \frac{(\alpha-1)(\frac{c}{\alpha}+\frac{1}{c})}{2r + \frac{c}{\alpha} - \frac{\alpha}{c}}\right)\\ &> \log\left(1 + \frac{(c-1)(\alpha-1)}{2(c+\alpha)}\right) - \log\left(1 + \frac{r(\alpha-1)(\frac{c}{\alpha}+\frac{1}{c})}{2r + \frac{c}{\alpha} - \frac{\alpha}{c}}\right)\\ &= \log\left(1 + \frac{(c-1)(\alpha-1)}{2(c+\alpha)}\right) - \log\left(1 + \frac{r(\alpha-1)(c+\frac{\alpha}{c})}{2\alpha r + c - \frac{\alpha^2}{c}}\right)\\ &> \log\left(1 + \frac{(c-1)(\alpha-1)}{2(c+2\delta)}\right) - \log\left(1 + \frac{r(\alpha-1)(c+\frac{2\delta}{c})}{2r + c - \frac{4\delta^2}{c}}\right),\end{aligned}$$

where the first inequality follows from Bernoulli's inequality, and the second inequality follows from $1 \leq \alpha < 2\delta$.

Next, note that

$$\lim_{c\to\infty} \frac{(c-1)}{2(c+2\delta)} = \frac{1}{2} > r = \lim_{c\to\infty} \frac{r(c+\frac{2\delta}{c})}{2r + c - \frac{4\delta^2}{c}}.$$

Therefore, there exists $\underline{c}_1$ such that

$$\frac{(c-1)}{2(c+2\delta)} > \frac{r(c+\frac{2\delta}{c})}{2r + c - \frac{4\delta^2}{c}}, \text{ for every } c > \underline{c}_1,$$

which in turn implies that $\eta(\alpha, c) > \eta(1, c)$ for every $c > \underline{c}_1$.

Case II: $\boldsymbol{\alpha \geq 2\delta}$. Recall that $\log(1 + \frac{rx}{3}) > r\log(1+x)$ for every $x > \delta$. Further, we have that $\frac{(\frac{c}{\alpha}+\frac{1}{c})}{2r+\frac{c}{\alpha}-\frac{\alpha}{c}} > \frac{\frac{c}{\alpha}}{1+\frac{c}{\alpha}} > \frac{1}{2}$, which in turn implies that

$$1 + \frac{(\alpha-1)(\frac{c}{\alpha}+\frac{1}{c})}{2r + \frac{c}{\alpha} - \frac{\alpha}{c}} > 1 + \frac{\alpha-1}{2} > \delta.$$

Therefore, we can obtain that

$$\begin{aligned}\eta(\alpha,c)-\eta(1,c) &= \log\left(1+\frac{(c-1)(\alpha-1)}{2(c+\alpha)}\right)-r\log\left(1+\frac{(\alpha-1)(\frac{c}{\alpha}+\frac{1}{c})}{2r+\frac{c}{\alpha}-\frac{\alpha}{c}}\right)\\ &> \log\left(1+\frac{(c-1)(\alpha-1)}{2(c+\alpha)}\right)-\log\left(1+\frac{r(\alpha-1)(\frac{c}{\alpha}+\frac{1}{c})}{3(2r+\frac{c}{\alpha}-\frac{\alpha}{c})}\right).\end{aligned}$$

It suffices to show that

$$\frac{r(\frac{c}{\alpha}+\frac{1}{c})}{3(2r+\frac{c}{\alpha}-\frac{\alpha}{c})}<\frac{c-1}{2(c+\alpha)},$$

for every $\alpha\in[2\delta,c)$ when c is sufficiently large. Note that

$$\frac{\frac{c}{\alpha}+\frac{1}{c}}{2r+\frac{c}{\alpha}-\frac{\alpha}{c}}-\frac{1+\frac{1}{c}}{2r}=-\frac{(c-\alpha)\left[(1+\frac{1}{c})(1+\frac{\alpha}{c})-2r\right]}{2r\alpha\left(2r+\frac{c}{\alpha}-\frac{\alpha}{c}\right)}<0. \tag{31}$$

Therefore, we can obtain that

$$\frac{r(\frac{c}{\alpha}+\frac{1}{c})}{3\left(2r+\frac{c}{\alpha}-\frac{\alpha}{c}\right)}<\frac{1+\frac{1}{c}}{6}<\frac{c-1}{4c}<\frac{c-1}{2\left(c+\alpha\right)},$$

where the first inequality follows from (31); the second inequality holds for $c>5$; and the third inequality follows from $\alpha<c$ and $c>1$.

In summary, if $c>\max\{\underline{c}_1,5\}$, then $\eta(\alpha,c)-\eta(1,c)>0$ for each $\alpha\in(1,c)$, which in turn implies that $\alpha^\star\le 1$. This completes the proof. ■

Proof of Proposition 3

Proof. The optimization problem can be simplified as follows:

$$\max_{\boldsymbol{p}\in\Delta^{n-1},\,k(\boldsymbol{p})\ge 2}\left(\sum_{i\in\mathcal{N}}\frac{p_i(1-p_i)}{c_i}\right)\times\frac{b-k(\boldsymbol{p})\gamma}{\sum_{i\in\mathcal{N}_+(\boldsymbol{p})}\left\{p_i(1-p_i)\right\}}.$$

Recall that we have shown (24) in the proof of Proposition 1, from which we can conclude that the maximum can be reached by an arbitrary profile of equilibrium winning probabilities $\boldsymbol{p}\in\Delta^{n-1}$ such that $\mathcal{N}_+(\boldsymbol{p})=\{1,2\}$. Therefore, we must have that $\mathcal{N}_+(\widehat{\boldsymbol{p}}^*)=\{1,2\}$.

Next, we solve for $\widehat{V}^*$ and $\widehat{\boldsymbol{\phi}}^*$. Because (17) must bind for all active firms, we have that

$$\widehat{\phi}_i^*=\widehat{p}_i^*\left[1-r(1-\widehat{p}_i^*)\right]\times\widehat{V}^*-\gamma,\text{ for }i\in\{1,2\}. \tag{32}$$

Plugging (32) into (18) yields that

$$\widehat{V}^* = b + \left[\widehat{p}_1^*(1 - r\widehat{p}_2^*) \times \widehat{V}^* - \gamma\right] + \left[\widehat{p}_2^*(1 - r\widehat{p}_1^*) \times \widehat{V}^* - \gamma\right],$$

which in turn implies that

$$\widehat{V}^* = \frac{b - 2\gamma}{1 - \widehat{p}_1^*(1 - r\widehat{p}_2^*) - \widehat{p}_2^*(1 - r\widehat{p}_1^*)} = \frac{b - 2\gamma}{2r\widehat{p}_1^*\widehat{p}_2^*}. \tag{33}$$

Substituting (33) into (32) yields that

$$\begin{aligned}(\widehat{\phi}_1^*, \widehat{\phi}_2^*) &= \left(\widehat{p}_1^*(1 - r\widehat{p}_2^*) \times \widehat{V}^* - \gamma, \widehat{p}_2^*(1 - r\widehat{p}_1^*) \times \widehat{V}^* - \gamma\right) \\ &= \left(\frac{b(1 - r\widehat{p}_2^*) - 2\gamma}{2r\widehat{p}_2^*}, \frac{b(1 - r\widehat{p}_1^*) - 2\gamma}{2r\widehat{p}_1^*}\right).\end{aligned}$$

It is straightforward to verify that there exists at least one tuple $(\widehat{p}_1^*, \widehat{p}_2^*)$, with $\widehat{p}_1^* \geq 0$, $\widehat{p}_2^* \geq 0$, and $\widehat{p}_1^* + \widehat{p}_2^* = 1$, such that $\widehat{\phi}_1^* \geq 0$ and $\widehat{\phi}_2^* \geq 0$ if $b > \underline{b}^* \equiv \frac{4\gamma}{2-r}$. This concludes the proof. ■

Proof of Proposition 4

Proof. The optimization problem can be simplified as follows:

$$\max_{\boldsymbol{p} \in \Delta^{n-1}, k(\boldsymbol{p}) \geq 2} \frac{b - k(\boldsymbol{p})\gamma}{\left(\sum_{i \in \mathcal{N}_+(\boldsymbol{p})} \frac{c_i^r p_i^{1-r}}{(1-p_i)^r}\right) \left\{1 - \sum_{i \in \mathcal{N}_+(\boldsymbol{p})} p_i \left[1 - r(1 - p_i)\right]\right\}^r}. \tag{34}$$

By the same argument as in establishing (23), we have that

$$1 - \sum_{i \in \mathcal{N}_+(\boldsymbol{p})} p_i \left[1 - r(1 - p_i)\right] = r \times \sum_{i \in \mathcal{N}_+(\boldsymbol{p})} \left[p_i(1 - p_i)\right] \geq 2p_1(1 - p_1)r.$$

The above inequality, together with (29), implies that $\mathcal{N}_+(\widehat{\boldsymbol{p}}^\star) = \{1, 2\}$. The objective (34) can then be simplified as

$$\frac{b - 2\gamma}{(2r)^r \times \left[c_1^r p_1 + c_2^r(1 - p_1)\right]},$$

which strictly increases with p_1 if $c_1 < c_2$ and remains constant if $c_1 = c_2$. In the case in which $c_1 < c_2$, the constraint $\widehat{\phi}_2^\star \geq 0$ must bind, which gives (20) when $\frac{1}{r} - \frac{2\gamma}{rb} < 1$, or equivalently, when $b < \frac{2\gamma}{1-r}$. In the case in which $c_1 = c_2$, it is evident that any tuple (p_1, p_2) can achieve the maximum, given that entry fees for the two firms are nonnegative, and the analysis closely follows that of Proposition 3. This concludes the proof. ■

Proof of Corollary 2

Proof. It suffices to show that $p_1^\star < \widehat{p}_1^\star$. By (20), we have that $\widehat{p}_1^\star = \frac{1}{r} - \frac{2\gamma}{rb}$. By Assumption 1, (13) and (15), we can obtain that $\widehat{p}_1^\star > 1/2$ and $bp_2^\star(1 - rp_1^\star) \geq \gamma$. For the case in which $p_2^\star = \frac{1}{2}$, we have that $\widehat{p}_1^\star > \frac{1}{2} = p_1^\star$. For the case in which $p_2^\star < \frac{1}{2}$, we have that $\frac{b}{2}(1 - rp_1^\star) > bp_2^\star(1 - rp_1^\star) \geq \gamma$, which in turn implies that $p_1^\star < \frac{1}{r} - \frac{2\gamma}{rb} = \widehat{p}_1^\star$. This concludes the proof. ■

Appendix B: Examples of Equilibrium Nonexistence and Multiple Equilibria in Contests with Entry Fees

Example 1 shows that a pure-strategy equilibrium may fail to exist in the second-stage game and Example 2 demonstrates the possibility of multiple equilibria in pure strategy.

Example 1 ***(Nonexistence of Pure-strategy Equilibrium)*** *Consider a simple research contest with two firms. Set* $(r,\gamma)=(1,1/4)$, $(c_1,c_2)=(1,1)$, *and fix a contest rule* $(V,\phi,\boldsymbol{\alpha})=\big(1,1/4,(1/2,1/2)\big)$.

Next, we show that there exists no pure-strategy equilibrium in the second-stage research contest. Suppose, to the contrary, that there exists an equilibrium, which we denote by $\boldsymbol{x}^*:=(x_1^*,x_2^*)$. *It is straightforward to see that* $x_i^*>0$ *for* $i\in\{1,2\}$ *in the equilibrium. To see this, suppose one firm chooses zero effort. Then the opponent's best response is not well defined: By exerting an infinitesimal effort, the firm is able to secure a sure win and a positive payoff close to 3/4, while exerting zero effort (i.e., opting out) yields zero payoff to the firm.*

Fix $(x_1,x_2)>(0,0)$. *Firm i's expected payoff amounts to*

$$\pi(x_1,x_2)=p_i(\boldsymbol{x};\boldsymbol{\alpha})\cdot V-c_ix_i-\phi-\gamma=\frac{x_i}{x_1+x_2}-x_i-\frac{1}{2}.$$

Solving the first-order conditions $\partial\pi(x_1,x_2)/\partial x_i=0$, *with* $i\in\{1,2\}$, *yields* $x_1^*=x_2^*=1/4$. *Firm i's expected payoff in the hypothetical equilibrium* $\boldsymbol{x}^*$ *is then* $\pi_i(x_1^*,x_2^*)=-1/4$, *which is smaller than that of choosing* $x_i=0$. *A contradiction.*

Example 2 ***(Multiple Pure-strategy Equilibria)*** *Consider a research contest with three firms. Suppose that the profile of firms' marginal cost* (c_1,c_2,c_3) *satisfies* $0<c_1<c_2<c_3$ *and* $c_2(c_1+c_3)>(c_3)^2$. *An example that satisfies these condition is* $(c_1,c_2,c_3)=(1,2.5,3)$. *Set* $(r,\gamma)=\big(1,\frac{(c_1)^2}{2(c_1+c_3)^2}\big)$ *and fix a contest rule* $(V,\phi,\boldsymbol{\alpha})=\big(1,\frac{(c_1)^2}{2(c_1+c_3)^2},(\frac{1}{3},\frac{1}{3},\frac{1}{3})\big)$.

Next, we show that the effort profile $\boldsymbol{x}^*\equiv(x_1^*,x_2^*,x_3^*)=\big(\frac{c_2}{(c_1+c_2)^2},\frac{c_1}{(c_1+c_2)^2},0\big)$ *constitutes a pure-strategy equilibrium of the constructed research contest. Fixing* (x_2^*,x_3^*), *it is straightforward to verify that choosing* $x_1=x_1^*$ *satisfies the first-order condition (4) and leads to a positive payoff to firm 1. Therefore, firm 1 has no incentive to deviate from* $x_1=x_1^*$. *Similarly, we can show that* $x_2=x_2^*$ *is firm 2's best response to its opponents' effort profile* (x_1^*,x_3^*). *It remains to verify that* $x_3=0$ *is optimal for firm 3. Fixing* $\boldsymbol{x}_{-3}^*\equiv(x_1^*,x_2^*)$, *simple algebra would verify that firm 3's expected payoff is*

$$\pi_3(x_3,\boldsymbol{x}_{-3}^*;V,\phi,\boldsymbol{\alpha})=\begin{cases}\frac{x_3}{x_3+\frac{1}{c_1+c_2}}-\frac{c_1^2}{(c_1+c_3)^2}-c_3x_3, & \text{if } x_3>0,\\ 0, & \text{if } x_3=0.\end{cases}$$

If firm 3 chooses to exert a positive amount of effort $x_3 > 0$, the effort level is pinned down by the first-order condition $\partial\pi_3/\partial x_3 = 0$, from which we can obtain

$$\frac{\frac{1}{c_1+c_2}}{(x_3+\frac{1}{c_1+c_2})^2} = c_3 \Rightarrow x_3 = \sqrt{\frac{1}{c_1+c_2}} \times \left(\sqrt{\frac{1}{c_3}} - \sqrt{\frac{1}{c_1+c_2}}\right) > 0;$$

and the firm's expected payoff of choosing the above effort level is

$$\begin{aligned} \pi_3(x_3, \boldsymbol{x}^*_{-3}; V, \phi, \boldsymbol{\alpha}) &= \left(1 - \sqrt{\frac{c_3}{c_1+c_2}}\right)^2 - \frac{c_1^2}{(c_1+c_3)^2} \\ &< \left(1 - \sqrt{\frac{c_2}{c_1+c_3}}\right)^2 - \frac{c_1^2}{(c_1+c_3)^2} < 0 = \pi_3(0, \boldsymbol{x}^*_{-3}; V, \phi, \boldsymbol{\alpha}), \end{aligned}$$

where the first inequality follows from $c_1 < c_2 < c_3$ and the second from the postulated $c_2(c_1+c_3) > (c_3)^2$. Therefore, fixing (x_1^*, x_2^*), it is optimal for firm 3 to opt out—i.e., exert zero effort.

Similarly, we can verify that the effort profile $\boldsymbol{x}^{**} \equiv (x_1^{**}, x_2^{**}, x_3^{**}) = \left(\frac{c_3}{(c_1+c_3)^2}, 0, \frac{c_1}{(c_1+c_3)^2}\right)$—in which firm 2 chooses to opt out—also constitutes a pure-strategy equilibrium of the constructed research contest. In the equilibrium, firm 1 receives a positive payoff and firms 2 and 3 zero payoff. In summary, the constructed research contest possesses (at least) two equilibria in pure strategy. In fact, we can show that there exist no other pure-strategy equilibria in this game.