

On Leveling the Playing Field in Networked Contests*

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Abstract

Heterogeneous players are matched into interconnected pairwise contests across multiple battlefields. Each organizer independently sets her contest rules to maximize effort provision on her respective battlefield. The conventional wisdom of leveling the playing field may fail in this environment. We identify sufficient conditions—concerning contest technologies and network structure—that mitigate network externalities and restore the level-playing-field principle, such that each organizer prefers a fully balanced contest regardless of others’ choices. However, even if the level-playing-field principle fails and an organizer may not unconditionally prefer a fully balanced contest, we demonstrate that an even-odds equilibrium always exists, in which all contests are resolved with equal winning probabilities, for every network structure and admissible contest technologies. Further, any subgame perfect Nash equilibrium of the game yields the same unique even-odds equilibrium outcome.

Keywords: Competing Contests; Contest Design; Networks.

JEL Classification Codes: C72, D74, D85.

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1 Introduction

Economic agents often compete across multiple fronts and are connected, directly or indirectly, through networks. One’s action on a battlefield not only influences local outcomes but also generates spillover remotely and globally. Consider a tech firm that participates in multiple R&D challenges. If the firm assigns its top researchers to a prioritized project, its competitor on a parallel project may face less pressure and reallocate resources elsewhere; the firm’s choice may thus affect competition in contests in which it is not directly involved and against rivals it does not confront head-to-head. A similar dynamic would arise, for instance, when professional athletes plan their seasons, with each conserving stamina for preferred tournaments. These interactions take place among multiple agents across multiple battlefields, forming a network of interconnected contests with complex externalities.

Significant scholarly effort has been devoted to examining the strategic interactions over networks and exploring how the nature of the underlying game (e.g., strategic substitutability or complementarity) determines the equilibrium and optimal intervention (Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv, 2010; Galeotti, Golub, and Goyal, 2020). Networked contests deserve to be examined systematically, given the nonmonotone best responses often inherent in contest games (Dixit, 1987; Kirkegaard, 2026), which yield important implications for strategic analysis of the game and optimal design of contest rules.¹ This property, for instance, underpins the conventional wisdom of leveling the playing field: A more balanced contest—one that encourages the underdog to challenge the favorite—incentivizes greater effort, which sheds light on a plethora of practices that aim to promote closer competitions. Consider, for instance, the handicap systems in golf tournaments and horse racing, as well as the various measures to support small and medium-sized enterprises (SME) set by the U.S. Small Business Administration in federal procurement.²

The economics literature has formally established that, to incentivize effort, a contest should prevent the emergence of dominant players in equilibrium: No contender can win with a probability exceeding $1/2$, which implies that a two-player contest must be resolved with equal winning odds.³ However, this level-playing-field principle was established in

¹As Dixit (1987) demonstrates, players’ best responses are often nonmonotone in contest games: In contrast to Cournot or Bertrand competitions, one’s effort choice is a strategic complement to that of his opponent when he is in the lead, while it is a strategic substitute when he is behind.

²Many horse-racing tournaments—e.g., the Grand National or the Melbourne Cup—require that horses with higher initial ratings carry heavier weight. A similar mechanism—aero handicap—is implemented in Formula One (F1) championships, in which teams that performed better in previous seasons are allocated reduced aerodynamic testing time.

³See, e.g., Fu and Wu (2020), who show in a broad context that the optimal two-player contest always

standalone contests. Imagine instead an organizer who sets the contest rules on her own turf to attract effort and attention from contenders who face competing opportunities elsewhere; e.g., a firm running an R&D challenge to seek an innovative technical solution when other projects are available to a pool of overlapped solvers. She takes into account contenders' strategic trade-offs across multiple contests, anticipates the choices of competing organizers, and understands how her local choice is transmitted through the network. This complexity casts doubt on such conventional wisdom.

Consider a simple case with each competing player subject to a resource constraint. Suppose that an organizer handicaps the frontrunner. This generates a direct *local effect* that intensifies competitions in her own contest. Meanwhile, a player's increased effort on her battlefield may come at the expense of reduced effort elsewhere. This causes an indirect *network effect*, because the local shift in effort alters contenders' marginal benefits and the cost of effort across all other contests.⁴ The global impact requires that all contenders reset their effort choices everywhere. These spillovers may feed back to the original battlefield through the network, and the overall outcome is a priori unclear. Section 2.2 provides an intuitive example in which the indirect network effect prevails and an organizer prefers an imbalanced contest over an even race.

We examine the level-playing-field principle in a network of interconnected contests. Understanding these intertwined decisions requires a comprehensive account of the network externalities and nuanced strategic interdependence at and across two different levels of strategic interactions—i.e., the networked contests and interactions among competing organizers who set rules for subsequent contests.

A two-stage game is set up for the analysis. In the first stage, multiple organizers independently set the contest rules for their respective battlefields; in the second, economic agents, whom we refer to as “players,” compete in pairwise contests. A player may participate in multiple parallel contests, and the overlapped competitions link organizers indirectly and cause their choices of contest rules to intertwine.

More specifically, each player either incurs a convex cost based on the aggregate effort exerted across all contests he participates in (a “pure-cost” case) or is subject to a budget

yields equal winning probabilities in equilibrium; when the contest involves three or more players, the optimum requires that each player's equilibrium winning odds be strictly below 1/2.

⁴König, Rohner, Thoenig, and Zilibotti (2017) and Amarasinghe, Raschky, Zenou, and Zhou (2026), for example, provide empirical evidence of spillovers from changes in local conditions in networked conflicts. Cortes-Corrales and Gorny (2025) examine the unintended knock-on effects of varying one player's strength throughout a network.

constraint (a “pure-budget” case). As a result, increasing effort on one battlefield limits the player’s ability to contribute to others. Players differ in their contest technologies, cost functions, or resource endowments. The organizer imposes a multiplicative bias on each player’s output, which varies players’ relative competitiveness and tilts the balance of the contest. Such biases encapsulate diverse mechanisms observed in real-world competition. It can be a weighted scoring rule, such as handicap systems in golf tournaments. It can also reflect the various measures of actual productivity interventions, including technical support (e.g., mentorship in Mozilla’s Open Innovation Challenge); capacity-building for SMEs (e.g., counseling and training in federal procurement bids); and preferential industrial policies that target prioritized firms (Frank, Kanzow, Leininger, and Schwartz, 2013).

This paper presents the first formal analysis of decentralized contest design in a networked contest game to explore the boundaries of the level-playing-field principle. Our analysis addresses two main questions. We first investigate under what conditions an organizer prefers an even race irrespective of the rules on parallel contests, such that the level-playing-field principle continues to hold in a network. We then explore whether imbalanced contests may arise in the equilibrium, given that an organizer may not level the playing field unconditionally, as Section 2.2 shows.

Our main results can be summarized as follows. First, we provide sufficient conditions under which each individual organizer prefers to fully level the playing field—i.e., setting her contest rule to induce even winning odds in equilibrium—regardless of the contest rules on other battlefields (Theorem 1). This follows if either (i) each player’s impact function (i.e., the function that maps effort onto contest output) is sufficiently concave or (ii) the network is acyclic. These conditions mitigate the intricate indirect effects of a rule change on a battlefield: The former limits spillovers across battlefields, while the latter shuts down feedback loops.

Second, a formal equilibrium analysis establishes the existence of a subgame perfect Nash equilibrium (SPNE): All contests are resolved with equal winning probabilities in equilibrium, and a unique set of contest rules exists that induces this outcome (Theorem 2). We name this the *even-odds equilibrium* and further show that this equilibrium outcome is unique. An organizer may not always prefer to level the playing field (see, e.g., Section 2.2); however, their simultaneous and decentralized actions always lead to this unique outcome. The analysis unfolds in two layers. First, it verifies that when every other contest is balanced, no organizer gains by unbalancing her own, which establishes equilibrium existence. Second, consider an arbitrary set of contest rules that involve at least one imbalanced battle.

We demonstrate that the organizer of a least balanced battle—i.e., one with the largest gap between the two players’ equilibrium winning probabilities—has a profitable deviation, which rules out any equilibrium that involves unequal winning probabilities. This verifies the uniqueness of the even-odds outcome.

Our two main results play distinct roles. Theorem 1 establishes the conditions under which leveling the playing field is an individual organizer’s uniform optimal choice against any arbitrary rules set by others. It nevertheless does not inform whether noneven winning odds may arise in equilibrium when these conditions fail. Theorem 2, in contrast, reinstates the level-playing-field principle from an equilibrium perspective: It does not concern an individual organizer’s optimal choices; it simply pins down the unique equilibrium as the outcome of decentralized actions.

We consider two extensions. First, we analyze a sequential-move version of the game in which organizers set contest rules in a predetermined order. Whenever the sufficient conditions for the level-playing-field principle hold in the baseline model, the equilibrium outcome remains invariant to the timing of moves: An SPNE exists with a unique equilibrium outcome, replicating the equilibrium rule profile of the simultaneous-move game (Theorem 3). Second, we consider a centralized organizer who sets contest rules for all battlefields. The unique optimal contest rules induce equal winning probabilities across contests (Theorem 4). This yields a useful insight: Decentralized rule-setting by self-interested organizers may achieve the same outcome as centralized planning, which suggests the potential efficiency of delegation in contest design.

Link to the Literature Our paper belongs to the extensive literature on strategic interactions among economic agents within networks, such as Ballester, Calvó-Armengol, and Zenou (2006); Bramoullé, Kranton, and D’amours (2014); Galeotti, Goyal, Jackson, Vega-Redondo, and Yariv (2010); and Galeotti, Golub, and Goyal (2020). A growing body of literature focuses on contests and conflicts within networks (Dziubiński, Goyal, and Vigier, 2016). These studies vary significantly in their assumptions regarding the mechanisms that link players and structure contests or conflicts.

Dziubiński, Goyal, and Minarsch (2021), for instance, examine a conflict network in which a ruler attacks connected “nodes” successively to acquire and accumulate resources. Dziubiński, Goyal, and Zhou (2025) assume that each player’s effort in one battlefield generates spillovers and helps him win the battles on neighboring battlefields. Goyal and Vigier (2014) study an attacker-defender game, in which the attacker and defender allocate combative efforts across nodes. König, Rohner, Thoenig, and Zilibotti (2017) assume that

each player’s entry in his (single) contest is determined by his own effort, positive spillovers from allies, and negative spillovers from enemies. In contrast, [Hiller \(2017\)](#) allows players to form either positive links (alliances that aid in conflict) or negative links (direct conflicts).

Our paper more closely relates to [Franke and Öztürk \(2015\)](#); [Xu, Zenou, and Zhou \(2022\)](#); [Cortes-Corrales and Gorny \(2025\)](#); and [Li and Zhou \(2025\)](#), in which multiple players are matched into contests across a network and allocate effort among battlefields. [Franke and Öztürk \(2015\)](#) assume bilateral contests on each battlefield and focus on specific network structures such as regular, complete bipartite, and star-shaped configurations. They adopt convex cost functions, in which increased effort on one battlefield raises effort costs elsewhere. Based on a conflict network à la [Franke and Öztürk \(2015\)](#), [Cortes-Corrales and Gorny \(2025\)](#) demonstrate how a change in a player’s strength triggers spillovers throughout the network. [Xu et al. \(2022\)](#) generalize the framework of [Franke and Öztürk \(2015\)](#) by allowing multilateral contests, budget constraints, and unrestricted network structures. They employ variational inequalities to overcome technical challenges, establish equilibrium existence, and provide conditions for equilibrium uniqueness. Under quadratic costs and bilateral contests on each battlefield, [Li and Zhou \(2025\)](#) examine comparative statics within acyclic networks, and analyze how shocks propagate throughout the network. Despite the lack of closed-form solutions, they remarkably demonstrate that comparative statics can be pinned down using sign functions.^{5,6}

Our paper differs from this literature in two respects. First, our model incorporates strategic interactions across two layers: Players compete within a network, while organizers indirectly interact through players’ strategic effort choices in response to independently set contest rules. Second, we specifically focus on organizers’ strategic choices of contest rules. Contest design is not considered by [Franke and Öztürk \(2015\)](#) or [Xu et al. \(2022\)](#). [Li and Zhou’s \(2025\)](#) comparative statics offer useful insights for contest design. However, their analysis primarily concerns externalities that arise from interventions on an individual battlefield. Our analysis accommodates both decentralized and centralized contest designs, and enables simultaneous and sequential rule-setting across all battlefields. [Dziubiński,](#)

⁵[Matros and Rietzke \(2024\)](#) and [Sun, Xu, and Zhou \(2023\)](#) also contribute to this strand of the literature. Unlike [Franke and Öztürk \(2015\)](#), [Matros and Rietzke \(2024\)](#) require that each player commit to a single effort level that applies identically across all contests they participate in. [Sun et al. \(2023\)](#) analyze both constrained (uniform effort) and unconstrained effort allocation regimes. They show that, for Tullock contest success functions in semi-symmetric networks, the two regimes produce the same total effort and equilibrium payoffs.

⁶In addition to the theoretical contributions, several notable studies empirically examine conflicts in networks, including [Jackson and Nei \(2015\)](#), [Berman and Couttenier \(2015\)](#), [König et al. \(2017\)](#), [Harari and La Ferrara \(2018\)](#), [Berman et al. \(2021\)](#), and [Amarasinghe et al. \(2026\)](#).

Goyal, and Zhou (2025) explore the design of network structure to maximize efforts or utilities. They focus on a centralized design problem, while we assume a given network structure and let the contest rule on each battlefield be set by its respective organizer.

Our paper naturally connects to the literature on multi-battle contests (e.g., Kovenock and Roberson, 2012; Snyder, 1989; Klumpp and Polborn, 2006; Konrad and Kovenock, 2009; Fu, Lu, and Pan, 2015). In particular, this study is closely related to the research stream that examines players' allocation of scarce resources across battlefields, a line of work that dates back to Borel (1921) and Borel and Ville (1938). The discrete version of the game is known as the Colonel Blotto game, with notable contributions including Friedman (1958); Roberson (2006); Kvasov (2007); Kovenock and Roberson (2012); Roberson and Kvasov (2012); and Fu and Iyer (2019).⁷

Our paper distinguishes itself from these studies along three dimensions. First, we allow for multiple players to be matched in contests across a network, whereas the literature predominantly assumes that two players compete in every battlefield. Second, our model accommodates both a pure-cost case with strictly convex cost functions and a pure-budget case, as seen in most Colonel Blotto game studies. Third, and most importantly, the networked contests in our model are subgames that follow decentralized decisions by contest organizers; none of the aforementioned studies consider contest rule setting.⁸

Finally, each organizer in our model competes for players' effort investment on her own battlefield. This links our work to the literature on competing contests, with Azmat and Möller (2009, 2018) and Morgan, Sisak, and Várdy (2018) as leading contributions. Unlike our setting, these studies typically assume that each player chooses which contest to enter, so organizers compete for contestants' discrete entry decisions. Körpeoğlu, Korpeoglu, and Hafalır (2022) allow solvers to participate in multiple contests, but their focus lies in comparing exclusivity versus non-exclusivity in contest design.

⁷Friedman (1958) analyzes two firms that allocate fixed advertising budgets across multiple marketing areas. Roberson (2006) fully characterizes the equilibrium of a Colonel Blotto game. Kovenock and Roberson (2012) introduce asymmetric prize valuations. Kvasov (2007) and Roberson and Kvasov (2012) relax the zero-sum assumption and allow for alternative uses of resources. Fu and Iyer (2019) accommodate rent-augmenting investment other than rent-seeking efforts.

⁸Feng and Lu (2018) and Feng, Jiao, Kuang, and Lu (2024) also consider contest design. However, they adopt team-based contest structures as in Fu, Lu, and Pan (2015). Their focus lies in the decisions of a central planner who governs the entire contest architecture.

2 Model and Preliminaries

In this section, we set up our model, provide an example to illustrate the nuances caused by the network, and lay out the preliminaries that pave the way for the main analysis.

2.1 Model Setup

A finite set of risk-neutral players $\mathcal{N} \equiv \{1, 2, \dots, N\}$ compete within a connected network.⁹ We focus on bilateral contests. Each player $i \in \mathcal{N}$ competes head-to-head with another on at least one battlefield. Let $\mathcal{E} \equiv \{a, b, \dots\}$ denote the set of battlefields and $e \in \mathcal{E}$ an indicative battlefield. The network can then be represented by $\Gamma \subset \mathcal{N} \times \mathcal{E}$, where $(i, e) \in \Gamma$ if and only if player i is involved in the contest on battlefield e . Further, let $\mathcal{E}_i \equiv \{e \in \mathcal{E} : (i, e) \in \Gamma\}$ denote the set of battlefields with player i 's participation and $\mathcal{N}^e \equiv \{i \in \mathcal{N} : (i, e) \in \Gamma\}$ the set of players who compete on battlefield e , with $|\mathcal{N}^e| = 2$ for all $e \in \mathcal{E}$.

The bilateral contest network Γ described above can model a rich class of interconnected contest games. Figure 1 depicts three examples. In each subfigure, the network is represented as a multigraph, in which the vertices represent players and the edges between vertices represent battlefields. Figure 1a represents a stylized single-battle contest, in which players 1 and 2 fight on battlefield a ; Figure 1b depicts a triangular network structure in which three players are matched to three pairwise battles; Figure 1c represents a two-player multi-battle contest, in which two players compete against each other simultaneously on battlefields a and b .

Each battlefield $e \in \mathcal{E}$ is governed by an organizer and the game proceeds in two stages. In the first stage, organizers each set the rules for the contests on their own battlefields. In the second stage, having observed the rule set for each contest, players simultaneously exert their efforts to vie for wins.

Second Stage: Contests and Payoffs The contest between players i and j on battlefield e is modeled as a generalized Tullock contest. Put formally, fixing the profile of efforts $\mathbf{x}^e \equiv (x_i^e, x_j^e)$ the players exert on battlefield e , player i wins with a probability

⁹For disconnected networks, we can always decompose them into several connected components and our results remain intact.

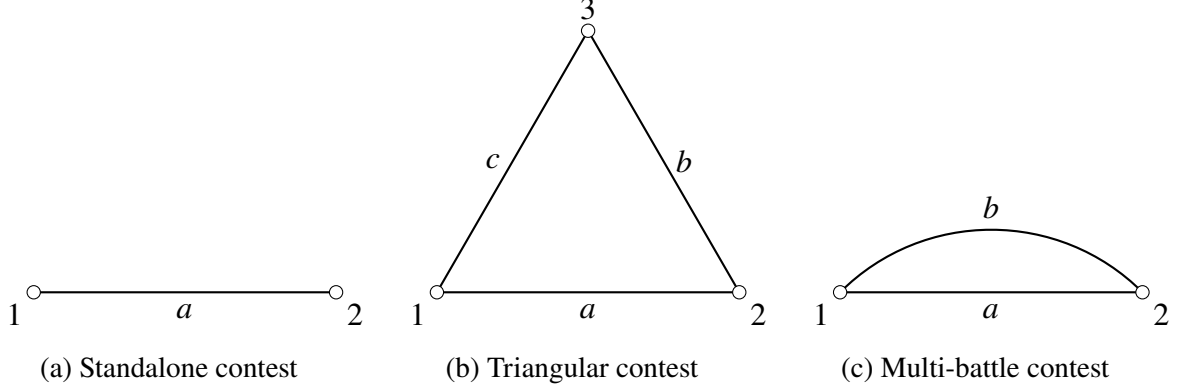


Figure 1: Examples of network structures.

$$p_i^e(x_i^e, x_j^e) = \begin{cases} \frac{\alpha_i^e f_i^e(x_i^e)}{\alpha_i^e f_i^e(x_i^e) + \alpha_j^e f_j^e(x_j^e)}, & x_i^e + x_j^e > 0, \\ \frac{1}{2}, & x_i^e + x_j^e = 0, \end{cases} \quad (1)$$

and player j wins with the complementary probability, i.e., $p_j^e(x_i^e, x_j^e) = 1 - p_i^e(x_i^e, x_j^e)$.

Following the tradition of the contest literature, we call $f_i^e(\cdot)$ player i 's *impact function* on battlefield e , which indicates the player's contest technology on this battlefield and satisfies $f_i^e(0) = 0$, $(f_i^e)'(\cdot) > 0$, and $(f_i^e)''(\cdot) \leq 0$. Further, the parameters α_i^e and α_j^e , with $\alpha_i^e, \alpha_j^e > 0$ and $\alpha_i^e + \alpha_j^e = 1$, are the multiplicative biases the organizer on battlefield e assigns to players i and j , respectively, which determine their relative competitiveness in the contest.

Fixing player i 's effort profile $\mathbf{x}_i \equiv (x_i^e)_{e \in \mathcal{E}_i}$ across all battlefields that involve him, let $X_i \equiv \sum_{e \in \mathcal{E}_i} x_i^e$ denote his total effort. The player bears a cost of $c_i(X_i)$. Players are subject to either resource constraints or regular cost functions.¹⁰ In the former (*pure-budget*) case, each player i 's effort cost can technically take the form of $c_i(X_i) = 0$ for all $X_i \in [0, \bar{X}_i]$ and $c_i(X_i) = +\infty$ for all $X_i \in (\bar{X}_i, +\infty)$, where $\bar{X}_i \in (0, +\infty)$ is the maximum effort at his disposal. In the latter (*pure-cost*) case, we set $\bar{X}_i$ to $+\infty$ and let $c_i(\cdot)$ be twice differentiable and satisfy $c_i(0) = 0$, $c_i'(\cdot) > 0$, and $c_i''(\cdot) > 0$.

A victory on battlefield $e \in \mathcal{E}$ yields a prize value of $v^e > 0$ to the winner. A player i 's

¹⁰This assumption is imposed for expositional convenience. Our analysis can easily be extended to the case in which some players are subject to a resource constraint while others have a regular effort cost function.

expected payoff in the game is thus

$$\pi_i(\mathbf{x}_i, \mathbf{x}_{-i}) = \sum_{e \in \mathcal{E}_i} v^e p_i^e(\mathbf{x}^e) - c_i(X_i),$$

where $\mathbf{x}_{-i} \equiv (\mathbf{x}_1, \dots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \dots, \mathbf{x}_N)$ is the profile of the effort strategies of all players other than i .

First Stage: Decentralized Contest Rule Setting in a Network In the beginning of the game, the organizer of each battlefield sets the rules for her battle. More formally, the organizer for each battlefield $e \in \mathcal{E}$ with $\mathcal{N}^e = \{i, j\}$ imposes multiplicative biases (α_i^e, α_j^e) on players' impact functions, with $\alpha_i^e, \alpha_j^e > 0$ and $\alpha_i^e + \alpha_j^e = 1$; they set the rules simultaneously, and all (α_i^e, α_j^e) become commonly known prior to the second stage of the game.

An organizer values the effort supplied by each player on her battlefield, so she chooses (α_i^e, α_j^e) to maximize an objective function $\Lambda^e(\mathbf{x}^e)$, which strictly increases with x_i^e for each $i \in \mathcal{N}^e$.

Summary The two-stage game can be described by $\mathcal{G} \equiv \langle \Gamma, (f_i^e(\cdot))_{(i,e) \in \Gamma}, (c_i(\cdot))_{i \in \mathcal{N}}, (\Lambda^e(\cdot))_{e \in \mathcal{E}} \rangle$, where Γ represents the network structure, $(f_i^e(\cdot))_{(i,e) \in \Gamma}$ the set of impact functions, $(c_i(\cdot))_{i \in \mathcal{N}}$ the set of players' effort cost functions, and $(\Lambda^e(\cdot))_{e \in \mathcal{E}}$ the set of organizers' objective functions. Organizers each set (α_i^e, α_j^e) in the first stage of the game, and players simultaneously sink their efforts afterward. We adopt subgame perfect Nash equilibrium (SPNE) in pure strategies as the solution concept.

2.2 An Illustrative Example

We now provide a simple example to show that leveling the playing field may not be optimal in a networked contest.

Example 1 (Optimality of Imbalanced Competitions in a Network) Suppose that $\mathcal{N} = \{1, 2, 3\}$, $\mathcal{E} = \{a, b, c\}$, and Γ is a bilateral contest network with triangular structure, as depicted in Figure 1b. Let $f_i^e(x) = x$ for all $(i, e) \in \Gamma$. Consider a pure-budget case with $(\bar{X}_1, \bar{X}_2, \bar{X}_3) = (22, 1, 1)$. The prizes for winning the battles are, respectively, $(v^a, v^b, v^c) = (1, 3, 1)$. Fix the biases on battlefields b and c at $\alpha^b = (\alpha_2^b, \alpha_3^b) = (0.1, 0.9)$ and $\alpha^c = (\alpha_3^c, \alpha_1^c) = (0.1, 0.9)$, respectively.

Suppose that the organizer on battlefield a seeks to maximize total effort in the battle, i.e., $\Lambda^a = x_1^a + x_2^a$. Figure 2c plots Λ^a as a function of the winning probability p_1^a induced by her

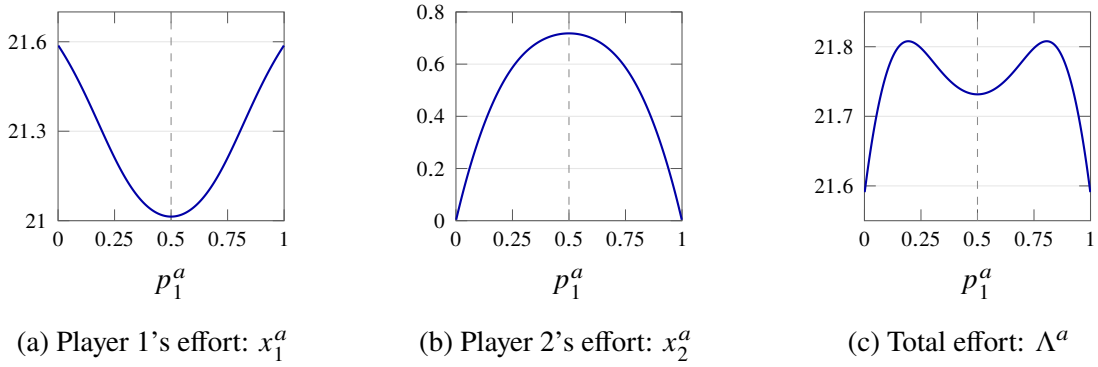


Figure 2: Equilibrium effort and organizer's objective on battlefield a as a function of p_1^a . Dashed lines indicate equal winning probabilities.

contest rule. A level playing field, with $p_1^a = p_2^a = 1/2$, yields total effort of approximately 21.732. By instead inducing $p_1^a = 1/4$, the organizer raises total effort to approximately 21.798. Thus, leveling the playing field is not optimal.

In this context, leveling the playing field maximizes player 2's effort (see Figure 2b) while minimizing player 1's (see Figure 2a). Given player 1's greater resource endowment, maximizing total effort on this battlefield primarily requires his input. When p_1^a rises from $1/4$ toward $1/2$, the decrease in his effort outweighs the increase in player 2's effort, so a more balanced contest reduces total effort.

As p_1^a increases and approaches $1/2$, two effects are triggered. First, a more level playing field intensifies competition on battlefield a , which prompts both players to increase their efforts: We term this a *direct local effect*, consistent with conventional wisdom. Second, this direct effect induces an *indirect network effect*: It propagates throughout the network, reshaping effort incentives on other battlefields; these shifts then feed back into battlefield a , further influencing x_1^a and x_2^a .

To see this, consider the following thought experiment. Fix the biases on battlefields b and c — $(\alpha_2^b, \alpha_3^b) = (\alpha_3^c, \alpha_1^c) = (0.1, 0.9)$ —and begin with $p_1^a = 1/4 < 1/2 < p_2^a$. Figure 3a shows players' relative standing on each battlefield under this set of biases. In this setting, player 2 is the frontrunner on battlefield a , since his winning probability exceeds $1/2$. Now suppose that the organizer adjusts her biases to increase p_1^a toward $1/2$. This change favors the underdog, player 1. The direct local effect encourages both players to intensify their efforts (see Figure 3b).

However, an increase in x_2^a would force player 2 to reduce his effort x_2^b on battlefield b , due to his budget constraint. By Figure 3a, player 2 is initially the underdog on battlefield

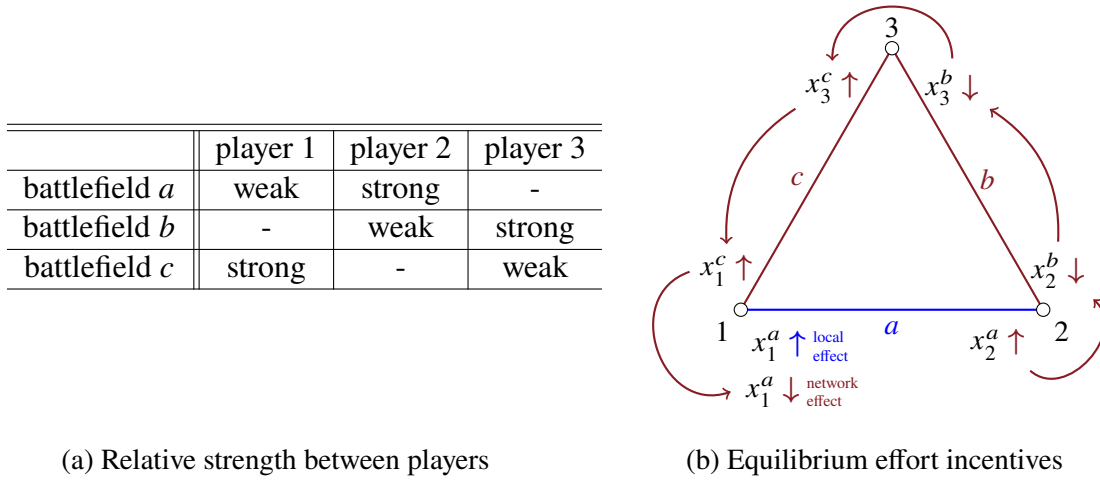


Figure 3: Illustration of the direct local effect and the indirect network effect.

b . A decrease in x_2^b gives the initial frontrunner on battlefield b —player 3—an easier win, which allows the player to scale back his effort x_3^b and redirect the saved resources to battlefield c . In turn, player 1—the initial leader on battlefield c —must respond to the more aggressive player 3 by raising his effort x_1^c . Ultimately, the increased demand on player 1’s resources devoted to battlefield c forces him to reduce his effort x_1^a on battlefield a , as shown in Figure 3b.

The indirect network effect counteracts the direct local effect in shaping player 1’s effort choice. As p_1^a increases and approaches $1/2$, the indirect effect dominates, leading to a lower equilibrium effort x_1^a on battlefield a (see Figure 2a). In contrast, the direct and indirect effects reinforce each other for player 2, resulting in an increase in his effort on battlefield a as p_1^a approaches $1/2$ (see Figure 2b).

These complexities are absent in standalone contests and render even winning odds suboptimal for a designer.

2.3 Preliminary Analysis: Second-stage Equilibrium

The second-stage game is a collection of bilateral contests interconnected through a network. The equilibrium existence in this setting has been established by [Xu, Zenou, and Zhou \(2022\)](#), and the result can be directly applied in our context.

Lemma 1 ([Xu, Zenou, and Zhou, 2022](#)) *Fixing a profile of contest rules $\alpha = \{\alpha^e\}_{e \in \mathcal{E}}$, there exists a Nash equilibrium in the second-stage game. Specifically, the equilibrium*

effort profile $\mathbf{x}^*(\boldsymbol{\alpha}) = \{\mathbf{x}^e(\boldsymbol{\alpha})\}_{e \in \mathcal{E}}$, together with a set of parameters $\{\lambda_i\}_{i \in \mathcal{N}}$, satisfies the following first-order conditions:

$$v^e p_i^e(\mathbf{x}^e) [1 - p_i^e(\mathbf{x}^e)] = \lambda_i g_i^e(x_i^e) \quad (2)$$

and

$$v^e \times \frac{\alpha_i^e (f_i^e)'(0)}{\alpha_j^e f_j^e(x_j^e)} \leq \lambda_i, \text{ whenever } x_i^e = 0, \quad (3)$$

where $g_i^e := f_i^e / (f_i^e)'$. In the pure-cost case, $\lambda_i = c_i'(X_i)$; in the pure-budget case, $X_i = \bar{X}_i$ and λ_i s are the Lagrangian multipliers for budget constraints.

By [Xu et al. \(2022\)](#), if the second-stage equilibrium is interior—i.e., when each player exerts a positive effort in every contest he participates in—then the equilibrium must be unique. As a result, a unique equilibrium always emerges in the pure-cost case, since no player would completely forgo a contest. However, in the pure-budget case, multiple equilibria may arise.

Despite this possibility, such multiplicity does not arise *on the equilibrium path* of any SPNE. We relegate the details to [Lemma A1](#) in the Appendix and only briefly explain the logic. As stated above, multiple equilibria are unique to pure-budget cases, and some players exert zero effort on certain battlefields in these equilibria. However, this is at odds with the organizers' objective of eliciting effort. In the first stage of the game, an organizer will strategically adjust the contest rules to avoid some player's complete concession. This ensures a unique second-stage outcome on the equilibrium path.

Let $\mathcal{X}(\boldsymbol{\alpha})$ denote the set of all second-stage equilibria that correspond to a given $\boldsymbol{\alpha}$. In [Lemma A1\(ii\)](#), we further verify that an equilibrium outcome $(\boldsymbol{\alpha}^*, \mathbf{x}^*(\boldsymbol{\alpha}^*))$ is robust even to equilibrium selection *off the path*: A profile $(\boldsymbol{\alpha}^*, \mathbf{x}^*(\boldsymbol{\alpha}^*))$ can still be sustained as the equilibrium outcome of an SPNE even if multiple equilibria arise off-path, and for a profile of contest rules $\boldsymbol{\alpha}' \neq \boldsymbol{\alpha}^*$, an alternative second-stage equilibrium $\mathbf{x}(\boldsymbol{\alpha}') \neq \mathbf{x}^*(\boldsymbol{\alpha}')$ is selected. As a result, we can describe an SPNE simply by its equilibrium outcome $(\boldsymbol{\alpha}^*, \mathbf{x}^*(\boldsymbol{\alpha}^*))$ without loss of generality.

3 Analysis

Assuming a nondegenerate conflict network with $|\mathcal{E}| \geq 2$, we now examine organizers' choices of contest rules.¹¹ Section 3.1 identifies conditions under which each organizer prefers to fully level her playing field, regardless of the contest rules set by others. Section 3.2 then characterizes the SPNE of the game with all organizers simultaneously setting their respective rules; we establish that all contests are resolved with equal winning probabilities in the unique equilibrium (*even-odds equilibrium*).

We briefly remark on the technical nature of the analysis before we lay out our results. First, given a profile of biases set in the first stage, there is no closed-form solution for the second-stage equilibrium of the networked contest. Organizers' objective functions are only implicitly characterized by equilibrium conditions and cannot be directly used to construct explicit best-response mappings. Second, to establish a hypothetical bias rule as an organizer's optimal choice, we have to verify that it constitutes a *global* best response to others' chosen biases. However, organizers' payoffs are generally non-explicit and non-concave due to cross-battlefield externalities and strategic interdependencies. Standard first-order conditions are thus insufficient for proving global first-stage optimality. The lack of regularity in payoffs poses further challenges when we characterize the equilibrium, which requires that every organizer play the best response.

Our analysis instead compares even-odds contests with nonlocal deviation to verify whether a focused bias rule is optimal or whether an even-odds equilibrium would stand, without characterizing their best responses. In what follows, we provide the sketch of our analysis. For expositional efficiency, we focus on the pure-cost case; the proof for the pure-budget case is similar.

3.1 Level-playing-field Principle in Networked Contests

We present the following preliminaries to ease subsequent analysis and exposition. Fixing $e \in \mathcal{E}$, define $w^e := v^e p_i^e (1 - p_i^e) = v^e p_j^e (1 - p_j^e)$, with $i, j \in \mathcal{N}^e$. A larger w^e implies a more balanced playing field; it is maximized when competition on battlefield e is a perfectly even race, with $p_i^e = p_j^e = 1/2$ and $w^e = v^e/4$. Given the correspondence between w^e and (p_i^e, p_j^e) , we obtain the following.

¹¹The analysis for the case of $|\mathcal{E}| = 1$ is straightforward.

Lemma 2 (Reformulating Organizers' Design Problem) *The tuple $(\alpha^*, x^*(\cdot))$ constitutes an SPNE if and only if for each battlefield $e_0 \in \mathcal{E}$, with $N^{e_0} = \{i_0, j_0\}$, $(w^{e_0})^*$ solves the following maximization problem:*

$$\begin{aligned} & \max_{\{w^{e_0}, (x_1, \dots, x_N)\}} \Lambda^{e_0}(x_{i_0}^{e_0}, x_{j_0}^{e_0}) \\ & \text{s.t. (1) holds for } \alpha^* \text{ in all } e \neq e_0, \\ & \quad \text{(2) holds for all } (i, e) \in \Gamma, \\ & \quad p_{i_0}^{e_0}(1 - p_{i_0}^{e_0}) = w^{e_0}/v^{e_0}. \end{aligned} \tag{4}$$

Lemma 2 establishes an equivalence between the decision problem of an organizer on battlefield e_0 who sets biases α^{e_0} and that of the organizer who chooses w^{e_0} .¹² When $w^{e_0} < v^{e_0}/4$, there exist two probabilities $p_{i_0}^{e_0}$ that satisfy (4); so the mapping between α^{e_0} —which determine $p_{i_0}^{e_0}$ —and w^{e_0} is not one-to-one. However, this nuance does not affect our analysis: A fully balanced contest requires $w^{e_0} = v^{e_0}/4$, which can only be attained when $p_{i_0}^{e_0} = 1/2$. We simply examine whether it is optimal for an organizer to induce $w^{e_0} = v^{e_0}/4$, given the contest rules on other battlefields.

Example 1 demonstrates the complications introduced by the indirect network effect in an organizer contest-rule decision. Specifically, changes to the contest rules on a single battlefield may affect the equilibrium behavior on others, which in turn reflexively influence the original one. This feedback loop casts doubt on the level-playing-field principle.

In what follows, we examine the extent to which the conventional wisdom holds in a network. We introduce two assumptions. The first concerns the impact function $f_i^e(\cdot)$ and the second the network topology Γ .

Assumption 1 *For each $(i, e) \in \Gamma$, f_i^e is 2-concave—i.e., $(f_i^e)^2$ is concave.*

Assumption 2 *The multigraph Γ reduces to a tree after we replace any set of multiple edges in Γ with a single edge.*

Assumption 1 requires sufficient concavity on the impact functions, which corresponds to a sufficiently noisy contest on each battlefield. Intuitively, greater concavity implies that a player's winning odds become less responsive to effort differentials. That is, outcomes depend more on random factors than on players' actions, so changes in efforts have limited

¹²The reformulation was first introduced by Fu and Wu (2020) to characterize an optimal contest under a centralized organizer, and can naturally be adapted to our setting.

influence on winning probabilities and therefore players' marginal benefits of effort. Assumption 2 requires that the network be acyclic. This prevents recursive feedback loops and dampens indirect network effects: While changes on one battlefield may spill over to others and vice versa, the absence of cycles guarantees that these effects cannot feed back to their origin. We obtain the following.

Theorem 1 (*The Level-playing-field Principle in a Conflict Network*) *Suppose that Assumption 1 or 2 holds. Fixing an arbitrary battlefield $e \in \mathcal{E}$ and a bias profile α^{-e} , it is optimal for the battlefield organizer to set α^e to induce $w^e = v^e/4$ —i.e., a fully level playing field, so that players win with equal probability—regardless of the contest rules set for other contests.*

Theorem 1 establishes sufficient conditions under which the level-playing-field principle can be sustained in networked contests, such that an organizer always sets her rules to induce even winning odds irrespective of others' choices.

We now present a sketch proof of the theorem to illustrate the logic underlying our result. For this purpose, we first introduce the term $\frac{d\lambda_i}{dw^{e_0}} \frac{dx_i^e}{dw^{e_0}}$, which we call the *ripple effect* for a player i on a battlefield e . Recall (i) that w^{e_0} measures the degree of competitive balance on battlefield e_0 ; and (ii) $\lambda_i = c'_i(X_i)$ is the marginal cost of effort in the pure-cost case and the Lagrangian multiplier for budget constraints in the pure-budget case.

The two components, $\frac{d\lambda_i}{dw^{e_0}}$ and $\frac{dx_i^e}{dw^{e_0}}$, respectively capture how a change in the competitive balance on e_0 affects player i 's total effort and his effort on an arbitrary battlefield e . When $\frac{d\lambda_i}{dw^{e_0}} \frac{dx_i^e}{dw^{e_0}} < 0$, player i 's total effort (as indicated by λ_i) and his effort on battlefield e move in opposite directions in response to a change in w^{e_0} . Conversely, when $\frac{d\lambda_i}{dw^{e_0}} \frac{dx_i^e}{dw^{e_0}} > 0$, the two forces are aligned.

We present two lemmata about the general properties of ripple effects, with neither requiring Assumption 1 or 2.

Lemma 3 (*Ripple Effects on Competing Battlefields*) *Fix a battlefield $e_0 \in \mathcal{E}$, with $N^{e_0} = \{i_0, j_0\}$. The following holds when competition on battlefield e_0 becomes more balanced (as w^{e_0} increases):*

(i) *The ripple effect for each battlefield $e \neq e_0$ is non-positive, i.e.,*

$$\frac{d\lambda_i}{dw^{e_0}} \frac{dx_i^e}{dw^{e_0}} + \frac{d\lambda_j}{dw^{e_0}} \frac{dx_j^e}{dw^{e_0}} \leq 0, \text{ with } N^e = \{i, j\}. \quad (5)$$

(ii) The ripple effect for each player $i \in \mathcal{N}$ is non-negative, i.e.,

$$\sum_{e \in \mathcal{E}_i} \frac{d\lambda_i}{dw^{e_0}} \frac{dx_i^e}{dw^{e_0}} \geq 0. \quad (6)$$

Lemma 3(i) suggests that balancing the competition on battlefield e_0 —i.e., increasing w^{e_0} —induces a negative aggregate ripple effect for each other contest. In contrast, by Lemma 3(ii), the aggregate ripple effect for each player is positive. The claim in Lemma 3(ii) follows directly from the definition. Recall that $\mathcal{E}_i$ denotes the set of battles that involve player i . The aggregate ripple effect for player i can be expressed as

$$\sum_{e \in \mathcal{E}_i} \frac{d\lambda_i}{dw^{e_0}} \frac{dx_i^e}{dw^{e_0}} = \frac{d\lambda_i}{dw^{e_0}} \sum_{e \in \mathcal{E}_i} \frac{dx_i^e}{dw^{e_0}} = \frac{d\lambda_i}{dw^{e_0}} \frac{dX_i}{dw^{e_0}},$$

where X_i denotes player i 's total effort across all contests. This expression must be positive, because λ_i and X_i move in the same direction under convex cost functions.

We now develop a key thought experiment for our proof. Consider battlefield e_0 , in which players i_0 and j_0 compete under rules that initially yield equal winning probabilities. Recall that our goal is to establish Assumption 1 or 2 as the sufficient condition under which the level-playing-field principle holds in a network. For this purpose, we examine a hypothetical scenario in which the organizer could get better off by unilaterally tilting the competitive balance on battlefield e_0 . The next lemma presents the properties of ripple effects under this circumstance. Together with Lemma 3, it leads to contradiction when either of the two assumptions is satisfied.

Lemma 4 (Ripple Effects in the Deviating Battlefield) Fix a battlefield $e_0 \in \mathcal{E}$, with $\mathcal{N}^{e_0} = \{i_0, j_0\}$, and a bias profile α that leads to $w^{e_0} = v^{e_0}/4$. Suppose that the organizer of battlefield e_0 can benefit from resetting her current contest rule α^{e_0} , given the bias profile α^{-e_0} for others. There always exists a degree of competitive balance $\tilde{w}^{e_0} < v^{e_0}/4$, such that the following holds: Letting all derivatives be evaluated at $w^{e_0} = \tilde{w}^{e_0}$ and assuming $\frac{dx_{i_0}^{e_0}}{dw^{e_0}} \geq \frac{dx_{j_0}^{e_0}}{dw^{e_0}}$ without loss of generality,

- (i) $\frac{dx_{i_0}^{e_0}}{dw^{e_0}} \geq 0 \geq \frac{dx_{j_0}^{e_0}}{dw^{e_0}};$
- (ii) $\frac{w^{e_0}}{\lambda_{j_0}} \frac{d\lambda_{j_0}}{dw^{e_0}} \geq 1 \geq \frac{w^{e_0}}{\lambda_{i_0}} \frac{d\lambda_{i_0}}{dw^{e_0}} \geq 0;$
- (iii) $\frac{dx_{i_0}^{e_0}}{dw^{e_0}} \frac{d\lambda_{i_0}}{dw^{e_0}} \geq 0 \geq \frac{dx_{j_0}^{e_0}}{dw^{e_0}} \frac{d\lambda_{j_0}}{dw^{e_0}};$

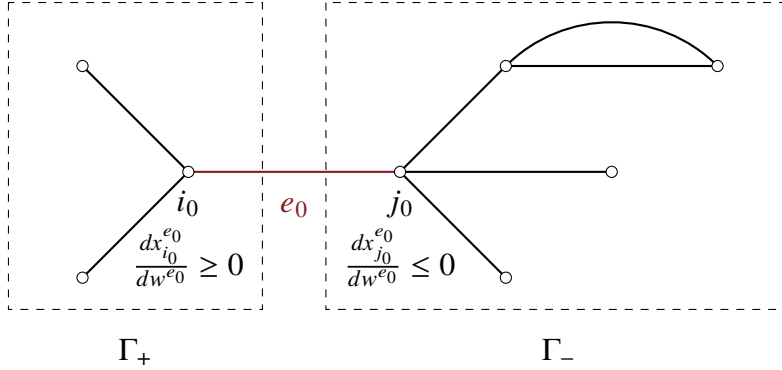


Figure 4: Network structure under Assumption 2.

(iv) for $e' \neq e_0$, if $\mathcal{N}^{e'} = \mathcal{N}^{e_0} = \{i_0, j_0\}$, $\frac{dx_{j_0}^{e'}}{dw^{e_0}} \leq 0$ and $\frac{dx_{j_0}^{e'}}{dw^{e_0}} \frac{d\lambda_{j_0}}{dw^{e_0}} \leq 0$.

In summary, whenever the organizer of battlefield e_0 can benefit from an imbalanced contest, we can identify some value $\tilde{w}^{e_0}$ for which the resulting ripple effects satisfy the properties in Lemma 4. It is worth noting that $\tilde{w}^{e_0}$ does not have to be a profitable deviation from $w^{e_0} = v^{e_0}/4$ for the given contest rules α^{-e_0} on other battlefields.

Lemma 4(iii), which follows directly from (i) and (ii), establishes that the ripple effects at $\tilde{w}^{e_0}$ diverge in sign between players i_0 and j_0 : positive for i_0 but negative for j_0 . Moreover, Lemma 4(iv) demonstrates that when i_0 and j_0 compete on other battlefields, j_0 's ripple effects remain negative on those battlefields as well.

We are now ready to prove Theorem 1. Suppose that the hypothetical scenario described above does exist. We derive a contradiction through Lemmata 3 and 4 under either Assumption 1 or 2.

Case (a): Assumption 1 holds. Let s denote the player with the highest value of $\left| \frac{w^{e_0}}{\lambda_i} \frac{d\lambda_i}{dw^{e_0}} \right|$, i.e., $\left| \frac{w^{e_0}}{\lambda_s} \frac{d\lambda_s}{dw^{e_0}} \right| = \max_{i \in \mathcal{N}} \left| \frac{w^{e_0}}{\lambda_i} \frac{d\lambda_i}{dw^{e_0}} \right|$. By Lemma 4(ii), $\frac{w^{e_0}}{\lambda_{j_0}} \frac{d\lambda_{j_0}}{dw^{e_0}} \geq \frac{w^{e_0}}{\lambda_{i_0}} \frac{d\lambda_{i_0}}{dw^{e_0}} \geq 0$. Therefore, either $s \notin \{i_0, j_0\}$ or we can set $s = j_0$ without loss of generality.

Next, fix a battlefield $e \in \mathcal{E}_s$ and consider the associated ripple effect. The case with $e = e_0$ and $s = j_0$ is straightforward. Lemma 4(iii) leads to $\frac{d\lambda_{j_0}}{dw^{e_0}} \frac{dx_{j_0}^{e_0}}{dw^{e_0}} \leq 0$.

If $e \neq e_0$, we prove in the Appendix that the ripple effect for player s on battlefield e is negative, i.e., $\frac{d\lambda_s}{dw^{e_0}} \frac{dx_s^e}{dw^{e_0}} < 0$. It is worth noting that Assumption 1 plays a critical role in verifying this inequality. Combining the two cases, we obtain $\sum_{e \in \mathcal{E}_s} \frac{d\lambda_s}{dw^{e_0}} \frac{dx_s^e}{dw^{e_0}} \leq 0$, which contradicts Lemma 3(ii).¹³

¹³It can be verified that $\sum_{e \in \mathcal{E}_s} \frac{d\lambda_s}{dw^{e_0}} \frac{dx_s^e}{dw^{e_0}} = 0$ is impossible.

Case (b): Assumption 2 holds. Under Assumption 2, the multigraph Γ decomposes into two connected components upon removal of all edges between i_0 and j_0 . For notational convenience, denote by Γ_+ and Γ_- the two connected components that contain i_0 and j_0 (see Figure 4 for a graphical illustration). Further, let $\mathcal{N}_-$ and $\mathcal{E}_-$ represent the player and battlefield sets in Γ_- , respectively. The sets $\mathcal{N}_+$ and $\mathcal{E}_+$ can be similarly defined.

Following Lemma 4(i), we can assume $\frac{dx_{i_0}^{e_0}}{dw^{e_0}} \geq 0 \geq \frac{dx_{j_0}^{e_0}}{dw^{e_0}}$ without loss of generality. Next, consider the aggregate ripple effect in Γ_- , i.e.,

$$\mathcal{I}_- := \sum_{i \in \mathcal{N}_-} \sum_{e \in \mathcal{E}_i} \frac{d\lambda_i}{dw^{e_0}} \frac{dx_i^e}{dw^{e_0}}.$$

By Lemma 3(ii), the ripple effect for each player across all of his battlefields must be non-negative. Consequently, $\sum_{e \in \mathcal{E}_i} \frac{d\lambda_i}{dw^{e_0}} \frac{dx_i^e}{dw^{e_0}} \geq 0$ for all $i \in \mathcal{N}_-$, which in turn implies that $\mathcal{I}_- \geq 0$. Meanwhile, $\mathcal{I}_-$ can alternatively be expressed as

$$\mathcal{I}_- = \sum_{e \in \mathcal{E}_-} \sum_{i \in \mathcal{N}^e} \frac{d\lambda_i}{dw^{e_0}} \frac{dx_i^e}{dw^{e_0}} + \sum_{e \in \mathcal{E}_{i_0} \cap \mathcal{E}_{j_0}} \frac{d\lambda_{j_0}}{dw^{e_0}} \frac{dx_{j_0}^e}{dw^{e_0}}.$$

From Lemma 3(i), $\sum_{i \in \mathcal{N}^e} \frac{d\lambda_i}{dw^{e_0}} \frac{dx_i^e}{dw^{e_0}} \leq 0$ for each battlefield $e \in \mathcal{E}_-$. By Lemma 4(iii) and (iv), $\frac{d\lambda_{j_0}}{dw^{e_0}} \frac{dx_{j_0}^e}{dw^{e_0}} \leq 0$ for all $e \in \mathcal{E}_{i_0} \cap \mathcal{E}_{j_0}$. We can then conclude $\mathcal{I}_- \leq 0$, which is a contradiction.¹⁴

Theorem 1 establishes that when either Assumption 1 or 2 holds, each organizer's optimal strategy is to maintain equal winning probabilities on her battlefield, independent of other battlefields' contest rules. This revives the level-playing-field principle in a network. The result reveals the respective roles played by contest technologies and prevailing network structure in shaping the equilibrium outcome. Both serve to limit the indirect network effect caused by the change in the competitive balance on one battlefield, so each organizer can focus on the direct local effect when setting her contest rule. We now elaborate on the respective roles they play.

First, Assumption 1 requires strongly concave impact functions $f_i^e(\cdot)$. For simplicity, our discussion focuses on the pure-cost case. The first-order conditions (2) that determine

¹⁴It can be verified that $\mathcal{I}_- = 0$ is impossible.

the equilibrium can be written as

$$\frac{(f_i^e)'}{f_i^e} w^e = \lambda_i.$$

The left-hand side indicates the marginal benefit of a player i 's effort on battlefield e , while the right-hand side gives the marginal cost. Suppose that the competitive balance of a battlefield e_0 varies. Its spillover alters players' marginal benefits and marginal costs of efforts on all other battlefields. This requires that players adjust their efforts everywhere to rebalance their costs and benefits. A strongly concave impact function, as previously noted, limits the impact of a change in effort on winning probability and, therefore, the impact on the marginal benefit of effort.

In the proof for Case (a), we identify a player s with $\left| \frac{w^{e_0}}{\lambda_s} \frac{d\lambda_s}{dw^{e_0}} \right| = \max_{i \in \mathcal{N}} \left| \frac{w^{e_0}}{\lambda_i} \frac{d\lambda_i}{dw^{e_0}} \right|$. The player, when being evaluated at $w^{e_0} = \widetilde{w}^{e_0}$, is most significantly affected by the change on battlefield e_0 in terms of marginal effort cost: $\left| \frac{w^{e_0}}{\lambda_i} \frac{d\lambda_i}{dw^{e_0}} \right|$ is simply the elasticity of λ_i with respect to w^{e_0} . Our analysis verifies that the required rebalancing is impossible for player s : The significant change in marginal effort cost cannot be matched by the limited change in the marginal benefit of the player's effort. Contradiction with Lemma 3(ii) thus ensues, because the aggregate ripple effect is negative, i.e., $\sum_{e \in \mathcal{E}_s} \frac{d\lambda_s}{dw^{e_0}} \frac{dx_s^e}{dw^{e_0}} \leq 0$. This implies that such a $\widetilde{w}^{e_0}$ does not exist, and hence the above hypothetical scenario is impossible. More intuitively, strongly concave impact functions restrain the spillover caused by a change in the competitive balance on a battlefield. This limits the indirect network effect that would feed back to the origin and prevents it from reversing the direct local effect.

Second, Assumption 2 requires an acyclic network structure. As stated in the sketch proof and illustrated in Figure 4, the multigraph Γ can be split into two connected components if the edges between two vertices—i.e., players i_0 and j_0 —are removed. The decomposition enables us to separate the respective impact of a change in the contest rule for battlefield e_0 on players i_0 and j_0 , which paves the way for our analysis. Suppose that Assumption 2 is violated (see, e.g., Figure 1b). Their effort choices would be entangled: They not only engage in the direct competition on battlefield e_0 , but are also connected via various indirect paths that traverse other players and battlefields (see, e.g., Figure 3b). The direct local effect on x_{i_0} and x_{j_0} caused by a change in α^{e_0} triggers indirect network effects that reflexively affect the choices of x_{i_0} and x_{j_0} , which causes complications. An acyclic network severs the linkages and keeps x_{i_0} and x_{j_0} immune to the shock of the indirect network effects.

3.2 Equilibrium

Theorem 1 concerns an individual organizer's optimal choice against arbitrary contest rules set by others. We now turn to the equilibrium that emerges when all organizers choose their rules simultaneously.

We focus on *even-odds equilibrium*, i.e., an equilibrium in which every contest is resolved with equal winning probabilities. The following ensues.

Theorem 2 (Existence and Uniqueness of the Even-odds Equilibrium) *Fix a game $\mathcal{G}$. An even-odds equilibrium always exists. There is a unique profile of contest rules α^{**} that induces equal winning probabilities on every battlefield, and the associated outcome $(\alpha^{**}, x^{**}(\alpha^{**}))$ is unique for pure-strategy SPNEs of game $\mathcal{G}$.*

Theorem 2 establishes that all contests are resolved with equal winning probabilities in equilibrium, without additional requirements on contest technologies or network topology. The result differs from the individual optimality established in Theorem 1: An organizer may not unconditionally prefer a level playing field against given rules on other battlefields, but no imbalanced contest can be sustained in equilibrium when all organizers choose their rules simultaneously.¹⁵

We next present a proof sketch to illustrate the main logic of our analysis. For the sake of brevity, we again focus on the pure-cost case; the proof for the pure-budget case is similar, although it requires additional arguments, which are provided in the Appendix.

Existence of the Even-odds Equilibrium We verify in the Appendix that a unique profile of biases α^{**} exists that induces the even-odds outcome in second-stage contests; it thus remains to verify that α^{**} constitutes a first-stage equilibrium.

Suppose, to the contrary, that α^{**} does not constitute an equilibrium. Then there exists a battlefield $e_0 \in \mathcal{E}$ whose organizer can set $\tilde{\alpha}^{e_0} \neq (\alpha^{e_0})^{**}$ to induce greater effort from at least one player on her battlefield. Let $\tilde{x}$, $\tilde{\lambda}$, and $\tilde{p}$ denote the second-stage equilibrium efforts, marginal effort costs, and winning probabilities under the bias profile $\tilde{\alpha} \equiv (\tilde{\alpha}^{e_0}, (\alpha^{-e_0})^{**})$, respectively. Similarly, we denote equilibrium variables under the bias profile α^{**} with double asterisks. The following lemma predicts how each player's equilibrium total effort changes in response to the organizer of battlefield e_0 's deviation from $(\alpha^{e_0})^{**}$.

¹⁵It is worth noting that the SPNEs of the game may not be unique: Multiple equilibria may emerge on off-equilibrium paths; multiple SPNEs may exist, depending on the prevailing equilibrium selection rules on off-equilibrium paths, although they yield the same equilibrium outcome.

Lemma 5 (Individual Player's Overall Effort Incentive) For each $i \in \mathcal{N}$, $\tilde{\lambda}_i \leq \lambda_i^{**}$.

Under α^{**} , the competition on every battlefield is perfectly balanced. A deviation on battlefield e_0 disrupts not only the competitive balance of e_0 itself but also that of interconnected battlefields. This deviation generates negative overall incentive effects on all players across the network. Lemma 5 confirms and formalizes this logic: Every player's equilibrium marginal effort cost λ_i weakly decreases following the deviation. Given the strict convexity of the effort cost functions, their respective equilibrium total efforts in the contest also weakly decrease accordingly.

The next lemma concerns the spillover of the deviation to players' efforts on other individual battlefields.

Lemma 6 (Individual Player's Incentive on a Battlefield) Fix an arbitrary battlefield $e \neq e_0$ with $\mathcal{N}^e = \{i, j\}$, and suppose $\tilde{\lambda}_i/\lambda_i^{**} \leq \tilde{\lambda}_j/\lambda_j^{**}$. Then $\tilde{x}_i^e \geq (x_i^e)^{**}$.

The deviation on battlefield e_0 triggers complex spillovers to interconnected contests, which alters players' marginal benefits and the marginal costs of efforts on other battlefields. While each player's total effort unambiguously decreases, it is unclear how an individual adjusts effort on a specific battlefield other than e_0 . Intuitively, the universal decline in marginal cost induced by the spillover tends to encourage players to increase effort on other battlefields. The ratio $\tilde{\lambda}_i/\lambda_i^{**}$ captures the impact on player i 's marginal cost: A lower ratio indicates a larger reduction in marginal cost and therefore a more significant increase in effort incentive from the cost side. Lemma 6 confirms that the player who experiences the greater cost reduction will indeed increase effort in response.

We are ready to verify that an even-odds equilibrium will stand. Let s denote the player with the lowest ratio $\tilde{\lambda}_i/\lambda_i^{**}$ among all $i \in \mathcal{N}$. By Lemma 6, following the organizer's deviation on battlefield e_0 , his efforts on all other battlefields weakly increase, i.e., $\tilde{x}_s^e \geq (x_s^e)^{**}$ for all $e \in \mathcal{E}_s$ and $e \neq e_0$. Two possible cases arise: Player s is either involved in the contest on battlefield e_0 or not.

Case (a): $s \notin \mathcal{N}^{e_0}$. Lemma 6 implies that player s 's effort weakly increases in all contests he participates in (he is not involved in the contest on battlefield e_0). Therefore, his total effort weakly increases following the deviation on battlefield e_0 . Meanwhile, Lemma 5 predicts that his total effort would weakly decrease. We can then conclude that his total effort remains unchanged, with $\tilde{\lambda}_s = \lambda_s^{**}$. By definition, player s has the

lowest ratio $\tilde{\lambda}_s/\lambda_s^{**} = 1$; together with Lemma 5, we can establish that the marginal effort costs of all players remain unchanged, i.e., $\tilde{\lambda}_i/\lambda_i^{**} = 1$ for all $i \in \mathcal{N}$.

Let i_0 and j_0 denote the players on battlefield e_0 . Recall that players win with equal probability and thus exert positive effort in all their contests; therefore, the first-order condition (2)—i.e., $v^e p_i^e(\mathbf{x}^e) [1 - p_i^e(\mathbf{x}^e)] = \lambda_i g_i^e(x_i^e)$ —holds for all $i \in \mathcal{N}$ and $e \in \mathcal{E}$ under the bias profile α^{**} . It follows that $\tilde{x}_{i_0}^{e_0} \leq (x_{i_0}^{e_0})^{**}$ and $\tilde{x}_{j_0}^{e_0} \leq (x_{j_0}^{e_0})^{**}$, because $\tilde{p}_{i_0}^{e_0}(1 - \tilde{p}_{i_0}^{e_0}) \leq 1/4 = (p_{j_0}^{e_0})^{**}(1 - (p_{j_0}^{e_0})^{**})$. This contradicts the assumption that setting $\tilde{\alpha}^{e_0}$ is a profitable deviation for the organizer on battlefield e_0 .

Case (b): $s \in \mathcal{N}^{e_0}$. Assume, for contradiction, that setting $\tilde{\alpha}^{e_0}$ is a profitable deviation for the organizer on battlefield e_0 . If $\tilde{x}_s^{e_0} > (x_s^{e_0})^{**}$, then we must have $\tilde{X}_s > X_s^{**}$ by Lemma 6. Consequently, $\tilde{\lambda}_s = c'_s(\tilde{X}_s) > c'_s(X_s^{**}) = \lambda_s^{**}$, which contradicts Lemma 5.

Alternatively, suppose $\tilde{x}_s^{e_0} \leq (x_s^{e_0})^{**}$. Then the other player on battlefield e_0 , denoted j_0 , must exert strictly more effort, i.e., $\tilde{x}_{j_0}^{e_0} > (x_{j_0}^{e_0})^{**} > 0$. Moreover, since $\tilde{x}_s^{e_0} \leq (x_s^{e_0})^{**}$, we have

$$\tilde{g}_s^{e_0} \equiv \frac{f_s^{e_0}(\tilde{x}_s^{e_0})}{(f_s^{e_0})'(\tilde{x}_s^{e_0})} \leq \frac{f_s^{e_0}((x_s^{e_0})^{**})}{(f_s^{e_0})'((x_s^{e_0})^{**})} \equiv (g_s^{e_0})^{**}.$$

Similarly, we can obtain $\tilde{g}_{j_0}^{e_0} > (g_{j_0}^{e_0})^{**}$. Together, these imply

$$\frac{\tilde{\lambda}_s}{\lambda_s^{**}} \geq \frac{\tilde{\lambda}_s \tilde{g}_s^{e_0}}{\lambda_s^{**} (g_s^{e_0})^{**}} = \frac{\tilde{\lambda}_{j_0} \tilde{g}_{j_0}^{e_0}}{\lambda_{j_0}^{**} (g_{j_0}^{e_0})^{**}} > \frac{\tilde{\lambda}_{j_0}}{\lambda_{j_0}^{**}},$$

where the equality follows from the first-order condition (2). This contradicts the definition whereby player s has the lowest $\tilde{\lambda}_i/\lambda_i^{**}$ among all players.

This argument demonstrates that any unilateral deviation from α^{**} would not render an organizer better off. Hence, the bias profile α^{**} constitutes a first-stage equilibrium of the game $\mathcal{G}$. We thus establish that leveling the playing field can always be sustained as an SPNE, even in the networked environment.

Next, we verify that no imbalanced contests can be sustained in equilibrium, so every SPNE must be an even-odds equilibrium.

Uniqueness of the Equilibrium Outcome Suppose that an equilibrium exists and involves at least one imbalanced contest. Our analysis examines how players' efforts in a

least balanced contest—i.e., one with the largest gap between the two players' equilibrium winning probabilities—would respond if its organizer deviated and leveled her playing field.

Specifically, the least balanced contest must minimize $p_i^e p_j^e$ among all $e \in \mathcal{E}$ in the hypothetical equilibrium, and we denote this battlefield e_0 with a slight abuse of notation, with $\mathcal{N}^{e_0} = \{i, j\}$. Let the organizer of battlefield e_0 reset her bias to induce equal winning probabilities while all other rules remain fixed.¹⁶ As in the analysis of equilibrium existence, we use tildes to indicate equilibrium variables after the deviation. Set $q := 1/(4p_i^{e_0} p_j^{e_0}) > 1$. Recall $w^e = v^e p_i^e p_j^e$. The following must hold:

$$\frac{\tilde{w}^e}{w^e} = \frac{\tilde{p}_i^e \tilde{p}_j^e}{p_i^e p_j^e} \leq \frac{1}{4p_i^e p_j^e} \leq q \quad \text{for every } e \in \mathcal{E},$$

with $\tilde{w}^{e_0}/w^{e_0} = q$ by definition. Clearly, following this deviation, battlefield e_0 exhibits the largest increase from w^e to $\tilde{w}^e$ as a proportion of $\tilde{w}^e$ among all $e \in \mathcal{E}$.

Further, by the definition of w^e , the first-order condition (2) can be written as

$$w^e = \lambda_i g_i^e(x_i^e).$$

We are ready to present the next two key lemmata. The first lemma caps the proportional increase in each player's marginal effort cost on this battlefield.

Lemma 7 (Overall Effort Incentive for Individual Players on Battlefield e_0) *In the pure-cost case, leveling the playing field on battlefield e_0 yields $\tilde{\lambda}_s/\lambda_s < q$ for each $s \in \mathcal{N}^{e_0}$.*

To prove Lemma 7, suppose, for contradiction, that the marginal effort cost of one player on this battlefield, say s , increases relative to λ_s by a factor of at least q , i.e., $\tilde{\lambda}_s/\lambda_s \geq q$. Since $\tilde{w}^e/w^e \leq q$ on every battlefield, the first-order condition would force his efforts on all battlefields to weakly decrease, i.e., $g_s^e(\tilde{x}_s^e) \leq g_s^e(x_s^e)$ for all $e \in \mathcal{E}_s$, given that $g_s^e(\cdot)$ is strictly increasing. His total effort must then weakly decrease. As a result, $\tilde{\lambda}_s/\lambda_s = c'_s(\tilde{X}_s)/c'_s(X_s) \leq 1$, which contradicts $\tilde{\lambda}_s/\lambda_s \geq q > 1$.

Lemmata 7 and 5 rely on the same logic: An increase in marginal cost cannot align with a weak decrease in effort on every battlefield. In Lemma 5, a deviation from the even-odds bias profile cannot increase w^e on any battlefield, so every player's marginal effort cost weakly decreases. In Lemma 7, leveling the least balanced contest raises w^{e_0} by a factor of $q > 1$. Marginal cost may increase accordingly, but its proportional increase must be smaller than that in w^{e_0} .

¹⁶Lemma A6 in the Appendix establishes that such a bias exists.

The next lemma concerns how this change in marginal cost affects each player's effort on battlefield e_0 itself.

Lemma 8 (*Effort Response on the Least Balanced Battlefield*) *In the pure-cost case, leveling the playing field on battlefield e_0 strictly increases both players' efforts on this battlefield, i.e., $\tilde{x}_s^{e_0} > x_s^{e_0}$ for each $s \in \mathcal{N}^{e_0}$.*

To see this, rearrange the first-order condition, which yields

$$\frac{g_s^{e_0}(\tilde{x}_s^{e_0})}{g_s^{e_0}(x_s^{e_0})} = \frac{q}{\tilde{\lambda}_s/\lambda_s} > 1, \quad \text{for } s \in \mathcal{N}^{e_0},$$

where the inequality follows from Lemma 7. Since $g_s^{e_0}$ is strictly increasing, both players' efforts on e_0 strictly increase.

Lemma 8 verifies that both players on e_0 exert more effort after the deviation. Its organizer strictly benefits from this deviation and is always tempted by this profitable deviation, which dissolves this hypothetical equilibrium. As a result, all contests must be fully balanced in any SPNE: This rules out any equilibrium that involves an imbalanced contest and establishes the uniqueness of an even-odds outcome.

Intuitively, leveling the least balanced contest generates the largest proportional increase in effort incentives among all battlefields. Although the deviation may also raise players' marginal effort costs, Lemma 7 demonstrates that any such increase is proportionally smaller than the increase in w^{e_0} , which requires effort to increase in order to rebalance marginal benefit against marginal cost.

This argument sheds further light on Example 1, which demonstrates that leveling an arbitrary battlefield could reduce a player's effort against an arbitrary bias profile on other battlefields off the equilibrium path.¹⁷ The discussion here implies that this is possible only if the concerned contest is not the least balanced one. In this example, when $p_1^a = 1/4$, the other two contests yield winning probabilities $p_2^b \approx 0.0461$ and $p_3^c \approx 0.0070$, respectively; both b and c are less balanced than a . Leveling a reduces player 1's effort from approximately 21.211 to 21.014. In contrast, battlefield c here is the least balanced, and its organizer can benefit if she levels her playing field, holding fixed the rules on a and b : This would raise x_3^c from about 0.050 to 0.444 and x_1^c from about 0.789 to 13.62.

¹⁷Although the analysis here focuses on the pure-cost case, the weak-inequality version of Lemma 8 also holds in the pure-budget case when the initial second-stage equilibrium is interior. See the discussion following the proof of Lemma 8 in the Appendix for details and proof.

4 Extensions

We explore two extensions. Section 4.1 considers organizers who set contest rules in a predetermined order, and Section 4.2 examines whether decentralized rule-setting replicates the contest rules a central organizer would choose.

4.1 Sequential Moves

Consider an alternative setting in which organizers act sequentially. Specifically, organizers are partitioned into T groups, and the first stage of the game unfolds over $T \geq 1$ period(s) accordingly. In each period, one group of organizers chooses their contest rules simultaneously, with their choices observable to later movers. Fix $t \in \{1, \dots, T\}$, and let $\mathcal{E}^t$ denote the set of battlefields whose organizers act in period t . Further, let $\alpha^{<t} := (\alpha^e)_{e \in \mathcal{E}^\tau, 1 \leq \tau \leq t-1}$ denote the bias profile chosen prior to period t . The following result ensues.

Theorem 3 (*Equilibrium with Sequential Moves*) *Suppose that Assumption 1 or 2 holds. Then an SPNE exists, and its equilibrium outcome is unique. Every organizer perfectly levels the playing field. That is, for each $t \in \{1, \dots, T\}$, each bias profile $\alpha^{<t}$, and each battlefield $e \in \mathcal{E}^t$, the organizer of battlefield e chooses $\alpha^e(\alpha^{<t})$ such that $p_i^e = p_j^e = 1/2$. As a result, (α^{**}, x^{**}) is the unique equilibrium outcome.*

By Theorem 3, the conditions in Theorem 1 ensure that (α^{**}, x^{**}) remains the unique equilibrium outcome under sequential moves. Thus, whenever the level-playing-field principle holds regardless of other organizers' choices, the equilibrium outcome is invariant to the timing of moves.

To see the logic, consider a simple example with two battlefields, as illustrated in Figure 1c. Let the organizer of battlefield a move first. The first-stage game can be solved by backward induction. The organizer on battlefield b —the second mover—will fully balance the playing field regardless of the contest rule for battlefield a , as implied by Theorem 1.

Now consider the organizer of battlefield a , the first mover. She anticipates that the late mover will fully level the playing field. In other words, the contest rule for battlefield b is endogenously pegged to her choice for battlefield a in order to induce a fully balanced competition on battlefield b . This dynamic strategic linkage neutralizes the indirect network

effect caused by her own choice, which allows her to focus on the direct local effect. Thus, she would also set a contest rule to fully level the playing field on her battlefield.

4.2 Centralized Contest Design

We now let a central organizer set contest rules $\alpha = \{\alpha^e\}_{e \in \mathcal{E}}$ for all battlefields. Upon observing the contest rules, players simultaneously exert their efforts in the second stage. We focus on the pure-cost case.

We assume that the organizer maximizes an objective function determined by the profile of players' individual total efforts, i.e., $\Lambda(\mathbf{x}) := \Lambda(X_1, X_2, \dots, X_N)$. The central organizer is only concerned about each player's individual total effort X_i instead of his entire effort profile $\mathbf{x}_i$. This assumption ensures the existence of an optimum.¹⁸ Moreover, we assume that the objective function $\Lambda(\mathbf{x})$ is strictly increasing in X_i for all $i \in \mathcal{N}$. That is, the organizer strictly benefits from each player's effort contribution. A simple example is the aggregate effort over the network—i.e., $\Lambda(\mathbf{x}) = \sum_{i \in \mathcal{N}} X_i$. The following result ensues.

Theorem 4 (Centralized Contest Design within a Network) *In the pure-cost case, suppose that the central organizer's objective function $\Lambda(\mathbf{x}) = \Lambda(X_1, \dots, X_N)$ is strictly increasing in X_i for all $i \in \mathcal{N}$. The optimal profile of contest rules is unique: The organizer sets $\alpha = \alpha^{**}$ and players win with equal probability on every battlefield.*

Theorem 4 shows that a central organizer always benefits from leveling the playing fields. Her choices of α internalize the externalities each battlefield's contest rule imposes on the others. Leveling the playing field maximizes every player's effort incentives. Recall that α^{**} is the unique bias profile that induces even winning odds on all battlefields. Theorem 2 shows that the same profile is the unique equilibrium rule profile under decentralized design. Together, the two results imply the following.

Corollary 1 (Centralization versus Decentralization) *In the pure-cost case, the optimal contest rules for the central designer, α^{**} , coincide with the unique equilibrium rule profile of the decentralized contest design game in which the organizer on each battlefield maximizes total effort in her own battle.*

¹⁸Otherwise, an optimum may not exist. To see this, consider a setting with $\mathcal{N} = \{1, 2\}$, $\mathcal{E} = \{a, b\}$, and $\Gamma = \{(1, a), (1, b), (2, a), (2, b)\}$, as in Figure 1c. Set $f_i^e(x_i^e) = x_i^e$ for all $(i, e) \in \Gamma$, $c_i(X_i) = (X_i)^2$, and $(v^a, v^b) = (1, 1)$. Suppose that the organizer's objective is to maximize total effort on battlefield a , i.e., $\Lambda = x_1^a + x_2^a$. We can verify that no optimal biases exist. The organizer can generate total effort arbitrarily close to the supremum—equal to $1/\sqrt{2}$ —by setting $\alpha^a = (1/2, 1/2)$ and $\alpha^b = (\varepsilon, 1 - \varepsilon)$, where ε is a positive parameter tending to zero.

Corollary 1 illustrates the potential efficiency of delegation in contest design. The central organizer’s interests are not aligned with the organizers in our original decentralized contest design game. Nevertheless, Corollary 1 predicts that if the central organizer simply delegates the task of setting contest rules to a set of self-interested agents—each managing a single battlefield—the unique equilibrium outcome replicates the centrally determined optimum.

5 Conclusion

In this paper, we analyze a game of decentralized contest design in which multiple players engage in pairwise contests within a network. Each battlefield is managed by an organizer who sets contest rules to incentivize effort supply for her own contest.

We investigate the SPNE of the game and examine the extent to which the canonical level-playing-field principle established in standalone contests continues to hold in this networked setting, given the complex externalities that arise when contest rules are set independently for individual battlefields. We first identify sufficient conditions under which the level-playing-field principle remains valid—i.e., conditions under which each organizer unconditionally prefers a fully balanced contest. We then show that an even-odds equilibrium—in which the contest on every battlefield is won by either side with equal probability—always exists and is the unique equilibrium outcome, even when these conditions are not satisfied and a fully balanced contest is not necessarily an organizer’s unconditional best response.

Our paper is the first to analyze decentralized contest design in a networked context. The analysis sheds new light on the game-theoretic structure of networked contest games and contributes novel insights to our understanding of the conventional wisdom in the contest literature of leveling the playing field.

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Appendix: Proofs

Lemma A1 (Equilibrium Property) Let $(\alpha^*, x^*(\cdot))$ be an SPNE. Then:

- (i) The second-stage equilibrium under α^* is unique.
- (ii) For any selection $x(\cdot)$ satisfying $x(\alpha) \in X(\alpha)$ for every α , $(\alpha^*, x(\cdot))$ is also an SPNE.

Proof of Lemma A1. By Xu, Zenou, and Zhou (2022), the second-stage equilibrium is unique for any first-stage bias profile in the pure-cost case. It therefore suffices to consider the pure-budget case. We use the following intermediate results.

Lemma A2 Fix α and $e \in \mathcal{E}$, with $N^e = \{i, j\}$. If there exists $x \in X(\alpha)$ such that $x_j^e > 0$, then $(x_i^e)' = x_i^e$ for all $x' \in X(\alpha)$. As a corollary, fixing $x \in X(\alpha)$, if $x_i^e = 0$ for some (i, e) —which implies $x_j^e > 0$ —then $(x_i^e)' = 0$ for all $x' \in X(\alpha)$.

Proof. Suppose, to the contrary, that there exists $e_0 \in \mathcal{E}$, with $N^{e_0} = \{i_0, j_0\}$, and two second-stage equilibria $x', x'' \in X(\alpha)$, with $(x_{i_0}^{e_0})' \neq (x_{i_0}^{e_0})''$ and $(x_{j_0}^{e_0})' > 0$.

Note that equilibrium requires that $\frac{\partial \pi_i(x')}{\partial x_i^e} \leq \lambda_i(x')$, where the inequality holds with equality if $(x_i^e)' > 0$. Similarly, $\frac{\partial \pi_i(x'')}{\partial x_i^e} \leq \lambda_i(x'')$, where the inequality holds with equality if $(x_i^e)'' > 0$. Together, these indicate that

$$\sum_{(i,e) \in \Gamma} ((x_i^e)' - (x_i^e)'') \times \frac{\partial \pi_i(x')}{\partial x_i^e} \geq 0, \text{ and } \sum_{(i,e) \in \Gamma} ((x_i^e)' - (x_i^e)'') \times \frac{\partial \pi_i(x'')}{\partial x_i^e} \leq 0.$$

Combining the above two inequalities yields

$$\sum_{(i,e) \in \Gamma} ((x_i^e)' - (x_i^e)'') \times \left[\frac{\partial \pi_i(x')}{\partial x_i^e} - \frac{\partial \pi_i(x'')}{\partial x_i^e} \right] \geq 0. \quad (\text{A1})$$

Define $x(z) := zx' + (1-z)x''$, with $z \in [0, 1]$, and

$$\omega(z) := \sum_{(i,e) \in \Gamma} ((x_i^e)' - (x_i^e)'') \times \frac{\partial \pi_i(x(z))}{\partial x_i^e}.$$

Evidently, (A1) is equivalent to $\omega(1) \geq \omega(0)$. Meanwhile, $\omega(z)$ can be rewritten as

$$\begin{aligned}\omega(z) &= \sum_{e \in \mathcal{E}, N^e = \{i, j\}} v^e \times \left[((x_i^e)' - (x_i^e)'') \times \frac{\partial p_i^e(\mathbf{x}(z))}{\partial x_i^e} + ((x_j^e)' - (x_j^e)'') \times \frac{\partial p_j^e(\mathbf{x}(z))}{\partial x_j^e} \right] \\ &= \sum_{e \in \mathcal{E}, N^e = \{i, j\}} v^e \times \left[((x_i^e)' - (x_i^e)'') \times \frac{\partial p_i^e(\mathbf{x}(z))}{\partial x_i^e} - ((x_j^e)' - (x_j^e)'') \times \frac{\partial p_i^e(\mathbf{x}(z))}{\partial x_j^e} \right],\end{aligned}$$

which yields that

$$\begin{aligned}\omega'(z) &= \sum_{e \in \mathcal{E}, N^e = \{i, j\}} v^e \times \left[((x_i^e)' - (x_i^e)'')^2 \times \frac{\partial^2 p_i^e(\mathbf{x}(z))}{(\partial x_i^e)^2} \right. \\ &\quad \left. + ((x_j^e)' - (x_j^e)'')^2 \times \frac{\partial^2 p_j^e(\mathbf{x}(z))}{(\partial x_j^e)^2} \right].\end{aligned}$$

Recall that $\frac{\partial p_i^e}{\partial x_i^e} = p_i^e(1 - p_i^e) \frac{(f_i^e)'}{f_i^e}$, which in turn implies that

$$\begin{aligned}\frac{\partial^2 p_i^e}{(\partial x_i^e)^2} &= (1 - 2p_i^e)p_i^e(1 - p_i^e) \left[\frac{(f_i^e)'}{f_i^e} \right]^2 + p_i^e(1 - p_i^e) \frac{(f_i^e)'' f_i^e - (f_i^e)'(f_i^e)'}{(f_i^e)^2} \\ &= \frac{p_i^e(1 - p_i^e)}{(f_i^e)^2} \times \left[(1 - 2p_i^e)(f_i^e)'(f_i^e)' + (f_i^e)'' f_i^e - (f_i^e)'(f_i^e)' \right] \\ &= \frac{\alpha_i^e \alpha_j^e f_j^e}{(\alpha_i^e f_i^e + \alpha_j^e f_j^e)^3} \times \left[(f_i^e)''(\alpha_i^e f_i^e + \alpha_j^e f_j^e) - 2\alpha_i^e (f_i^e)'(f_i^e)' \right] \leq 0,\end{aligned}$$

where the equality holds if and only if $f_j^e = 0$, or equivalently, $x_j^e = 0$. Similarly, we have that $\frac{\partial^2 p_j^e}{(\partial x_j^e)^2} \leq 0$. Together, these indicate that $\omega'(z) \leq 0$. Moreover, from $(x_{i_0}^{e_0})' \neq (x_{i_0}^{e_0})''$ and the postulated $(x_{j_0}^{e_0})' > 0$, we have that

$$((x_{i_0}^{e_0})' - (x_{i_0}^{e_0})'')^2 \times \frac{\partial^2 p_{i_0}^{e_0}(\mathbf{x}(z))}{(\partial x_{i_0}^{e_0})^2} < 0.$$

Therefore, $\omega'(z) < 0$ for all $z \in (0, 1)$, which implies that $\omega(1) < \omega(0)$. This contradicts (A1). ■

Lemma A3 Fix α , $\mathbf{x}' \in X(\alpha)$, and battlefield e_1 , with $N^{e_1} = \{i_0, j_1\}$. If $(x_{j_1}^{e_1})' = 0$, then for any $\mathbf{x}'' \in X(\alpha)$, we have $(x_{j_u}^{e_u})'' = 0$ for all $e_u \in \mathcal{E}_{i_0}$, with $N^{e_u} = \{i_0, j_u\}$. Moreover,

$$\lambda_{i_0}(\mathbf{x}'') = 0.$$

Proof. Suppose $(x_{j_1}^{e_1})' = 0$. By Lemma A2, $(x_{j_1}^{e_1})'' = 0$ for all $\mathbf{x}'' \in X(\alpha)$. Evidently, each battlefield has at least one active player in the second-stage equilibrium, which implies $(x_{i_0}^{e_1})'' > 0$.

Suppose, to the contrary, that $(x_{j_u}^{e_u})'' > 0$ for some $e_u \in \mathcal{E}_{i_0}$, with $\mathcal{N}^{e_u} = \{i_0, j_u\}$. Then player i_0 has a profitable deviation. Specifically, suppose that he slightly reduces $(x_{i_0}^{e_1})''$ and increases $(x_{i_0}^{e_u})''$ by the same amount. This does not change his winning odds on battlefield e_1 but strictly increases his winning odds on battlefield e_u . Therefore, for all $\mathbf{x}'' \in X(\alpha)$, $(x_{j_u}^{e_u})'' = 0$ for all $e_u \in \mathcal{E}_{i_0}$.

It remains to show $\lambda_{i_0}(\mathbf{x}'') = 0$. Thus far, we have shown that $(x_{j_1}^{e_1})'' = 0$ and $(x_{i_0}^{e_1})'' > 0$; together with (1) and (2), we can conclude that $\lambda_{i_0}(\mathbf{x}'') = 0$. ■

Lemma A4 Fixing α , $\lambda_i(\mathbf{x}') = \lambda_i(\mathbf{x}'') =: \lambda_i$ for all $\mathbf{x}', \mathbf{x}'' \in X(\alpha)$ and all $i \in \mathcal{N}$.

Proof. Evidently, each battlefield has at least one active player in the second-stage equilibrium. There are two cases:

- (a) player i has an inactive opponent in some battlefield in one equilibrium. In this case, by Lemma A3, we have that $\lambda_i = 0$ across all equilibria.
- (b) player i has an active opponent in all his battlefields in all equilibria. Note that, for the pure-budget case, it is impossible that player i remains inactive in all his battlefields in any equilibrium. This implies that in any equilibrium, player i must be active in at least one battlefield. Fix an equilibrium and consider one such battlefield—say battlefield e , with $\mathcal{N}^e = \{i, j\}$. It follows immediately that $x_i^e, x_j^e > 0$. By Lemma A2, both players i and j exert the same effort—i.e., x_i^e and x_j^e —in all equilibria. Note that λ_i is uniquely determined by (2) once we know x_i^e and x_j^e . This implies that λ_i must be the same across all equilibria.

This concludes the proof. ■

We now prove part (i) of Lemma A1. Let $(\alpha^*, \mathbf{x}^*(\cdot))$ be an SPNE and let $\mathbf{x}' := \mathbf{x}^*(\alpha^*)$. Suppose, to the contrary, that there exists $\mathbf{x}'' \neq \mathbf{x}'$ such that $\mathbf{x}'' \in X(\alpha^*)$. Then there exists $i_0 \in \mathcal{N}$ and $e_1 \in \mathcal{E}_{i_0}$ such that $(x_{i_0}^{e_1})' \neq (x_{i_0}^{e_1})''$. By Lemma A2, $(x_{j_1}^{e_1})' = 0$ and $(x_{j_1}^{e_1})'' = 0$, where $j_1 \in \mathcal{N}^{e_1}$ and $j_1 \neq i_0$; otherwise, player i_0 would choose the same effort in the battlefield across all equilibria, which contradicts $(x_{i_0}^{e_1})' \neq (x_{i_0}^{e_1})''$. Further, by Lemma A3, $(x_{j_u}^{e_u})' = (x_{j_u}^{e_u})'' = 0$ for all $e_u \in \mathcal{E}_{i_0}$, with $\mathcal{N}^{e_u} = \{i_0, j_u\}$, and $\lambda_{i_0}(\mathbf{x}') = \lambda_{i_0}(\mathbf{x}'') = 0$.

Consider an arbitrary battlefield $e_u \in \mathcal{E}_{i_0}$ and player i_0 's opponent, player j_u . By Lemma A4, λ_{j_u} is the same across all equilibria; together with the fact that $(x_{j_u}^{e_u})' = (x_{j_u}^{e_u})'' = 0$, we can conclude that

$$\lambda_{j_u} \geq v^{e_u} \frac{(\alpha_{j_u}^{e_u})^* (f_{j_u}^{e_u})'(0)}{(\alpha_{i_0}^{e_u})^* f_{i_0}^{e_u}(x_{i_0}^{e_u})},$$

where $x_{i_0}^{e_u}$ is player i_0 's equilibrium effort on battlefield e_u . By the monotonicity of $f_{i_0}^{e_u}(\cdot)$, there exists a unique $x_{i_0}^{e_u}$ such that the above inequality holds with equality, which we denote $\hat{x}_{i_0}^{e_u}$. It follows immediately that $\hat{x}_{i_0}^{e_u} \leq (x_{i_0}^{e_u})'$ and $\hat{x}_{i_0}^{e_u} \leq (x_{i_0}^{e_u})''$, which in turn implies that

$$\sum_{e_u \in \mathcal{E}_{i_0}} \hat{x}_{i_0}^{e_u} \leq \sum_{e_u \in \mathcal{E}_{i_0}} (x_{i_0}^{e_u})' \leq \bar{X}_{i_0} \text{ and } \sum_{e_u \in \mathcal{E}_{i_0}} \hat{x}_{i_0}^{e_u} \leq \sum_{e_u \in \mathcal{E}_{i_0}} (x_{i_0}^{e_u})'' \leq \bar{X}_{i_0}.$$

If $\sum_{e_u \in \mathcal{E}_{i_0}} \hat{x}_{i_0}^{e_u} = \bar{X}_{i_0}$, then all inequalities above hold with equality, which implies that $(x_{i_0}^{e_u})' = (x_{i_0}^{e_u})'' = \hat{x}_{i_0}^{e_u}$ for all $e_u \in \mathcal{E}_{i_0}$. This contradicts the postulated $(x_{i_0}^{e_1})' \neq (x_{i_0}^{e_1})''$. If $\sum_{e_u \in \mathcal{E}_{i_0}} \hat{x}_{i_0}^{e_u} < \bar{X}_{i_0}$, we consider the following alternative strategy for player i_0 : $x_{i_0} = (x_{i_0}^{e_1}, (\hat{x}_{i_0}^{e_u})_{e_u \in \mathcal{E}_{i_0} \setminus \{e_1\}})$, where $x_{i_0}^{e_1} = \bar{X}_{i_0} - \sum_{e_u \in \mathcal{E}_{i_0} \setminus \{e_1\}} \hat{x}_{i_0}^{e_u} \geq 0$. It follows immediately that $x_{i_0}^{e_1} \geq (x_{i_0}^{e_1})'$ and $x_{i_0}^{e_1} \geq (x_{i_0}^{e_1})''$. Recall the postulated $(x_{i_0}^{e_1})' \neq (x_{i_0}^{e_1})''$. We can thus assume $x_{i_0}^{e_1} > (x_{i_0}^{e_1})'$ without loss.

Set $\alpha^{e_1} = (\alpha_{i_0}^{e_1}, \alpha_{j_1}^{e_1})$ such that $\lambda_{j_1} = v^{e_1} \frac{\alpha_{j_1}^{e_1} (f_{j_1}^{e_1})'(0)}{\alpha_{i_0}^{e_1} f_{i_0}^{e_1}(x_{i_0}^{e_1})}$. It is straightforward to verify that $(x_{i_0}, (x_{-i_0})')$ satisfies (2) and (3) in Lemma 1 under $(\alpha^{e_1}, \{(\alpha^e)^*\}_{e \in \mathcal{E} \setminus \{e_1\}})$, and thus constitutes a second-stage equilibrium. Following a similar argument as in the previous analysis, we can conclude that for any second-stage equilibrium under this bias profile, player i_0 's equilibrium effort on battlefield e_1 is $x_{i_0}^{e_1}$. Note that $x_{i_0}^{e_1} > (x_{i_0}^{e_1})'$. Therefore, fixing $\{(\alpha^e)^*\}_{e \in \mathcal{E} \setminus \{e_1\}}$, the organizer of battlefield e_1 is better off by deviating from $(\alpha^{e_1})^*$ to α^{e_1} , which contradicts the postulate that α^* constitutes a first-stage equilibrium.

Next, we prove part (ii) of Lemma A1. Suppose, to the contrary, that there exists $x(\cdot)$, with $x(\alpha) \in \mathcal{X}(\alpha)$, such that $(\alpha^*, x(\cdot))$ is not an SPNE. Therefore, fixing $x(\cdot)$ and $(\alpha^{-e})^*$, there exists a battlefield e such that setting $(\alpha^e)^*$ is suboptimal to its organizer. Denote the most profitable deviation for the organizer $(\alpha^e)'$ and let $\alpha' = ((\alpha^e)', (\alpha^{-e})^*)$. By assumption, $(\alpha^*, x^*(\cdot))$ is an SPNE. Therefore, the organizer of battlefield e is better off in $x^*(\alpha^*)$ than in $x^*(\alpha')$. Thus, she is strictly better off in $x(\alpha')$ than in $x^*(\alpha')$.

By Lemma A2, it is impossible that the two players on battlefield e are active. Otherwise, the equilibrium efforts on battlefield e under $x(\alpha')$ coincide with those under $x^*(\alpha')$.

Meanwhile, it is evident that at least one player exerts positive effort on each battlefield in the second stage. Therefore, one player remains active and the other inactive on battlefield e , and the active player's effort is strictly higher in $\mathbf{x}(\alpha')$ than in $\mathbf{x}^*(\alpha')$. Following a similar argument as in the proof of part (i), this is impossible given that $(\alpha^e)'$ is the most profitable deviation for the organizer of battlefield e . This concludes the proof. ■

Proof of Lemma 2. It suffices to show that for any α^{e_0} , the organizer of battlefield e_0 can choose w^{e_0} to induce the same equilibrium effort profile $\mathbf{x}$ and vice versa.

First, fix an arbitrary α^{e_0} and a second-stage equilibrium $\mathbf{x}^*$, which yields $(w^{e_0})^*$. Evidently, the organizer can set $w^{e_0} = (w^{e_0})^*$ to induce $\mathbf{x}^*$. Second, fixing an arbitrary $w^{e_0} \leq v^{e_0}/4$ —which induces $\mathbf{x}^*$ —the winning probability on battlefield e_0 can be solved from (4). The corresponding biases α^{e_0} can then be derived from (1). ■

Lemma A5 Fix a battlefield $e_0 \in \mathcal{E}$, with $\mathcal{N}^{e_0} = \{i_0, j_0\}$. The following statements hold in the second-stage equilibrium:

(i) Fix $e \neq e_0$, with $\mathcal{N}^e = \{i, j\}$. If $x_i^e, x_j^e > 0$, then

$$\frac{dx_i^e}{dw^{e_0}} = -\frac{w^e m_i^e}{\lambda_i} \times \frac{[1 - (2p_i^e - 1)m_j^e] \frac{1}{\lambda_i} \frac{d\lambda_i}{dw^{e_0}} + (2p_i^e - 1)m_j^e \frac{1}{\lambda_j} \frac{d\lambda_j}{dw^{e_0}}}{1 + (m_i^e - m_j^e)(p_i^e - p_j^e)}, \quad (\text{A2})$$

where $g_i^e := g_i^e(x_i^e)$ and $m_i^e := \frac{(f_i^e)'(x_i^e)^2}{(f_i^e)'(x_i^e)^2 - f_i^e(x_i^e)(f_i^e)''(x_i^e)} \in [0, 1]$.

(ii) For battlefield e_0 , we have that

$$\frac{dx_{i_0}^{e_0}}{dw^{e_0}} = g_{i_0}^{e_0} m_{i_0}^{e_0} \left[\frac{1}{w^{e_0}} - \frac{1}{\lambda_{i_0}} \frac{d\lambda_{i_0}}{dw^{e_0}} \right] \text{ and } \frac{dx_{j_0}^{e_0}}{dw^{e_0}} = g_{j_0}^{e_0} m_{j_0}^{e_0} \left[\frac{1}{w^{e_0}} - \frac{1}{\lambda_{j_0}} \frac{d\lambda_{j_0}}{dw^{e_0}} \right]. \quad (\text{A3})$$

(iii) For each $i \in \mathcal{N}$, we have that

$$\delta_i \frac{d\lambda_i}{dw^{e_0}} = \sum_{e \in \mathcal{E}_i} \frac{dx_i^e}{dw^{e_0}}, \quad (\text{A4})$$

where $\delta_i = 0$ in the pure-budget case and $\delta_i = \frac{1}{c_i''(x_i)}$ in the pure-cost case.

Proof. We first prove part (i). Fix $e \neq e_0$, with $\mathcal{N}^e = \{i, j\}$. By (1), we have $\frac{p_i^e}{1-p_i^e} = \frac{\alpha_i^e f_i^e}{\alpha_j^e f_j^e}$.

Taking the logarithm of both sides and differentiating with respect to w^{e_0} gives

$$\frac{1}{p_i^e(1-p_i^e)} \frac{dp_i^e}{dw^{e_0}} = \frac{1}{g_i^e} \frac{dx_i^e}{dw^{e_0}} - \frac{1}{g_j^e} \frac{dx_j^e}{dw^{e_0}}. \quad (\text{A5})$$

Suppose $x_i^e, x_j^e > 0$ at w^{e_0} . Then, by continuity, $x_i^e, x_j^e > 0$ in a neighborhood of w^{e_0} . Therefore, the first-order condition (2) holds in the neighborhood, which gives $v^e p_i^e(1-p_i^e) = \lambda_i g_i^e = \lambda_j g_j^e$. Again, taking the logarithm of both sides and differentiating with respect to w^{e_0} gives

$$\frac{1-2p_i^e}{p_i^e(1-p_i^e)} \frac{dp_i^e}{dw^{e_0}} = \frac{1}{\lambda_i} \frac{d\lambda_i}{dw^{e_0}} + \frac{1}{g_i^e m_i^e} \frac{dx_i^e}{dw^{e_0}} = \frac{1}{\lambda_j} \frac{d\lambda_j}{dw^{e_0}} + \frac{1}{g_j^e m_j^e} \frac{dx_j^e}{dw^{e_0}}. \quad (\text{A6})$$

Combining (A5) and (A6), we have that

$$\begin{aligned} \frac{1}{\lambda_i} \frac{d\lambda_i}{dw^{e_0}} &= \frac{1-2p_i^e}{p_i^e(1-p_i^e)} \frac{dp_i^e}{dw^{e_0}} - \frac{1}{g_i^e m_i^e} \frac{dx_i^e}{dw^{e_0}} \\ &= - \left[1 - (1-2p_i^e)m_i^e \right] \frac{1}{g_i^e m_i^e} \frac{dx_i^e}{dw^{e_0}} - (1-2p_i^e)m_j^e \frac{1}{g_j^e m_j^e} \frac{dx_j^e}{dw^{e_0}}. \end{aligned} \quad (\text{A7})$$

Similarly, we have that

$$\frac{1}{\lambda_j} \frac{d\lambda_j}{dw^{e_0}} = - \left[1 + (1-2p_i^e)m_j^e \right] \frac{1}{g_j^e m_j^e} \frac{dx_j^e}{dw^{e_0}} + (1-2p_i^e)m_i^e \frac{1}{g_i^e m_i^e} \frac{dx_i^e}{dw^{e_0}}. \quad (\text{A8})$$

Combining (A7) and (A8) yields that

$$\frac{dx_i^e}{dw^{e_0}} = -g_i^e m_i^e \frac{\left[1 - (2p_i^e - 1)m_j^e \right] \frac{1}{\lambda_i} \frac{d\lambda_i}{dw^{e_0}} + (2p_i^e - 1)m_j^e \frac{1}{\lambda_j} \frac{d\lambda_j}{dw^{e_0}}}{1 + (m_i^e - m_j^e)(p_i^e - p_j^e)}.$$

Substituting (2) into the above equation gives (A2).

Next, we prove part (ii). The first-order condition (2) on battlefield e_0 becomes $w^{e_0} = \lambda_{i_0} g_{i_0}^{e_0} = \lambda_{j_0} g_{j_0}^{e_0}$. Note that this condition holds in a neighborhood of w^{e_0} . Taking the logarithm of both sides and differentiating with respect to w^{e_0} gives (A3).

Last, we prove part (iii). For the pure-budget case, it is evident that the left-hand side of (A4) is zero because $\delta_i = 0$. Further, $\sum_{e \in \mathcal{E}_i} x_i^e = \bar{X}_i$ implies that the right-hand side—i.e., $\sum_{e \in \mathcal{E}_i} \frac{dx_i^e}{dw^{e_0}}$ —is also zero, and thus (A4) holds. For the pure-cost case, we have that

$\lambda_i = c'_i(X_i) = c'_i(\sum_{e \in \mathcal{E}_i} x_i^e)$. Differentiating both sides with respect to w^{e_0} gives (A4). This completes the proof. ■

Proof of Lemma 3. For part (i), first consider the case of $x_i^e, x_j^e > 0$. By (A2), we have that

$$\begin{aligned} & \frac{d\lambda_i}{dw^{e_0}} \frac{dx_i^e}{dw^{e_0}} + \frac{d\lambda_j}{dw^{e_0}} \frac{dx_j^e}{dw^{e_0}} \\ &= -w^e \frac{m_i^e [1 - (2p_i^e - 1)m_j^e] (\frac{1}{\lambda_i} \frac{d\lambda_i}{dw^{e_0}})^2 + m_j^e [1 + (2p_i^e - 1)m_i^e] (\frac{1}{\lambda_j} \frac{d\lambda_j}{dw^{e_0}})^2}{1 + (m_i^e - m_j^e)(p_i^e - p_j^e)}. \end{aligned}$$

Note that $m_i^e \in [0, 1]$. Moreover, $x_i^e, x_j^e > 0$ implies that $p_i^e \in (0, 1)$. Therefore, $1 - (2p_i^e - 1)m_j^e > 0$, $1 + (2p_i^e - 1)m_i^e > 0$, and $1 + (m_i^e - m_j^e)(p_i^e - p_j^e) > 0$, which implies (5).

Next, suppose $x_i^e = 0$ and $x_j^e > 0$. If $x_i^e = 0$ in a neighborhood of w^{e_0} , then by Lemma A3, $\lambda_j = 0$ in this neighborhood. Therefore, $\frac{dx_i^e}{dw^{e_0}} = \frac{d\lambda_j}{dw^{e_0}} = 0$, which also implies (5). Otherwise, if $x_i^e > 0$ in a neighborhood of w^{e_0} , then (5) holds in the neighborhood and is satisfied at w^{e_0} by continuity.

Next, we prove part (ii) of the lemma. By (A4), we have that $\sum_{e \in \mathcal{E}_i} \frac{d\lambda_i}{dw^{e_0}} \frac{dx_i^e}{dw^{e_0}} = \delta_i (\frac{d\lambda_i}{dw^{e_0}})^2 \geq 0$. This concludes the proof. ■

Proof of Lemma 4. For part (i), we first show that there exists $\tilde{w}^{e_0} < v^{e_0}/4$ such that $\frac{dx_{j_0}^{e_0}}{dw^{e_0}} \Big|_{w^{e_0}=\tilde{w}^{e_0}} \leq 0$. By assumption, there exists $\hat{w}^{e_0} < v^{e_0}/4$ such that $\Lambda^{e_0}(x_{i_0}^{e_0}, x_{j_0}^{e_0}) \Big|_{w^{e_0}=\hat{w}^{e_0}} \geq \Lambda^{e_0}(x_{i_0}^{e_0}, x_{j_0}^{e_0}) \Big|_{w^{e_0}=v^{e_0}/4}$. Recall that Λ^{e_0} is strictly increasing in $x_{i_0}^{e_0}$ and $x_{j_0}^{e_0}$. There exists $\tilde{w}^{e_0} \in (\hat{w}^{e_0}, v^{e_0}/4)$ to satisfy $\frac{dx_{i_0}^{e_0}}{dw^{e_0}} \Big|_{w^{e_0}=\tilde{w}^{e_0}} \leq 0$ or $\frac{dx_{j_0}^{e_0}}{dw^{e_0}} \Big|_{w^{e_0}=\tilde{w}^{e_0}} \leq 0$. Further, $\frac{dx_{i_0}^{e_0}}{dw^{e_0}} \Big|_{w^{e_0}=\tilde{w}^{e_0}} \geq \frac{dx_{j_0}^{e_0}}{dw^{e_0}} \Big|_{w^{e_0}=\tilde{w}^{e_0}}$ by assumption. Therefore, $\frac{dx_{j_0}^{e_0}}{dw^{e_0}} \Big|_{w^{e_0}=\tilde{w}^{e_0}} \leq 0$. In the rest of the proof, all derivatives are evaluated at $w^{e_0} = \tilde{w}^{e_0}$.

Next, we show that $\frac{dx_{i_0}^{e_0}}{dw^{e_0}} \geq 0$. Suppose, to the contrary, that $\frac{dx_{i_0}^{e_0}}{dw^{e_0}} < 0$. By (A3), we have that $\frac{d\lambda_{i_0}}{dw^{e_0}} > 0$ and $\frac{d\lambda_{j_0}}{dw^{e_0}} > 0$. Therefore,

$$\frac{d\lambda_{i_0}}{dw^{e_0}} \frac{dx_{i_0}^{e_0}}{dw^{e_0}} + \frac{d\lambda_{j_0}}{dw^{e_0}} \frac{dx_{j_0}^{e_0}}{dw^{e_0}} < 0. \quad (\text{A9})$$

Note that

$$\mathcal{I} = \sum_{i \in \mathcal{N}} \sum_{e \in \mathcal{E}_i} \frac{d\lambda_i}{dw^{e_0}} \frac{dx_i^e}{dw^{e_0}} \geq 0, \quad (\text{A10})$$

where the inequality follows from Lemma 3(ii). However, (5) and (A9) imply that

$$\mathcal{I} = \sum_{e \neq e_0} \sum_{i \in \mathcal{N}^e} \frac{d\lambda_i}{dw^{e_0}} \frac{dx_i^e}{dw^{e_0}} + \frac{d\lambda_{i_0}}{dw^{e_0}} \frac{dx_{i_0}^{e_0}}{dw^{e_0}} + \frac{d\lambda_{j_0}}{dw^{e_0}} \frac{dx_{j_0}^{e_0}}{dw^{e_0}} < 0,$$

which is a contradiction.

Next, we prove part (ii). The first two inequalities in part (ii) follow immediately from part (i) and (A3), and it remains to prove $\frac{d\lambda_{i_0}}{dw^{e_0}} \geq 0$. Suppose, to the contrary, that $\frac{d\lambda_{i_0}}{dw^{e_0}} < 0$; together with $\frac{dx_{i_0}^{e_0}}{dw^{e_0}} > 0$ as shown in part (i), we can obtain (A9). From (A9) and Lemma 3(ii), we have that $\mathcal{I} < 0$, which contradicts (A10).

Part (iii) follows immediately from parts (i) and (ii), and it remains to prove part (iv). It suffices to show $\frac{dx_{j_0}^{e'}}{dw^{e_0}} \leq 0$; $\frac{dx_{j_0}^{e'}}{dw^{e_0}} \frac{d\lambda_{j_0}}{dw^{e_0}} \leq 0$ can be implied by $\frac{dx_{j_0}^{e'}}{dw^{e_0}} \leq 0$ and (ii).

Fix $e' \neq e_0$, with $\mathcal{N}^{e'} = \{i_0, j_0\}$. By (A2), we have that

$$\frac{dx_{j_0}^{e'}}{dw^{e_0}} = -\frac{w^{e'} m_{j_0}^{e'}}{\lambda_{j_0}} \times \frac{\left[1 - (2p_{j_0}^{e'} - 1)m_{i_0}^{e'}\right] \frac{1}{\lambda_{j_0}} \frac{d\lambda_{j_0}}{dw^{e_0}} + (2p_{j_0}^{e'} - 1)m_{i_0}^{e'} \frac{1}{\lambda_{i_0}} \frac{d\lambda_{i_0}}{dw^{e_0}}}{1 + (m_{i_0}^{e'} - m_{j_0}^{e'})(p_{i_0}^{e'} - p_{j_0}^{e'})}.$$

Recall that we have shown $1 + (m_{i_0}^{e'} - m_{j_0}^{e'})(p_{i_0}^{e'} - p_{j_0}^{e'}) > 0$ in Lemma 3. Further, simple algebra would verify that

$$\begin{aligned} & \left[1 - (2p_{j_0}^{e'} - 1)m_{i_0}^{e'}\right] \frac{1}{\lambda_{j_0}} \frac{d\lambda_{j_0}}{dw^{e_0}} + (2p_{j_0}^{e'} - 1)m_{i_0}^{e'} \frac{1}{\lambda_{i_0}} \frac{d\lambda_{i_0}}{dw^{e_0}} \\ &= \frac{1}{\lambda_{j_0}} \frac{d\lambda_{j_0}}{dw^{e_0}} - (2p_{j_0}^{e'} - 1)m_{i_0}^{e'} \left[\frac{1}{\lambda_{j_0}} \frac{d\lambda_{j_0}}{dw^{e_0}} - \frac{1}{\lambda_{i_0}} \frac{d\lambda_{i_0}}{dw^{e_0}} \right] \\ &\geq \frac{1}{\lambda_{j_0}} \frac{d\lambda_{j_0}}{dw^{e_0}} - \left[\frac{1}{\lambda_{j_0}} \frac{d\lambda_{j_0}}{dw^{e_0}} - \frac{1}{\lambda_{i_0}} \frac{d\lambda_{i_0}}{dw^{e_0}} \right] \geq 0, \end{aligned}$$

where the first inequality follows from $p_{j_0}^{e'} \in [0, 1]$, $m_{i_0}^{e'} \in [0, 1]$, and $\frac{1}{\lambda_{j_0}} \frac{d\lambda_{j_0}}{dw^{e_0}} - \frac{1}{\lambda_{i_0}} \frac{d\lambda_{i_0}}{dw^{e_0}} \geq 0$, and the second inequality follows from part (ii) of the lemma. Therefore, $\frac{dx_{j_0}^{e'}}{dw^{e_0}} \leq 0$, which concludes the proof. ■

Proof of Theorem 1. We first show that an organizer can induce equal winning probabilities on her battlefield while holding all other biases fixed.

Lemma A6 Fix a bias profile α and a battlefield e_0 , with $\mathcal{N}^{e_0} = \{i, j\}$. Holding α^{-e_0} fixed, the organizer of e_0 can choose a bias profile $(\alpha^{e_0})'$ such that every second-stage

equilibrium under $\alpha' \equiv ((\alpha^{e_0})', \alpha^{-e_0})$ satisfies

$$(p_i^{e_0})' = (p_j^{e_0})' = \frac{1}{2} \quad \text{and} \quad (x_i^{e_0})', (x_j^{e_0})' > 0.$$

Proof. Fix $e_0 = \{i, j\}$ and α^{-e_0} . Construct two identical copies of the entire network. Each player s has copies $s^{(1)}$ and $s^{(2)}$, each with his original cost function or budget. Every battlefield other than e_0 is replicated within each copy and retains its prize, impact functions, and biases. Remove e_0 from both copies and replace it with two battlefields: one between $i^{(1)}$ and $i^{(2)}$ and the other between $j^{(1)}$ and $j^{(2)}$. Each new battlefield has prize v^{e_0} and equal biases, and each participant retains his impact function from e_0 .

By [Xu, Zenou, and Zhou \(2022\)](#), this auxiliary contest game admits an equilibrium. Exchanging the two copies produces another equilibrium. In the pure-cost case, equilibrium uniqueness implies equal efforts by the two copies of each target player. In the pure-budget case, each new battlefield has at least one active player. By [Lemma A2](#), his opponent's effort must be identical across the original and the exchanged equilibria, so the two copies exert the same positive effort. Denote the common efforts on the two new battlefields $y_i > 0$ and $y_j > 0$.

Retain the effort profile in the first copy and restore the original battlefield e_0 , assigning efforts y_i and y_j to its participants. Set

$$(\alpha_i^{e_0})' = \frac{f_j^{e_0}(y_j)}{f_i^{e_0}(y_i) + f_j^{e_0}(y_j)}, \quad (\alpha_j^{e_0})' = 1 - (\alpha_i^{e_0})'. \quad (\text{A11})$$

These biases induce equal winning probabilities on e_0 . For each $s \in \{i, j\}$, the marginal prize payoff on the restored battlefield is

$$\frac{v^{e_0}}{4} \frac{(f_s^{e_0})'(y_s)}{f_s^{e_0}(y_s)},$$

exactly as in the auxiliary game. All other marginal prize payoffs, effort costs, and budget allocations remain unchanged. Hence the retained effort profile satisfies the second-stage equilibrium conditions in [Lemma 1](#) under $\alpha' = ((\alpha^{e_0})', \alpha^{-e_0})$.

Finally, second-stage uniqueness in the pure-cost case, and [Lemma A2](#) applied to both positive target efforts in the pure-budget case, imply that every equilibrium under α' has the same effort pair on e_0 . Thus every continuation induces equal winning probabilities on the target battlefield. ■

By Lemma A6, the organizer can attain $w^{e_0} = v^{e_0}/4$.

It remains to show that under Assumption 1, $\frac{d\lambda_s}{dw^{e_0}} \frac{dx_s^e}{dw^{e_0}} < 0$ for each $e \in \mathcal{E}_s$, with $e \neq e_0$. We first show that $\frac{d\lambda_s}{dw^{e_0}} \neq 0$. Suppose, to the contrary, that $\frac{d\lambda_s}{dw^{e_0}} = 0$. By the definition of s , $\frac{d\lambda_i}{dw^{e_0}} = 0$ for each $i \in \mathcal{N}$; together with (A3), we have that $\frac{dx_{i_0}^{e_0}}{dw^{e_0}} > 0$ and $\frac{dx_{j_0}^{e_0}}{dw^{e_0}} > 0$, which contradicts Lemma 4(i).

Next, suppose $\frac{d\lambda_s}{dw^{e_0}} > 0$ (the analysis for the case with $\frac{d\lambda_s}{dw^{e_0}} < 0$ follows analogously). Fix $e \in \mathcal{E}_s$, with $e \neq e_0$ and $\mathcal{N}^e = \{s, j\}$. By (A2), we have that

$$\frac{dx_s^e}{dw^{e_0}} = -\frac{w^e m_s^e}{\lambda_s} \times \frac{\left[1 - (2p_s^e - 1)m_j^e\right] \frac{1}{\lambda_s} \frac{d\lambda_s}{dw^{e_0}} + (2p_s^e - 1)m_j^e \frac{1}{\lambda_j} \frac{d\lambda_j}{dw^{e_0}}}{1 + (m_s^e - m_j^e)(p_s^e - p_j^e)}.$$

Assumption 1 implies that $(f_j^e)'(0) = (f_s^e)'(0) = +\infty$ and thus $p_s^e \in (0, 1)$. Further, the assumption implies $m_j^e \in [0, 1/2]$. Carrying out the algebra, we can obtain that

$$\begin{aligned} & \left[1 - (2p_s^e - 1)m_j^e\right] \frac{1}{\lambda_s} \frac{d\lambda_s}{dw^{e_0}} + (2p_s^e - 1)m_j^e \frac{1}{\lambda_j} \frac{d\lambda_j}{dw^{e_0}} \\ &= \frac{1}{\lambda_s} \frac{d\lambda_s}{dw^{e_0}} - (2p_s^e - 1)m_j^e \left[\frac{1}{\lambda_s} \frac{d\lambda_s}{dw^{e_0}} - \frac{1}{\lambda_j} \frac{d\lambda_j}{dw^{e_0}} \right] \\ &> \frac{1}{\lambda_s} \frac{d\lambda_s}{dw^{e_0}} - \frac{1}{2} \times \left[\frac{1}{\lambda_s} \frac{d\lambda_s}{dw^{e_0}} - \frac{1}{\lambda_j} \frac{d\lambda_j}{dw^{e_0}} \right] \geq 0, \end{aligned}$$

where the first inequality follows from $m_j^e \in [0, 1/2]$ and $p_s^e \in (0, 1)$, and the last inequality follows from $\frac{1}{\lambda_s} \frac{d\lambda_s}{dw^{e_0}} \geq \left| \frac{1}{\lambda_j} \frac{d\lambda_j}{dw^{e_0}} \right|$. This concludes the proof. ■

Proof of Theorem 2.

Part I: Existence We first show that there is a unique profile of contest rules α^{**} that leads to the equilibrium outcome of equal winning odds on every battlefield. For the pure-cost case, we first solve for the second-stage equilibrium profile x^{**} that leads to equal winning odds in all battlefields. Plugging $(p_i^e)^{**} = 1/2$ into (2) yields $\frac{v^e}{4} = \lambda_i^{**} g_i^e \left((x_i^e)^{**} \right)$; together with the monotonicity of $g_i^e \equiv f_i^e / (f_i^e)'$, we have $(x_i^e)^{**} = (g_i^e)^{-1} \left(\frac{v^e}{4\lambda_i^{**}} \right)$ and

$$\lambda_i^{**} = c_i'(X_i^{**}) = c_i' \left(\sum_{e \in \mathcal{E}_i} (g_i^e)^{-1} \left(\frac{v^e}{4\lambda_i^{**}} \right) \right), \quad (\text{A12})$$

from which we can solve for λ_i^{**} and pin down $\mathbf{x}^{**}$. The first-stage bias profile $\boldsymbol{\alpha}^{**}$ is uniquely determined by (1).

The proof of the pure-budget case closely follows that of the pure-cost case after we replace (A12) with $\bar{X}_i = \sum_{e \in \mathcal{E}_i} (g_i^e)^{-1} \left(\frac{v^e}{4\lambda_i^{**}} \right)$. This completes the construction.

We prove the two lemmata used in the existence argument in Section 3.2.

Proof of Lemma 5. For the pure-cost case, similar to (A12), we can obtain that

$$\tilde{\lambda}_i = c'_i \left(\sum_{e \in \mathcal{E}_i} (g_i^e)^{-1} \left(\frac{v^e \tilde{p}_i^e (1 - \tilde{p}_i^e)}{\tilde{\lambda}_i} \right) \right) \leq c'_i \left(\sum_{e \in \mathcal{E}_i} (g_i^e)^{-1} \left(\frac{v^e}{4\tilde{\lambda}_i} \right) \right),$$

where the inequality follows from the monotonicity of c'_i and $(g_i^e)^{-1}$ and $\tilde{p}_i^e (1 - \tilde{p}_i^e) \leq 1/4$. The above inequality, together with (A12), implies $\tilde{\lambda}_i \leq \lambda_i^{**}$. The proof for the pure-budget case is similar and omitted for brevity. ■

Proof of Lemma 6. For notational convenience, define $(f_i^e)^{**} := f_i^e((x_i^e)^{**})$, $(g_i^e)^{**} := g_i^e((x_i^e)^{**})$, $\tilde{f}_i^e := f_i^e(\tilde{x}_i^e)$, and $\tilde{g}_i^e := g_i^e(\tilde{x}_i^e)$. Suppose, to the contrary, that $\tilde{x}_i^e < (x_i^e)^{**}$. By (2), we have

$$\lambda_i^{**} (g_i^e)^{**} = \lambda_j^{**} (g_j^e)^{**} = \frac{v^e}{4} \text{ and } \tilde{\lambda}_i \tilde{g}_i^e = \tilde{\lambda}_j \tilde{g}_j^e = v^e \tilde{p}_i^e (1 - \tilde{p}_i^e). \quad (\text{A13})$$

Evidently, at least one player is active on battlefield e in the equilibrium. We consider three cases:

- (a) Suppose $\tilde{x}_i^e = 0$ and $\tilde{x}_j^e > 0$. The postulated $\tilde{\lambda}_i/\lambda_i^{**} \leq \tilde{\lambda}_j/\lambda_j^{**}$ implies that $\tilde{\lambda}_i = 0$. Meanwhile, it follows from (3) that $\tilde{\lambda}_i \geq v^e \frac{(\alpha_i^e)^{**} (f_i^e)'(0)}{(\alpha_j^e)^{**} \tilde{f}_j^e} > 0$. A contradiction.
- (b) Suppose $\tilde{x}_j^e = 0$ and $\tilde{x}_i^e > 0$. From (3), we have $\tilde{\lambda}_j \geq v^e \frac{(\alpha_j^e)^{**} (f_j^e)'(0)}{(\alpha_i^e)^{**} \tilde{f}_i^e}$. Further, $(x_j^e)^{**} > 0$ implies $\lambda_j^{**} < v^e \frac{(\alpha_j^e)^{**} (f_j^e)'(0)}{(\alpha_i^e)^{**} (f_i^e)^{**}}$. Note that $\tilde{\lambda}_j \leq \lambda_j^{**}$ by Lemma 5. Together, these indicate that $(f_i^e)^{**} < \tilde{f}_i^e$, which implies that $(x_i^e)^{**} < \tilde{x}_i^e$. A contradiction.
- (c) Now suppose $\tilde{x}_j^e > 0$ and $\tilde{x}_i^e > 0$. Let $a_i := \tilde{f}_i^e / (f_i^e)^{**}$ and $a_j := \tilde{f}_j^e / (f_j^e)^{**}$. Note that $g_i^e(\cdot) \equiv f_i^e(\cdot) / (f_i^e(\cdot))'$ is strictly increasing; together with the postulated $\tilde{x}_i^e < (x_i^e)^{**}$, we have that $\tilde{g}_i^e < (g_i^e)^{**}$. Further, by (A13), we have $\frac{\tilde{\lambda}_i \tilde{g}_i^e}{\lambda_i^{**} (g_i^e)^{**}} = \frac{\tilde{\lambda}_j \tilde{g}_j^e}{\lambda_j^{**} (g_j^e)^{**}}$; together with $\tilde{g}_i^e < (g_i^e)^{**}$ and the postulated $\tilde{\lambda}_i/\lambda_i^{**} \leq \tilde{\lambda}_j/\lambda_j^{**}$, we can obtain that $\tilde{g}_j^e < (g_j^e)^{**}$ and $\tilde{x}_j^e < (x_j^e)^{**}$, which implies that $0 < a_i, a_j < 1$.

By (1), we have that $1 = \frac{(p_i^e)^{**}}{(p_j^e)^{**}} = \frac{(\alpha_i^e)^{**}(f_i^e)^{**}}{(\alpha_j^e)^{**}(f_j^e)^{**}}$, which implies $\frac{\tilde{p}_i^e}{\tilde{p}_j^e} = \frac{(\alpha_i^e)^{**}\tilde{f}_i^e}{(\alpha_j^e)^{**}\tilde{f}_j^e} = \frac{\tilde{f}_i^e/(f_i^e)^{**}}{\tilde{f}_j^e/(f_j^e)^{**}} = \frac{a_i}{a_j}$ and thus $(\tilde{p}_i^e, \tilde{p}_j^e) = (\frac{a_i}{a_i+a_j}, \frac{a_j}{a_i+a_j})$; together with (A13), we have that $\frac{\tilde{g}_i^e}{(g_i^e)^{**}} = 4\tilde{p}_i^e(1 - \tilde{p}_i^e) \times \frac{\lambda_i^{**}}{\lambda_i} \geq \frac{4a_i a_j}{(a_i+a_j)^2}$, where the inequality follows from Lemma 5.

Further, from the concavity of $f_i^e(\cdot)$ and the postulated $\tilde{x}_i^e < (x_i^e)^{**}$, we have

$$a_i = \frac{\tilde{f}_i^e}{(f_i^e)^{**}} = \frac{\tilde{g}_i^e}{(g_i^e)^{**}} \times \frac{(f_i^e)'(\tilde{x}_i^e)}{(f_i^e)'((x_i^e)^{**})} \geq \frac{\tilde{g}_i^e}{(g_i^e)^{**}} \geq \frac{4a_i a_j}{(a_i + a_j)^2},$$

which implies $\frac{4a_j}{(a_i+a_j)^2} \leq 1$. Similarly, we can obtain that $\frac{4a_i}{(a_i+a_j)^2} \leq 1$. Summing the two inequalities yields $a_i + a_j \geq 2$, which contradicts $a_i, a_j < 1$.

This completes the proof. ■

Together with the construction of the even-odds outcome, these two lemmata complete the existence argument in Section 3.2: No organizer can profitably deviate from α^{**} , regardless of the second-stage equilibrium selected after the deviation.

Part II: Uniqueness We use the following comparison result.

Lemma A7 *Compare an interior second-stage equilibrium (w, λ, x) with an arbitrary equilibrium $(\tilde{w}, \tilde{\lambda}, \tilde{x})$ following a unilateral balancing deviation. Fix a player i with $\tilde{\lambda}_i > 0$. If, for some $q > 1$,*

$$\frac{\tilde{w}^h}{w^h} \leq q \quad \text{for every } h \in \mathcal{E}_i, \quad (\text{A14})$$

then $\tilde{\lambda}_i/\lambda_i < q$ in the pure-cost case and $\tilde{\lambda}_i/\lambda_i \leq q$ in the pure-budget case. In the latter case, equality implies

$$\tilde{x}_i^h = x_i^h \quad \text{and} \quad \frac{\tilde{w}^h}{w^h} = q \quad \text{for every } h \in \mathcal{E}_i. \quad (\text{A15})$$

Proof. At every incidence with positive comparison effort, the first-order conditions before and after the deviation give

$$g_i^h(\tilde{x}_i^h) = \frac{\tilde{w}^h/w^h}{\tilde{\lambda}_i/\lambda_i} g_i^h(x_i^h). \quad (\text{A16})$$

Recall that $g_i^h = f_i^h/(f_i^h)'$ is strictly increasing, since

$$(g_i^h)' = 1 - \frac{f_i^h(f_i^h)''}{((f_i^h)')^2} \geq 1.$$

A zero comparison effort is strictly below the positive base effort.

In the pure-cost case, suppose that $\tilde{\lambda}_i/\lambda_i \geq q$. By (A16), every effort of player i weakly decreases. Therefore $\tilde{X}_i \leq X_i$ and

$$\tilde{\lambda}_i = c'_i(\tilde{X}_i) \leq c'_i(X_i) = \lambda_i,$$

which contradicts $\tilde{\lambda}_i/\lambda_i \geq q > 1$.

In the pure-budget case, if $\tilde{\lambda}_i/\lambda_i > q$, every effort of player i strictly decreases. This contradicts the fact that $\tilde{\lambda}_i > 0$ causes the comparison budget constraint to bind. Hence $\tilde{\lambda}_i/\lambda_i \leq q$. If equality holds, every comparison effort is weakly below its base value. Both budgets bind, so the equality of their sums forces equality at every incidence. Equation (A16) then yields (A15). ■

Proof of Lemma 7. In the pure-cost case, the initial second-stage equilibrium is interior. Since $\tilde{w}^h/w^h \leq q$ on every battlefield and $q > 1$, Lemma A7 gives $\tilde{\lambda}_s/\lambda_s < q$ for each $s \in \mathcal{N}^{e_0}$. ■

Proof of Lemma 8. On the target battlefield, the first-order conditions give

$$\frac{g_s^{e_0}(\tilde{x}_s^{e_0})}{g_s^{e_0}(x_s^{e_0})} = \frac{q}{\tilde{\lambda}_s/\lambda_s} > 1 \quad \text{for each } s \in \mathcal{N}^{e_0},$$

where the inequality follows from Lemma 7. Strict monotonicity of $g_s^{e_0}$ implies $\tilde{x}_s^{e_0} > x_s^{e_0}$ for both players. ■

For the pure-budget counterpart, suppose the initial second-stage equilibrium is interior. After the balancing deviation supplied by Lemma A6, both target efforts are positive and $\tilde{\lambda}_s > 0$ for $s \in \mathcal{N}^{e_0}$. Lemma A7 gives $\tilde{\lambda}_s/\lambda_s \leq q$. The same first-order comparison then implies $\tilde{x}_s^{e_0} \geq x_s^{e_0}$ for both players.

It suffices to show that every equilibrium outcome has equal winning odds on every battlefield. The construction of the even-odds outcome uniquely determines the associated effort and bias profiles.

Suppose, to the contrary, that an alternative equilibrium outcome (α, x) contains an imbalanced battlefield. We consider the pure-cost and pure-budget cases separately.

Pure-cost case. Choose a least balanced battlefield e_0 . Since it is imbalanced, Lemma 8 gives a unilateral deviation that strictly raises both target efforts. This contradicts the organizer's first-stage optimality.

Pure-budget case. We first show that every equilibrium outcome is interior. Suppose instead that $x_i^e = 0 < x_j^e$ on some battlefield. By Lemma A3, $\lambda_j = 0$, every opponent of player j is inactive against him on every incident battlefield, and player j is positive on each such battlefield. The last statement follows because both players cannot be inactive on the same battlefield: Either player could allocate an arbitrarily small effort there and obtain a discrete probability gain at a vanishing loss. Budget feasibility gives $\sum_{h \in \mathcal{E}_j} x_j^h \leq \bar{X}_j$.

Choose $e_0 \in \mathcal{E}_j$ to maximize

$$\frac{v^h}{4g_j^h(x_j^h)}.$$

Balance e_0 using Lemma A6. Both target efforts are positive after the deviation, so $\tilde{\lambda}_j > 0$ and player j 's budget binds. If

$$\tilde{\lambda}_j > \frac{v^{e_0}}{4g_j^{e_0}(x_j^{e_0})},$$

then, at every $h \in \mathcal{E}_j$ with $\tilde{x}_j^h > 0$,

$$g_j^h(\tilde{x}_j^h) = \frac{\tilde{w}^h}{\tilde{\lambda}_j} \leq \frac{v^h}{4\tilde{\lambda}_j} < g_j^h(x_j^h).$$

A zero comparison effort is also strictly smaller. Strict monotonicity of g_j^h would imply

$$\bar{X}_j = \sum_h \tilde{x}_j^h < \sum_h x_j^h \leq \bar{X}_j,$$

a contradiction. Hence $\tilde{\lambda}_j \leq v^{e_0}/[4g_j^{e_0}(x_j^{e_0})]$, and therefore

$$g_j^{e_0}(\tilde{x}_j^{e_0}) = \frac{v^{e_0}}{4\tilde{\lambda}_j} \geq g_j^{e_0}(x_j^{e_0}).$$

Player j 's target effort weakly rises, while his formerly inactive opponent becomes strictly active. The organizer of e_0 has a profitable deviation—again a contradiction. Thus every pure-budget equilibrium outcome is interior.

Return to the postulated imbalanced interior outcome and select e_0 to maximize $q = v^e/(4w^e) > 1$. After balancing e_0 , use tildes to denote an arbitrary second-stage continuation. Since $\tilde{w}^h \leq v^h/4$, we have

$$\frac{\tilde{w}^h}{w^h} \leq \frac{v^h}{4w^h} \leq q, \quad \frac{\tilde{w}^{e_0}}{w^{e_0}} = q. \quad (\text{A17})$$

By Lemma A7, both target efforts weakly increase. Neither can increase strictly, since the organizer's objective is strictly increasing in both efforts. Thus both target efforts remain unchanged. For each endpoint $s \in \mathcal{N}^{e_0}$, equation (A16) on e_0 yields $\tilde{\lambda}_s/\lambda_s = q$. The equality part of Lemma A7 then implies that every effort of player s is unchanged and

$$\frac{\tilde{w}^h}{w^h} = q \quad \text{for every } h \in \mathcal{E}_s.$$

Together with (A17), this gives

$$\frac{v^h}{4w^h} = q \quad \text{and} \quad \tilde{w}^h = \frac{v^h}{4} \quad \text{for every } h \in \mathcal{E}_s. \quad (\text{A18})$$

Repeating the same comparison at newly reached battlefields and using the connectedness of the network, (A18) extends to every battlefield. In particular, balancing any one battlefield leaves all efforts of its endpoints unchanged and renders every adjacent battlefield perfectly balanced.

It remains to show that this is impossible. Take two distinct adjacent battlefields e and h that share player i and balance e . Let k be the other player on h . Player i 's effort on h remains unchanged, while Lemma A7 implies $\tilde{x}_k^h \geq x_k^h$. Battlefield h becomes perfectly balanced although its bias is unchanged, so

$$\alpha_i^h f_i^h(x_i^h) = \alpha_k^h f_k^h(\tilde{x}_k^h) > \alpha_k^h f_k^h(x_k^h),$$

where the inequality is strict because h was imbalanced before the deviation. Thus player i must initially be the strict favorite on h .

For every player participating on at least two battlefields and every one of his incident battlefields, we can choose another incident battlefield as the target of the balancing deviation. Hence every such player must be the strict favorite on every incident battlefield. No battlefield can have two endpoints who each participate on at least two battlefields. Since the network is connected and $|\mathcal{E}| \geq 2$, it must therefore be a simple star.

Finally, balance one battlefield of the star and consider another battlefield h . The center's effort on h remains unchanged. The leaf on h participates in no other battlefield, so it allocates its entire budget to h both before and after the deviation; its effort also remains unchanged. Since the bias on h is unchanged, its winning probabilities cannot change. This contradicts (A18), which requires h to change from imbalanced to perfectly balanced. The postulated alternative equilibrium is impossible. Therefore every equilibrium has equal

winning odds on every battlefield, and the construction of the even-odds outcome pins down the unique equilibrium outcome. ■

Proof of Theorem 3. We prove Theorem 3 by induction on t .

Base case: Consider the last period $t = T$. Fixing $e \in \mathcal{E}^T$, the organizer chooses α^e to maximize $\Lambda^e(x^e)$, holding fixed α^{-e} . By Theorem 1, the organizer chooses α^e to induce $p_i^e = p_j^e = 1/2$, with $\mathcal{N}^e = \{i, j\}$, in the equilibrium.

Inductive step: For each $t \in \{1, \dots, T-1\}$, suppose that the statement holds for each $\tau > t$. We show that for each battlefield $e_0 \in \mathcal{E}^t$ and each $\alpha^{<t}$, the battlefield organizer chooses α^e to induce $p_{i_0}^e = p_{j_0}^e = 1/2$, with $\mathcal{N}^{e_0} = \{i_0, j_0\}$.

First, following a similar argument as in the proof of Lemma A5, we can show that (A2) holds for each $e \in \cup_{\tau \leq t} \mathcal{E}^\tau$ and $e \neq e_0$. Second, by the induction hypothesis, for each $e \in \cup_{\tau > t} \mathcal{E}^\tau$ and each $i \in \mathcal{N}^e$, we have that $p_i^e \equiv 1/2$ and thus $\frac{dp_i^e}{dw^{e_0}} = 0$. This implies that $\frac{dx_i^e}{dw^{e_0}} = -\frac{g_i^e m_i^e}{\lambda_i} \frac{d\lambda_i}{dw^{e_0}}$ and thus $\frac{dx_i^e}{dw^{e_0}} \frac{d\lambda_i}{dw^{e_0}} \leq 0$.

Fixing the bias profile up to period $t-1$ —i.e., fixing $\alpha^{<t}$ —the battlefield organizers in period t choose their biases, anticipating the biases set by their followers. Note that the condition $\frac{dx_i^e}{dw^{e_0}} \frac{d\lambda_i}{dw^{e_0}} \leq 0$ we prove above ensures that Lemmata 3 and 4 continue to hold. Following a similar argument as in the proof of Theorem 1, we can show that all battlefield organizers in period t will again choose biases to induce equal winning probabilities in their battlefields. This completes the inductive step.

Conclusion: By the principle of induction, for each $t \in \{1, \dots, T\}$, a bias profile $\alpha^{<t}$, and a battlefield $e \in \mathcal{E}^t$, the organizer chooses $\alpha^e(\alpha^{<t})$ such that $p_i^e = p_j^e = 1/2$.

The construction of the even-odds outcome then pins down the unique equilibrium effort and bias profiles. This completes the proof. ■

Proof of Theorem 4. Fix an arbitrary bias profile α and $x \in \mathcal{X}(\alpha)$. By (2), we have

$$X_i = \sum_{e \in \mathcal{E}_i} x_i^e = \sum_{e \in \mathcal{E}_i} (g_i^e)^{-1} \left(\frac{v^e p_i^e (1 - p_i^e)}{\lambda_i} \right) \leq \sum_{e \in \mathcal{E}_i} (g_i^e)^{-1} \left(\frac{v^e}{4c_i'(X_i)} \right). \quad (\text{A19})$$

Further, by the definition of α^{**} , we have that $(p_i^e)^{**} = 1/2$ for all $(i, e) \in \Gamma$. Similar to (A19), we have

$$X_i^{**} = \sum_{e \in \mathcal{E}_i} (g_i^e)^{-1} \left(\frac{v^e (p_i^e)^{**} (1 - (p_i^e)^{**})}{(\lambda_i)^{**}} \right) = \sum_{e \in \mathcal{E}_i} (g_i^e)^{-1} \left(\frac{v^e}{4c_i'(X_i^{**})} \right). \quad (\text{A20})$$

A closer look at (A19) and (A20) reveals that $X_i \leq X_i^{**}$ for every player i . Since the organizer's objective is strictly increasing in each X_i , it is maximized if and only if equality holds for every player, which requires $p_i^e = 1/2$ for all $(i, e) \in \Gamma$. By the construction of the even-odds outcome, we must then have $\alpha = \alpha^{**}$. This completes the proof. ■

Proof of Corollary 1. By Theorem 2, decentralized contest design yields the unique equilibrium bias profile α^{**} . By Theorem 4, the same profile uniquely maximizes total individual efforts in the pure-cost case. This proves the result. ■