Orchestrating Organizational Politics: Baron and Ferejohn Meet Tullock*

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Abstract

This paper examines the optimal organizational rules governing the process of dividing a fixed surplus. The process is modeled as a sequential multilateral bargaining game with costly recognition. The designer sets the voting rule—i.e., the minimum number of votes required to approve a proposal—and the mechanism for proposer recognition, which is modeled as a biased generalized lottery contest. We show that for diverse design objectives, the optimum can be achieved by a dictatorial voting rule, which simplifies the game into a standard biased contest model.

Keywords: Multilateral Bargaining; Costly Recognition; Contest Design

JEL Classification Codes: C70; C78; D72.

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1 Introduction

Organizations—firms, academic institutions, political parties, etc.—are political structures that "operate by distributing authority and setting a stage for the exercise of power (Zaleznik, 1970)." Organizational power grants individuals preferred access to scarce resources or wider oversight over vital activities (Black, Hollingsworth, Nunes, and Simon, 2022), which, in turn, motivates influencing efforts to acquire such power and leveraging it to influence the outcome of resource allocation within an organization. In this paper, we delve into the crafting of organizational rules that govern this process to promote the organization's interests.

We examine the interactions within an organization through the lens of a multilateral sequential bargaining model with costly recognition (Yildirim, 2007). A pool of agents—e.g., business units, academic departments, and R&D teams—divide a fixed surplus. In the beginning of each period, agents invest in nonrefundable efforts to earn the right to propose, securing an advantageous position in future resource allocation, which resembles a contest; the proposer then suggests how to split the surplus, and the rest vote. If the proposal is approved, the surplus is divided accordingly, and the game ends; otherwise, the process restarts.

This dynamic is commonplace in organizational life. Executives strive to climb corporate ladders, aiming for positions that offer significant decision-making influence. Within a company, more profitable business units generally receive priority in the allocation of corporate resources. Internal research teams may propose competing approaches to new product development; the favored team then guides the subsequent collective development effort (Deb, Kuvalekar, and Lipnowski, 2023). Furthermore, political activists who provide exceptional party services are often prioritized for electoral candidacy nominations (Mattozzi and Merlo, 2015).

The literature on multilateral bargaining has primarily assumed that the process is governed by exogenous organizational rules, which include a fixed voting rule and a fixed mechanism to recognize/select the proposer. Variations in the prevailing organizational rules can significantly affect agents' payoffs and reshape their incentives for strategic actions. This leads to a crucial question that organizations must confront: How should an organization strategically set its rules to induce agents' behavior that advance its interests? For instance, a firm's decision protocol—which can either limit or expand its key executives' power in influencing distributive outcome—can arguably be tailored according to its own objectives. Similarly, its evaluation and promotion mechanism—which selects the key executives—can also be adjusted strategically. This paper addresses these questions.

Snapshot of the Model The bargaining protocol is abstracted as a k-majority voting rule. A proposal needs at least $k-1$ favorable votes from the peers for approval. Following in the vein of Yildirim (2007), we model the mechanism to recognize the proposer as a generalized Tullock contest: Each agent's effort is transformed into an effective output and an individual's recognition probability is given by his output's proportion to the total. Agents are heterogeneous in that they differ in their effort costs, production technologies, and patience levels.

A designer can choose the voting rule k and set the recognition mechanism with two instruments. Specifically, she can impose a multiplicative bias to amplify or discount one's output; she can also create a headstart that adds to the output. Both of them can bias the competition for recognition in favor of a subset of the agents. A biased mechanism can intuitively mirror various measures that alter contenders' relative competitiveness. For instance, a preferred candidate in a company's succession process often receives significant positions—e.g., president or COO—which enhances their visibility to board members.

The design objective accommodates diverse preferences regarding the profile of agents' efforts and their recognition probabilities. We assume that she values effort contributions and thus the objective function (weakly) increases with each agent's effort. For instance, activists' services benefit a political party, enhancing its overall strategy and appeal. Similarly, a corporate executive's performance not only propels their own career forward but also contributes to the firm's value. Likewise, efforts of a research team in product development play a crucial role in the firm's success. The conventionally assumed objectives in the literature on optimal contest design—such as total effort maximization and expected winner's effort maximization—are both special cases of our model. Notably, our objective function accommodates concerns about the ex ante distribution of bargaining power, i.e., recognition probability profile. For instance, the designer may have fairness concerns and value more evenly distributed recognition opportunities.

Nature of the Problem The bargaining game with costly recognition can be viewed as a contest with an *endogenously determined prize.* Agents balance their potential payoffs from winning—being recognized—against those from losing. This difference effectively functions as *the prize spread* that motivates their efforts. The payoffs, and consequently the prize spread, ultimately hinge on their equilibrium continuation values. The winner offers a subset of peers—namely, agents in a winning coalition—their equilibrium continuation values to secure votes, whereas a loser receives his continuation value contingent on inclusion in a winning coalition, or nothing if excluded. However, agents' continuation values are endogenously and reflexively determined by their efforts: Continuation values and efforts, along with their recognition probabilities and the formation of winning coalitions, are jointly determined in the equilibrium.

This endogeneity substantially complicates the analysis of the game's equilibrium properties and the effects of varying organizational rules on its equilibrium outcome, which nullifies conventional approaches to optimal contest design. Suppose that we alter the voting rule—e.g., by increasing k—while keeping the recognition mechanism constant. Two effects emerge. First, there is a direct *prize effect*. An increase in k modifies each agent's continuation value and reshuffles all winning coalitions. Consequently, a proposer must secure more votes at varied costs, which in turn changes the payoff for successful recognition. Moreover, this alteration in continuation value impacts each agent's demand for his vote and the probability of being included in others' winning coalitions, thereby affecting his payoff when not recognized. Second, an indirect *rebalancing effect* occurs as a rise in k shifts the balance in the competition for recognition. This adjustment in prize spreads varies among heterogeneous agents, altering their incentives non-uniformly and potentially amplifying or reducing asymmetries in their inherent competencies, such as differences in production technologies and effort cost functions. The outcome is thus inherently uncertain.

Alternatively, modifying the recognition mechanism—which could bias the competition in favor of one subset of agents while keeping k fixed—also triggers a rebalancing effect and a prize effect, but through a different linkage. First, a direct rebalancing effect arises as a biased recognition mechanism reshapes agents' relative competitiveness. Second, an indirect prize effect emerges: The tilted playing field changes each agent's effort incentives and the equilibrium outcome, thus altering his continuation value. This, in turn, affects the agent's respective payoffs for winning and losing, and consequently, his prize spread.

Clearly, these intricate interactions, driven by the bargaining process, do not occur in contests with a fixed prize.

Summary of Results and Implications Despite the complications, our predictions are straightforward. We demonstrate that when the designer can deploy both sets of instruments (voting rule and recognition mechanism), the optimal approach always requires a *dictatorial voting rule* with $k = 1$: A proposal is accepted with the consent of only the proposer. As a result, each agent captures the entire surplus once recognized and receives nothing otherwise. The game essentially becomes a standard static (biased) contest. Furthermore, the designer sets the recognition mechanism without using additive headstarts, relying solely on multiplicative biases to tilt the playing field.

As stated earlier, varying either the voting rule or the recognition mechanism would trigger both a prize effect and a rebalancing effect, each with distinct underlying linkages. A dictatorial voting rule implies a winner-take-all distribution outcome, which generates the largest and an equalized prize spread for every agent. However, the impact of this equalized and fixed prize spread on the balance of the playing field remains indeterminate due to the agents' existing heterogeneity in terms of production technologies and effort costs, which constitutes the aforementioned rebalancing effect. Nevertheless, the optimal balance of the competition can always be adjusted by resetting the recognition mechanism—that is, the contest rules—as our results indicate. We show that properly set multiplicative biases can induce any desirable equilibrium outcome regarding effort supplies and the ex ante distribution of recognition opportunities, rendering additive headstarts redundant (Theorem 1).

It is worth noting that with $k = 1$, the bargaining game with costly recognition effectively becomes a standard contest. Adjusting the recognition mechanism tilts the playing field but no longer triggers the indirect prize effect. The voting rules and recognition mechanism play distinctly different roles in achieving the optimum: The designer leaves the rebalancing role exclusively to setting multiplicative biases, while maximizing incentives by setting k to 1.

To further illuminate the nature of the model, we consider two cases with restricted settings, each limiting the designer's choice to only one set of instruments. In the first case, an unbiased recognition mechanism is fixed, and the designer sets the voting rules. We demonstrate that a more inclusive voting rule—i.e., $k > 1$—may emerge to address a sufficiently strong fairness concern (Example 1): The designer adjusts k to induce more evenly distributed recognition opportunities. However, a unanimous voting rule could remain suboptimal even if the designer predominantly prefers an egalitarian ex ante distribution of recognition opportunities.

When the designer cannot adjust the voting rule but is allowed to alter the recognition mechanism, the optimum may involve a positive headstart that boosts an agent's effective output (Example 2). This observation stands in stark contrast to the findings in our main model and those obtained in the contest literature (e.g., Fu and Wu, 2020). We demonstrate that a headstart triggers a unique effect that does not occur in a standard contest with a constant prize.

The details will be discussed in depth after analytical results are presented.

Link to the Literature To the best of our knowledge, this paper provides the first formal analysis of the optimal rule design in multilateral bargaining with costly recognition.

An extensive body of literature has developed from the canonical framework established by Baron and Ferejohn (1989) to explore the process of distributive politics—e.g., Merlo and Wilson (1995, 1998); Banks and Duggan (2000); Eraslan (2002); Eraslan and Merlo (2002, 2017); Diermeier and Fong (2011); Diermeier, Prato, and Vlaicu (2015, 2016); and

Evdokimov (2023). The majority of this literature assumes that the proposer is exogenously and randomly selected from the agents.

A small, yet growing, strand of literature considers the selection of the proposer as an integral part of the political process, examining the endogenous formation of bargaining protocols. Yildirim (2007) models the process to select proposers as a contest in which agents exert costly effort to gain power, pioneering the integration of a contest model (generalized Tullock contest) with multilateral bargaining game to endogenize the recognition mechanism. He analyzes the the effect of the prevailing voting rule on the equilibrium total effort, and shows that with homogeneous agents, a more inclusive voting rule leads to lower total effort. Yildirim (2010) compares total efforts and distributive outcomes between persistent and transitory recognition procedures. Ali (2015) models the recognition process as an all-pay auction.

Our paper extends the effort to incorporate recognition mechanisms into a holistic distribution process and models the recognition process as an influencing competition. Our work is closely related to Yildirim (2007). Similar to Yildirim, we adopt a generalized Tullock contest, but we introduce heterogeneous production technologies with fewer restrictions, as well as nonlinear effort cost functions. Yildirim employs a *fixed* contest mechanism and assesses the impact of varying voting rules on total efforts in the context of symmetric agents. In contrast, we undertake a comprehensive exercise to design rules that govern the entire distribution process for fully heterogeneous agents—i.e., setting protocols for both the recognition process and voting—while assuming a general objective function concerned with effort and recognition probability profiles.

Several papers examine the endogenous formation of bargaining protocol without using contest approach. Diermeier, Prato, and Vlaicu (2015, 2016) employ a pre-bargaining process to determine proposal power in the bargaining over policy. In McKelvey and Riezman (1992, 1993); Muthoo and Shepsle (2014); and Eguia and Shepsle (2015), recognition probability is determined by seniority, which is endogenously voted on at the end of each session. Kim (2019) assumes that current and past proposers are excluded from the pool of eligible candidates when a round of bargaining fails to reach consensus. Jeon and Hwang (2022) assume that an agent's recognition probability and bargaining power depend on the previous bargaining outcome in a dynamic legislative bargaining model, leading to an oligopolistic outcome as the result of an evolutionary process. Agranov, Cotton, and Tergiman (2020) examine, both theoretically and experimentally, a repeated multilateral bargaining model in which the agenda setter can retain his power with the majoritarian support of other committee members.

Our paper is naturally linked to the literature on contest design and, particularly, that on optimally biased contests. We develop a technique similar to that of Fu and Wu (2020) and Fu, Wu, and Zhu (2023), who characterize the optimum without explicitly solving for the equilibrium. Our analysis complements these studies by embedding the contest in a multilateral sequential bargaining framework, which generates an endogenous prize spread.

The rest of this paper is structured as follows. Section 2 sets up the model and the design problem. Section 3 characterizes the equilibrium. Section 4 solves the optimal design problem and provides examples of the case with single instrument. Section 5 concludes. Proofs and derivation for equilibria in the examples are collected in the Appendix.

2 Model Setup

The game proceeds in two stages. A set of agents interact in the second stage to divide a fixed sum of surplus, which is modeled as a multilateral sequential bargaining process with costly recognition, à la Yildirim (2007, 2010) and Ali (2015); a designer sets the rules in the first stage that govern agents' subsequent interaction.

2.1 Multilateral Sequential Bargaining with Costly Recognition

A set of $n \geq 2$ agents, indexed by $\mathcal{N} := \{1, 2, \ldots, n\}$, decide how to divide a dollar. In each period $t = 0, 1, 2, \ldots$, one agent (proposer) makes a proposal $\boldsymbol{s}_t \in \triangle^{n-1} := \{(s_{1,t}, \ldots, s_{n,t}) : 0 \leq s_{i,t} \leq 1, \sum_{i \in \mathcal{N}} s_{i,t} = 1\}$, where $s_{i,t}$ denotes the share of the dollar each agent i is to secure under this proposal. Agents simultaneously vote in favor of or against the proposal. At the beginning of each period t, each agent exerts an effort $x_{i,t} \geq 0$ to vie for the proposing right, which incurs a cost $c_i(x_{i,t})$.

We assume a "k-majority" voting rule—with $1 \leq k \leq n$—for this sequential bargaining process: The proposal is approved if at least k agents accept it (including the proposer). Specifically, $k = n$ implies a unanimous rule wherein the proposal can be vetoed by any single dissident; $k = \lfloor n/2 \rfloor + 1$ refers to a simple majority rule; with $k = 1$, the proposer dictates the decision process.

Recognition Mechanism The proposer selection mechanism is modeled as a contest. In each period t, fixing an effort profile $\boldsymbol{x}_t := (x_{1,t}, \ldots, x_{n,t})$, the probability of an agent i's being recognized as the proposer is given by

$$p_i(\boldsymbol{x}_t) = \begin{cases} \dfrac{\tilde{f}_i(x_{i,t})}{\sum_{j\in\mathcal{N}} \tilde{f}_{j,t}(x_{j,t})}, & \sum_{j\in\mathcal{N}} \tilde{f}_j(x_{j,t}) > 0, \\ \dfrac{1}{n}, & \sum_{j\in\mathcal{N}} \tilde{f}_j(x_{j,t}) = 0. \end{cases} \tag{1}$$

The function $\tilde{f}_i(\cdot)$ is called impact function in the contest literature; it converts one's effort into his effective output in the competition, taking the form of

$$\tilde{f}_i(\cdot) := \alpha_i f_i(\cdot) + \beta_i,\ \forall\, i \in \mathcal{N}. \tag{2}$$

The function $f_i(\cdot)$ describes agent i's actual production technology, while the multiplicative bias $\alpha_i \geq 0$ and additive headstart $\beta_i \geq 0$ are set by the designer as a part of the game's rules and will be detailed later.

Preferences and Payoffs Each agent is risk neutral and has a discount factor $\delta_i \in (0, 1)$. Agents differ in the degrees of their patience. If a proposal is approved in period τ, an agent i's discounted payoff is[1]

$$\Pi_i := \delta_i^\tau s_{i,\tau} - \sum_{t=0}^{\tau} \delta_i^t c_i(x_{i,t}).$$

Solution Concept The bargaining game can be described as $\langle (\tilde{f}_i(\cdot))_{i\in\mathcal{N}}, (c_i(\cdot))_{i\in\mathcal{N}}, \boldsymbol{\delta}, k \rangle$, where $(\tilde{f}_i(\cdot))_{i\in\mathcal{N}}$ denotes the set of impact functions, $(c_i(\cdot))_{i\in\mathcal{N}}$ the set of effort cost functions, $\boldsymbol{\delta} := (\delta_1, \ldots, \delta_n)$ the set of discounting factors, and k the voting rule.

We assume that agents use stationary strategies whereby for each period t, agents' period-t actions are independent of the history (see Theorem 1 for the details of the strategies). We adopt the solution concept of the stationary subgame perfect equilibrium (SSPE) and drop the time subscript t throughout. A strategy profile is an SSPE if it is stationary and constitutes a subgame perfect equilibrium.

To ensure the equilibrium existence, we impose the following mild and standard regularity conditions:

Assumption 1 *For each $i \in \mathcal{N}$, $f_i(\cdot)$ and $c_i(\cdot)$ are twice differentiable in $(0, +\infty)$, satisfying $f_i(0) = 0$, $f_i'(\cdot) > 0$, $f_i''(\cdot) \leq 0$, $c_i(0) = 0$, $c_i'(\cdot) > 0$, and $c_i''(\cdot) \geq 0$.*

[1] If no agreement is reached, agent i's discounted payoff is $\Pi_i = -\sum_{t=0}^{+\infty} \delta_i^t c_i(x_{i,t})$.

2.2 Rule Design: Instruments and Objectives

We now lay out the design problem faced by the designer.

Design Instruments The designer sets the voting rule that governs the bargaining process, which is implemented by the choice of k, the minimum number of favourable votes required for the proposal's approval. Meanwhile, she can manipulate the mechanism of proposer recognition by setting the contest rules, which determine the probability of each agent's recognition for every given effort profile and, in turn, affect their effort incentives.

Recall that each agent i's impact function $\tilde{f}_i(\cdot)$ is given by (2). The designer imposes the multiplicative weights $\boldsymbol{\alpha} \in \mathbb{R}_+^n \setminus \{(0, \ldots, 0)\}$—which scale up or down one's output—and additive headstarts $\boldsymbol{\beta} \in \mathbb{R}_+^n$. One may view $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ as nominal scoring rules. Alternatively, they can be viewed as the organizational resources assigned to agents that alter their productivity or influence (see, e.g., Fu and Wu, 2022).

Both multiplicative weights $\boldsymbol{\alpha}$ and additive headstarts $\boldsymbol{\beta}$ are broadly adopted in modelling biased contests: Epstein, Mealem, and Nitzan (2011); and Franke, Kanzow, Leininger, and Schwartz (2014), for instance, consider the former; Konrad (2002); Siegel (2009, 2014); and Kirkegaard (2012) focus on the latter; Franke, Leininger, and Wasser (2018) and Fu and Wu (2020) allow for both. It is noteworthy that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ play subtly different roles in impacting the contest's outcome: $\boldsymbol{\alpha}$ alter the marginal returns of agents' efforts, while $\boldsymbol{\beta}$ directly add to their effective output regardless of their efforts.

Design Objectives As will be shown later in Theorem 1, there is no delay in each SSPE, and thus agents exert effort at most once on the equilibrium path. The designer chooses $(\boldsymbol{\alpha}, \boldsymbol{\beta}, k)$ to maximize an objective function $\Lambda(\boldsymbol{x}, \boldsymbol{p})$, where $\boldsymbol{x} := (x_1, \ldots, x_n)$ and $\boldsymbol{p} := (p_1, \ldots, p_n)$ denote the profiles of equilibrium efforts and agents' recognition probabilities, respectively. The following regularity condition is imposed.

Assumption 2 *Fixing $\boldsymbol{p}$, $\Lambda(\boldsymbol{x}, \boldsymbol{p})$ weakly increases with x_i for each $i \in \mathcal{N}$.*

By Assumption 2, we focus on the scenario in which agents' efforts are productive and accrue to the benefit of the designer. Consider, for example, internal R&D teams within a corporation crafting innovative proposals to compete for leadership roles in new product development, or executives enhancing their performance to climb the corporate ladder.

The objective function accommodates a diverse array of preferences. Consider, for example, $\Lambda(\boldsymbol{x}, \boldsymbol{p}) = \sum_{i \in \mathcal{N}} x_i - \lambda \sum_{i \in \mathcal{N}} \left| p_i - \frac{1}{n} \right|$, with $\lambda \geq 0$, which clearly satisfies Assumption 2. When $\lambda = 0$, this objective boils down to maximizing equilibrium total effort, which is conventionally assumed in the contest design literature. When $\lambda > 0$, the designer's payoff depends on the profile of agents' recognition probabilities. The term $\sum_{i\in\mathcal{N}}\left|p_i - \frac{1}{n}\right|$ is essentially the mean absolute deviation of $\boldsymbol{p}$, which increases in the dispersion of $\boldsymbol{p}$. The function thus depicts a preference for more equitable distribution of recognition opportunities, which compels the designer to set rules to reduce $\sum_{i\in\mathcal{N}}\left|p_i - \frac{1}{n}\right|$.[2]

Alternatively, consider $\Lambda(\boldsymbol{x}, \boldsymbol{p}) = \sum_{i\in\mathcal{N}} p_i x_i$, which is the expected winner's effort. Maximizing the expected winner's effort has gained increasing attention in the literature (e.g., Moldovanu and Sela, 2006; Barbieri and Serena, 2024). For instance, a firm often views its succession race as a process to develop managerial talent; the firm might benefit from the chosen successor's investment in their areas of expertise, as the losing candidates might pursue alternative career paths, especially in high-profile public firms. For instance, James McNerney joined and Robert Nardelli, respectively, joined 3M and Home Depot after they lost the race to succeed Jack Welch at General Electric.

3 Equilibrium Existence and Characterization

We now characterize the equilibrium. Let $\boldsymbol{v} := (v_i)_{i\in\mathcal{N}}$ be the set of agents' equilibrium expected payoffs and consider stage-undominated voting strategies, such that agents vote as if they were pivotal. Suppose that an agent is not recognized as the proposer, he accepts a proposal if his share exceeds the discounted continuation value—i.e., $s_i \geq \delta_i v_i$—and rejects it otherwise. The proposer, in contrast, needs to select $k-1$ agents to form the least costly winning coalition and offers to them their continuation values. His expected cost is

$$w_i = \sum_{j\neq i} \psi_{ij}\delta_j v_j,$$

where ψ_{ij} is the probability of agent i's including j in his offer. For each $j \in \mathcal{N}$, we further define $\mu_j := \sum_{i\neq j} \psi_{ij} p_i$ as agent j's probability of being included in others' winning coalitions before a proposer is recognized.

For each agent i, the expected payoff conditional on being the proposer is $1 - w_i$ and that when not selected is $\frac{\mu_i}{1-p_i}\delta_i v_i$. The agent's equilibrium effort x_i solves the maximization problem on the right-hand side of the following Bellman equation:

$$v_i = \max_{x_i\geq 0}\left\{p_i(x_i, \boldsymbol{x}_{-i})(1-w_i) + [1 - p_i(x_i, \boldsymbol{x}_{-i})] \times \frac{\mu_i}{1 - p_i(x_i, \boldsymbol{x}_{-i})}\delta_i v_i - c_i(x_i)\right\}. \quad (3)$$

[2] Eraslan and Merlo (2017) examine the distributive implications of voting rules. They show that unanimity may paradoxically lead to more unequal distributive outcome. It is noteworthy that in our context, the designer's fairness concern refers to her preference for ex ante distribution of bargaining power among agents—i.e., the recognition probability profile—instead of ex post distribution of the surplus.

The first-order condition with respect to x_i is

$$\underbrace{c_i'(x_i)}_{\text{marginal cost of effort}} \geq \underbrace{\frac{\tilde{f}_i'(x_i)}{\tilde{f}_i(x_i)} \times p_i(1-p_i) \times \overbrace{\left(1 - w_i - \frac{\mu_i}{1-p_i}\delta_i v_i\right)}^{\text{effective prize spread}}}_{\text{marginal benefit of effort}}. \tag{4}$$

Equations (3) and (4) depict the strategic nature of this game. The term $1 - w_i - \frac{\mu_i}{1-p_i}\delta_i v_i$ is the payoff differential between winning the competition for recognition and losing that; this is thus the effective prize spread that motivates each contender's effort. However, w_i, p_i, μ_i and v_i are all endogenously and jointly determined in the equilibrium, together with agents' effort profile $\boldsymbol{x} = (x_1, \ldots, x_n)$. These nuances complicate the analysis and differentiate the model from a standard contest with a fixed prize or a standard multilateral sequential bargaining game. Our analysis obtains the following.

Theorem 1 *Suppose that Assumption 1 holds. For each game $\langle (\tilde{f}_i(\cdot))_{i\in\mathcal{N}}, (c_i(\cdot))_{i\in\mathcal{N}}, \boldsymbol{\delta}, k\rangle$, there exists an SSPE characterized by $(\boldsymbol{x}, \boldsymbol{v})$ and $\{\psi_{ij}\}_{i\neq j}$. In the equilibrium, each agent $i \in \mathcal{N}$ exerts effort x_i in each period. If selected as the proposer, he forms a winning coalition of $k-1$ agents such that agent j is included with probability ψ_{ij} and offers the agent $\delta_j v_j$. Otherwise, he accepts a proposer's offer if and only if his share is no less than $\delta_i v_i$. The equilibrium is unique when $k = 1$.*

Theorem 1 establishes equilibrium existence of the game, which paves the way for optimal rule design. Assuming $\tilde{f}_i(0) = 0$, linear cost function, and weakly decreasing elasticity $x_i\tilde{f}_i'(x_i)/\tilde{f}_i(x_i)$ for each $i \in \mathcal{N}$, Yildirim (2007) verifies equilibrium uniqueness in the game. Our setting relaxes these restrictions, allowing for headstarts β_i which could lead to $\tilde{f}_i(0) \neq 0$, nonlinear cost functions $c_i(\cdot)$, and unrestricted elasticity conditions. The equilibria are, in general, nonunique in our context.

Because of the nuances caused by the endogenous payoff structure, a closed-form solution for the equilibrium is in general unavailable. This nullifies the usual implicit programming approach to optimal design commonly used in the contest literature. We develop a technique similar to Fu and Wu (2020), which enables us to characterize the optimum without explicitly solving for the equilibrium.

4 Optimal Rule Design

We now characterize the optimal organizational rules. We first present the main results—i.e., the optimum when the designer has full flexibility to adjust $(\boldsymbol{\alpha}, \boldsymbol{\beta}, k)$. We then consider two restricted cases, each limiting the designer's choice to one set of instruments, either the voting rule k or the recognition mechanism $(\boldsymbol{\alpha}, \boldsymbol{\beta})$. The observations stand in contrast to our main result and the conventional wisdom in the extant literature, which sheds further light on the nature of this model.

4.1 Main Result

The designer chooses $(\boldsymbol{\alpha}, \boldsymbol{\beta}, k)$ to maximize the objective function $\Lambda(\boldsymbol{x}, \boldsymbol{p})$. As previously noted, a change in either k or $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ catalyzes two effects.

A change in k alters each agent's prize spread, triggering a *prize effect*: Both w_i and $\frac{\mu_i}{1-p_i}\delta_i v_i$ depend on k, as demonstrated by condition (4). Imagine a dictatorial voting rule with $k = 1$, which yields $w_i = 0$ and $\delta_i v_i = 0$ since the proposer monopolizes the entire surplus, leaving others with nothing. In this case, each agent enters a winner-take-all contest with a prize spread of 1. Suppose that k increases to 2. This reduces the prize spread for all, as w_i must be strictly positive, allowing one to earn a positive payoff even when losing.

This prize effect is non-uniform across heterogeneous agents: Their w_i and $\frac{\mu_i}{1-p_i}\delta_i v_i$ may differ, which also respond to changes in k differently, implying a *rebalancing effect*. The asymmetric changes in prize spreads affect agents' relative competitiveness and incentives, which in turn influences the equilibrium outcome of the competition for recognition—i.e., agents' efforts and the recognition probability profile. To see this, suppose k increases from 1 to 2. Ceteris paribus, the most patient agent is unlikely to be included in any other agent's winning coalition, in which case his payoff for losing the competition remains unchanged, implying that his prize incentive decreases less than those of others. However, the impact on the balance of the competition remains ambiguous. The non-uniform decrease in prize spreads could either exacerbate or alleviate the asymmetry across agents caused by differences in their production technologies and effort cost functions. The indirect rebalancing effect emerges.

A change in the recognition mechanism $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ directly alters agents' relative competitiveness and the balance of the playing field. This, in turn, affects agents' continuation values, reshuffles their respective winning coalitions, and alters the costs of buying votes. Consequently, an indirect prize effect is triggered, and its effect on the equilibrium outcome is also ambiguous and complex.

The optimum moderates and reconciles the interaction of these forces. Our analysis concludes the following.

Theorem 2 *Suppose that Assumptions 1 and 2 hold. When the designer can flexibly choose $(\boldsymbol{\alpha}, \boldsymbol{\beta}, k)$, the optimum involves a dictatorial voting rule ($k = 1$) and zero headstart ($\boldsymbol{\beta} = \mathbf{0}$).*

By Theorem 2, the sequential bargaining game with costly recognition simplifies to a standard static contest with a prize spread of 1. A proposer does not need to form a winning coalition and relinquish his share. As a result, agents' patience levels no longer affect the equilibrium outcome.

The logic of these results can be interpreted in light of the interactions between the prize and rebalancing effects postulated earlier. A dictatorial voting rule ($k = 1$) results in a maximized prize spread, since both w_i and $\frac{\mu_i}{1-p_i}\delta_i v_i$ are zero. This provides the largest prize incentive to the agents and tempts them to strive for recognition. The ambiguous indirect effect on the balance of the playing field, however, can be corrected, if necessary, by adjusting the recognition mechanism $(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

As mentioned above, varying the recognition mechanism $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ also entails an ambiguous indirect prize effect, since a tilted competition affects agents' prize spreads. However, this effect is muted when k is set to 1. A dictatorial voting rule generates a fixed prize spread of 1, which allows the designer to set $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ for optimally balanced playing field while isolating further complications on agents' prize spreads.

Under a dictatorial voting rule $k = 1$, the bargaining game with costly recognition boils down to a standard contest. The findings from the contest literature (Fu and Wu, 2020) can be reinstated: The designer can induce any profile of equilibrium winning odds using $\boldsymbol{\alpha}$, and additive headstarts $\boldsymbol{\beta}$ are rendered redundant.

In summary, the two sets of instruments play different roles: The voting rule—with $k = 1$—maximizes the prize spread, while the multiplicative biases $\boldsymbol{\alpha}$ optimally exploit agents' heterogeneity in terms of innate abilities and sets the optimal competitive balance.

4.2 Further Discussions

Next, we examine two cases with restricted settings to shed further light on the nature of the model. Consider the objective function

$$\Lambda(\boldsymbol{x}, \boldsymbol{p}) = \sum_{i \in \mathcal{N}} x_i - \lambda \sum_{i \in \mathcal{N}} \left| p_i - \frac{1}{n} \right|, \text{ with } \lambda > 0. \tag{5}$$

We first consider a case in which the designer can only vary voting rule—i.e., choosing $k \in \{1, \dots, n\}$—while keeping the recognition mechanism neutral—i.e., $\boldsymbol{\alpha} = (1, \dots, 1)$ and $\boldsymbol{\beta} = (0, \dots, 0)$. We then examine a scenario in which the voting rule is fixed with $k = 2$ and the designer is allowed to set the contest rule $(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

Rule Design by Voting Rule under Fixed Recognition Mechanism Fix a neutral recognition mechanism $\boldsymbol{\alpha} = (1, \ldots, 1)$ and $\boldsymbol{\beta} = (0, \ldots, 0)$. We explore the optimal voting rule. We show that a dictatorial voting rule can be rendered suboptimal under this circumstance.

Example 1 *Suppose that $n = 3$, $f_i(x_i) = x_i$, $c_i(x_i) = c_i x_i$ with $\boldsymbol{c} = (\frac{49}{60}, \frac{48}{60}, \frac{47}{60})$, and $\boldsymbol{\delta} = (\frac{4}{7}, \frac{4}{8}, \frac{4}{9})$. The recognition mechanism is fixed and required to be neutral, with $\boldsymbol{\alpha} = (1, 1, 1)$ and $\boldsymbol{\beta} = (0, 0, 0)$. The designer chooses $k \in \{1, 2, 3\}$ to maximize her objective function (5). The equilibria under different voting rules are depicted in Table 1.*

	$k = 1$	$k = 2$	$k = 3$
Equilibrium efforts	$\frac{5}{6}(\frac{23}{72}, \frac{24}{72}, \frac{25}{72})$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	$\frac{405}{653}(\frac{23}{72}, \frac{24}{72}, \frac{25}{72})$
Total effort	$\frac{5}{6}$	$\frac{3}{4}$	$\frac{405}{653}$
Winning probability	$(\frac{23}{72}, \frac{24}{72}, \frac{25}{72})$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$(\frac{23}{72}, \frac{24}{72}, \frac{25}{72})$
Equilibrium payoff	$\left((\frac{23}{72})^2, (\frac{24}{72})^2, (\frac{25}{72})^2\right)$	$(\frac{7}{60}, \frac{8}{60}, \frac{9}{60})$	$\frac{486}{653}\left(\frac{7}{3}\left(\frac{23}{72}\right)^2, \frac{8}{4}\left(\frac{24}{72}\right)^2, \frac{9}{5}\left(\frac{25}{72}\right)^2\right)$
Designer payoff	$\frac{5}{6} - \frac{\lambda}{36}$	$\frac{3}{4}$	$\frac{405}{653} - \frac{\lambda}{36}$

Table 1: Equilibrium Outcomes in Example 1.

When $k = 1$, all three agents have an effective prize spread of 1. Given $f_i(x_i) = x_i$ for all $i \in \{1, 2, 3\}$, agents' relative competitiveness is determined entirely by the heterogeneity in their effort costs. Consequently, agents' equilibrium winning probabilities would be ranked as $p_1 < p_2 < p_3$, with agent 3 being the frontrunner given his lowest marginal effort cost ($c_3 < c_2 < c_1$).

Suppose instead that k increases to 2. As mentioned earlier, this change generates a prize effect and further triggers a rebalancing effect. Agents' patience δ_i now influences their valuations of the win: Ceteris paribus, the most patient agent is least likely to be included in a winning coalition when losing the competition, thus facing the largest prize incentive. The non-uniform change in prize spreads alters the balance of the playing field. In this specific example, a more patient agent also has a higher marginal effort cost ($c_3 < c_2 < c_1$ and $\delta_3 < \delta_2 < \delta_1$). As a result, setting $k = 2$ mitigates the imbalance caused by the heterogeneity in effort costs. The more inclusive voting rule—i.e., $k = 2$—reduces agents' prize valuations, which disincentivizes efforts but offsets the asymmetry caused by heterogeneous effort cost functions.

In this particular case, a voting rule $k = 2$ perfectly levels the playing field, enabling all agents to win with the same probability. Consequently, the designer would abandon dictatorship if λ is sufficiently large—i.e., with a strong preference for evenness—since this leads to the most equally distributed recognition probabilities.

This results stand in contrast to the observations obtained in the baseline model. When k and $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ are set together, the optimal balance of the contest is addressed by setting $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ to achieve desirable distribution outcome. However, this is infeasible in this example. The designer has to adjust the voting rule to exploit the indirect rebalancing effect to achieve a more even outcome while sacrificing effort supplies.

Rule Design with Recognition Mechanism under Fixed Voting Rule We then examine a case with fixed voting rule. We demonstrate that additive headstarts can be rendered a part of the optimal rules.

Example 2 *Suppose that $n = 3$, $k = 2$, $f_i(x_i) = x_i$, and $c_i(x_i) = c_i x_i$ with $(c_1, c_2, c_3) = (1, 1, c)$. Let $(\delta_1, \delta_2, \delta_3) = (\frac{3}{8}, \frac{1}{2}, \frac{12}{13})$. Assume that λ is sufficiently large and c is sufficiently small, with $\lambda \gg \frac{1}{c} \gg 1$.*

Intuitively, the optimal recognition mechanism requires $p_1 = p_2 = p_3 = 1/3$ when λ is sufficiently large (i.e., $\lambda \gg \frac{1}{c}$). Moreover, the total effort depends crucially on agent 3 when he is excessively strong (i.e., $\frac{1}{c} \gg 1$). These observations enable us to conclude $\boldsymbol{\alpha}^* = (\frac{62Y}{35}, \frac{62Y}{37}, \frac{62Yc}{39})$ and $\boldsymbol{\beta}^* = (0, \frac{17Y}{222}, 0)$, where $Y > 0$ is an arbitrary positive constant, in the optimal recognition mechanism. The game yields an equilibrium outcome of $\boldsymbol{x} = (\frac{70}{372}, \frac{57}{372}, \frac{78}{372c})$ and $\boldsymbol{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. The designer's payoff is $\Lambda = \frac{127}{372} + \frac{78}{372c}$.

	Agent 1	Agent 2	Agent 3
Equilibrium efforts	70/372	57/372	$78/(372c)$
Winning probability	1/3	1/3	1/3
Equilibrium payoff	56/372	72/372	39/372
Winning coalition	$\{1, 2\}$	$\{1, 2\}$	$\{1, 3\}$

Table 2: Equilibrium Outcomes in Example 2.

The prediction of Example 2 departs from the findings of the extant literature on contest design. Fu and Wu (2020), for instance, formally establish that headstart is suboptimal for a design objective function (5). This contrast reveals how the endogenous prize structure differentiates the game from a standard contest.

With $k = 2$, varying the recognition mechanism triggers both the direct rebalancing effect and the indirect prize effect. The designer values both efforts and an even distribution of recognition opportunities. The two concerns could be at odds, but the use of headstart provides an avenue to achieve both through the indirect prize effect.

Note that in this setting, $c_1 = c_2 > c_3$ and $\delta_1 < \delta_2 < \delta_3$: Agent 3 is ex ante the strongest contender, followed by agent 2, then agent 1. The designer would benefit if agent 3 can be sufficiently incentivized given his low effort cost. As the most patient agent, agent 3 will not be included in any other's winning coalition (see Table 2), which tends to amplify his prize spread and incentivize his effort. However, his prize spread also depends on the cost of buying votes, i.e., w_i. By Table 2, he would include agent 1 in his winning coalition, offering the latter's continuation value for his approval. To more effectively incentivize agent 3, the designer may reduce agent 1's continuation value—or equivalently, his expected payoff—to enlarge agent 3's prize spread.

Further, by Table 2, agent 1 would buy agent 2's vote upon being the proposer. To reduce agent 1's equilibrium payoff, the designer needs to increase agent 2's continuation value. This can be achieved by awarding either a positive headstart $\beta_2 > 0$ to agent 2 or assigning a larger multiplicative α_2. However, the former is effective than the latter in this regard. Both approaches increase agent 2's recognition probabilities and improve his payoffs. However, a larger α_2 increases the marginal benefit of effort, which promotes his effort supply; effort is costly and reduces agent 2's payoff. In contrast, a headstart increases agent 2's recognition probability without eliciting more effort, which more effectively boosts agent 2's payoff, while increasing agent 1's vote-buying cost and diminishing his continuation value.

These effects stem from the dynamic bargaining process and are absent in a simple static contest, in which case the prize spread is exogenous and independent of the rule of the recognition mechanism $(\boldsymbol{\alpha}, \boldsymbol{\beta})$. In a standard contest, the equilibrium is governed by the first-order condition

$$\underbrace{c_i'(x_i)}_{\text{marginal cost of effort}} \geq \underbrace{\frac{\tilde{f}_i'(x_i)}{\tilde{f}_i(x_i)} \times p_i(1-p_i) \times \text{fixed prize spread}}_{\text{marginal benefit of effort}}.$$

Fu and Wu (2020) verify that setting $\boldsymbol{\alpha}$ alone can induce any desirable contest outcome; multiplicative biases $\boldsymbol{\alpha}$ can more effectively motivate efforts than additive headstarts $\boldsymbol{\beta}$ because of the former's direct impact on marginal benefits of efforts, rendering $\boldsymbol{\beta}$ redundant. The same logic applies in the current framework with $k = 1$. However, with $k \geq 2$, in addition to the first-order condition (4), agents' behavior is also subject to the Bellman equation (3): An agent's equilibrium effort affects his expected payoff, as well as his continuation value, which in turn alters the overall equilibrium through its impact on others' prize valuations (effective prize spreads). In this particular example, varying $\boldsymbol{\beta}$ creates an opportunity for the designer to exploit the endogenous payoff structure of the game through the indirect prize effect.

5 Concluding Remarks

In this paper, we explore the design of optimal organizational rules that govern a sequential multilateral bargaining game with costly recognition, in which the right to propose a plan for dividing resources is determined by a contest. We consider two sets of design instruments: (i) the voting rule that governs how proposals are accepted or rejected; and (ii) the recognition mechanism that determines how the proposer is selected based on agents' efforts. When both sets of instruments are deployed together, the optimum always involves a dictatorial voting rule, which simplifies the bargaining game with costly recognition into a standard contest.

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Appendix: Proofs

Proof of Theorem 1

Proof. We first characterize the SSPE assuming its existence, then prove equilibrium existence.

Equilibrium Characterization Denote by V^Δ the k-th lowest continuation value. Let $\mathcal{N}_1 := \{i \in \mathcal{N} : \delta_i v_i < V^\Delta\}$, $\mathcal{N}_2 := \{i \in \mathcal{N} : \delta_i v_i = V^\Delta\}$, and $\mathcal{N}_3 := \{i \in \mathcal{N} : \delta_i v_i > V^\Delta\}$. Evidently, agent i, when becoming the proposer, buys out the votes of the cheapest "winning coalition"—i.e., $\mathcal{N}_1$ and a subset of $\mathcal{N}_2$, from which we can conclude

$$\psi_{ij} \begin{cases} = 0, & j \in \mathcal{N}_3 \text{ or } i = j, \\ = 1, & j \in \mathcal{N}_1 \text{ and } i \neq j, \\ \in [0,1], & \text{otherwise,} \end{cases} \quad \text{and } \mu_i \begin{cases} = 0, & i \in \mathcal{N}_3, \\ \in [0, 1-p_i], & i \in \mathcal{N}_2, \\ = 1 - p_i, & i \in \mathcal{N}_1. \end{cases} \tag{6}$$

Define

$$V_L := 1 - \sum_{j \in \mathcal{N}_1} \delta_j v_j - (k - |\mathcal{N}_1|) V^\Delta. \tag{7}$$

His expected cost is then

$$w_i = \begin{cases} 1 - V_L - \delta_i v_i, & i \in \mathcal{N}_1, \\ 1 - V_L - V^\Delta, & \text{otherwise.} \end{cases}$$

The effective prize spread $1 - w_i - \frac{\mu_i}{1-p_i}\delta_i v_i$ in (4) can be expressed as

$$1 - w_i - \frac{\mu_i}{1-p_i}\delta_i v_i = V_L + \frac{1-\mu_i-p_i}{1-p_i} V^\Delta = \begin{cases} V_L, & i \in \mathcal{N}_1, \\ V_L + \frac{1-p_i-\mu_i}{1-p_i} V^\Delta, & i \in \mathcal{N}_2, \\ V_L + V^\Delta, & i \in \mathcal{N}_3. \end{cases} \tag{8}$$

We are ready to lay out the conditions for equilibrium characterization. An SSPE can be characterized by $(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{p}, \boldsymbol{\mu}, V_L, V^\Delta)$. Combining (4) and (8) yields

$$\frac{c_i'(x_i)\tilde{f}_i(x_i)}{\tilde{f}_i'(x_i)} \geq p_i(1-p_i)\left(V_L + \frac{(1-p_i-\mu_i)V^\Delta}{1-p_i}\right). \tag{9}$$

Next, consider the expected payoff v_i. By (3), we have

$$v_i = p_i(1-w_i) + \mu_i\delta_i v_i - c_i(x_i) = \begin{cases} \frac{1}{1-\delta_i}\left(p_i V_L - c_i(x_i)\right), & i \in \mathcal{N}_1, \\ \frac{V^\Delta}{\delta_i}, & i \in \mathcal{N}_2, \\ p_i(V_L + V^\Delta) - c_i(x_i), & i \in \mathcal{N}_3. \end{cases} \tag{10}$$

Combining (3), (6), and (10) yields

$$\mu_i \begin{cases} = 1 - p_i, & i \in \mathcal{N}_1, \\ \in [0, 1-p_i] \text{ solves } \frac{V^\Delta}{\delta_i} = p_i V_L + (\mu_i + p_i) V^\Delta - c_i(x_i), & i \in \mathcal{N}_2, \\ = 0, & i \in \mathcal{N}_3. \end{cases} \tag{11}$$

Each agent chooses exactly $k-1$ agents in his winning coalition—i.e., $\sum_{j\in\mathcal{N}} \psi_{ij} = k-1$, $\forall i \in \mathcal{N}$. Therefore,

$$\sum_{i\in\mathcal{N}} \mu_i = \sum_{i\in\mathcal{N}} \sum_{j\in\mathcal{N}} \psi_{ji} p_j = \sum_{j\in\mathcal{N}} p_j \sum_{i\in\mathcal{N}} \psi_{ji} = \sum_{j\in\mathcal{N}} (k-1) p_j = k-1. \tag{12}$$

Last, (7) can be rewritten as

$$V_L + \sum_{i\in\mathcal{N}_1} (\delta_i v_i) + \left(k - |\mathcal{N}_1|\right) V^\Delta = 1. \tag{13}$$

To characterize an SSPE, it suffices to find $(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{p}, \boldsymbol{\mu}, V^\Delta, V_L)$ that satisfies (9)-(13).

Equilibrium Existence Let $Y := \sum_{i\in\mathcal{N}} \tilde{f}_i(x_i)$. By (1), we have $p_i = \tilde{f}_i(x_i)/Y$, which implies that

$$x_i = \tilde{f}_i^{-1}(Y p_i), \text{ for } p_i \in \left[\tilde{f}_i(0)/Y, 1\right], \tag{14}$$

and

$$\sum_{i\in\mathcal{N}} p_i = 1. \tag{15}$$

Substituting (14) into (9) yields

$$\frac{Y c_i'\left(\tilde{f}_i^{-1}(Y p_i)\right)}{\tilde{f}_i'\left(\tilde{f}_i^{-1}(Y p_i)\right)} \geq (1-p_i)(V_L + V^\Delta) - \mu_i V^\Delta, \text{with equality holding if } p_i > \frac{\tilde{f}_i(0)}{Y}. \tag{16}$$

Rewriting (11) and (12) and substituting (10) into (13) yield

$$\mu_i = \frac{1}{V^\Delta} \text{med} \left\{ 0, V^\Delta(1-p_i), \frac{V^\Delta}{\delta_i} - p_i(V_L + V^\Delta) + c_i\left(\tilde{f}_i^{-1}(Y p_i)\right) \right\}, \tag{17}$$

$$\sum_{i\in\mathcal{N}} \mu_i = k-1, \tag{18}$$

and

$$\sum_{i\in\mathcal{N}_1} \frac{\delta_i}{1-\delta_i} \left[p_i V_L - c_i\left(\tilde{f}_i^{-1}(Y p_i)\right) \right] + \left(k - |\mathcal{N}_1|\right) V^\Delta + V_L = 1, \tag{19}$$

where med$\{\cdot, \cdot, \cdot\}$ gives the median of the input.

To prove equilibrium existence, it suffices to show that there exists $(\boldsymbol{p}, \boldsymbol{\mu}, Y, V^\Delta, V_L)$ to satisfy conditions (15)-(19). The proof consists of four steps. First, fixing (Y, V^Δ, V_L), we show that there exists a unique $(\boldsymbol{p}, \boldsymbol{\mu})$ to satisfy (16) and (17). Second, fixing (V^Δ, V_L), there exists $Y \geq \sum_{i\in\mathcal{N}} \tilde{f}_i(0)$ to satisfy (15). Third, fixing V_L, there exists V^Δ to satisfy (18). Last, we show that there exists V_L to satisfy (19).

Step I Substituting (17) into (16) yields

$$\frac{Yc_i'\big(\tilde{f}_i^{-1}(Yp_i)\big)}{\tilde{f}_i'\big(\tilde{f}_i^{-1}(Yp_i)\big)} \geq \text{med}\left\{(1-p_i)(V_L+V^\Delta), (1-p_i)V_L, V_L+V^\Delta-\frac{V^\Delta}{\delta_i}-c_i\big(\tilde{f}_i^{-1}\big(Yp_i\big)\big)\right\}, \tag{20}$$

with equality holding if $p_i > \frac{\tilde{f}_i(0)}{Y}$.

Let

$$\phi(p_i) := \frac{Yc_i'\big(\tilde{f}_i^{-1}(Yp_i)\big)}{\tilde{f}_i'\big(\tilde{f}_i^{-1}(Yp_i)\big)} - \text{med}\left\{(1-p_i)(V_L+V^\Delta), (1-p_i)V_L, V_L+V^\Delta-\frac{V^\Delta}{\delta_i}-c_i\big(\tilde{f}_i^{-1}(Yp_i)\big)\right\}.$$

By Assumption 1 and (2), $\tilde{f}_i(\cdot)$ is increasing and concave, implying that $\phi(\cdot)$ strictly increases with p_i. Therefore, if $\phi\big(\frac{\tilde{f}_i(0)}{Y}\big) \geq 0$, or equivalently,

$$\frac{Yc_i'(0)}{\tilde{f}_i'(0)} \geq \text{med}\left\{\left(1-\frac{\tilde{f}_i(0)}{Y}\right)V_L, \left(1-\frac{\tilde{f}_i(0)}{Y}\right)(V_L+V^\Delta), V_L+V^\Delta-\frac{V^\Delta}{\delta_i}\right\}, \tag{21}$$

then $p_i = \frac{\tilde{f}_i(0)}{Y}$. Otherwise, if $\phi\big(\frac{\tilde{f}_i(0)}{Y}\big) < 0$, or equivalently,

$$\frac{Yc_i'(0)}{\tilde{f}_i'(0)} < \text{med}\left\{\left(1-\frac{\tilde{f}_i(0)}{Y}\right)V_L, \left(1-\frac{\tilde{f}_i(0)}{Y}\right)(V_L+V^\Delta), V_L+V^\Delta-\frac{V^\Delta}{\delta_i}\right\}, \tag{22}$$

then $p_i > \frac{\tilde{f}_i(0)}{Y}$; moreover, p_i is uniquely pinned down by $\phi(p_i) = 0$, or equivalently,

$$\frac{Yc_i'\big(\tilde{f}_i^{-1}(Yp_i)\big)}{\tilde{f}_i'\big(\tilde{f}_i^{-1}\big(Yp_i\big)\big)} = \text{med}\left\{(1-p_i)(V_L+V^\Delta), (1-p_i)V_L, V_L+V^\Delta-\frac{V^\Delta}{\delta_i}-c_i\big(\tilde{f}_i^{-1}(Yp_i)\big)\right\}. \tag{23}$$

Further, μ_i can be uniquely solved from (17). Therefore, fixing (Y, V^Δ, V_L), there exists unique pair (p_i, μ_i) to satisfy (16) and (17), which we denote by $\big(p_i(Y, V^\Delta, V_L), \mu_i(Y, V^\Delta, V_L)\big)$ with slight abuse of notation.

Step II We show that fixing (V^Δ, V_L) and $\{p_i(Y, V^\Delta, V_L), \mu_i(Y, V^\Delta, V_L))\}_{i\in\mathcal{N}}$, there exists $Y \geq \sum_{i\in\mathcal{N}} \tilde{f}_i(0)$ to satisfy (15). By definition of $p_i(Y, V^\Delta, V_L)$, $p_i(Y, V^\Delta, V_L) \geq \frac{\tilde{f}_i(0)}{Y}$, which implies

$$\sum_{i\in\mathcal{N}} p_i\left(\sum\nolimits_{j\in\mathcal{N}} \tilde{f}_j(0), V^\Delta, V_L\right) \geq 1.$$

Next, we claim that

$$\lim_{Y\to+\infty} \sum_{i\in\mathcal{N}} p_i(Y, V^\Delta, V_L) = 0.$$

Suppose $\tilde{f}_i'(0) < +\infty$. Then (21) holds as Y approaches infinity, in which case $p_i = \frac{\tilde{f}_i(0)}{Y}$ and

$$\lim_{Y\to+\infty} p_i(Y, V^\Delta, V_L) = \lim_{Y\to+\infty} \frac{\tilde{f}_i(0)}{Y} = 0.$$

Suppose $\tilde{f}_i'(0) = +\infty$. Then (22) holds for all Y and $p_i(Y, V^\Delta, V_L)$ solves (23). As Y approaches infinity, the right-hand side of (23) is finite; therefore, the left-hand side must be finite, indicating that $p_i(Y, V^\Delta, V_L)$ approaches 0.

By the intermediate value theorem, there exists $Y \geq \sum_{i\in\mathcal{N}} \tilde{f}_i(0)$ such that

$$\sum_{i\in\mathcal{N}} p_i(Y, V^\Delta, V_L) = 1.$$

In what follows, we denote the largest Y determined by (V^Δ, V_L) by $Y(V^\Delta, V_L)$.

Step III Fixing V_L, $Y(V^\Delta, V_L)$ and $\{p_i(Y, V^\Delta, V_L), \mu_i(Y, V^\Delta, V_L))\}_{i\in\mathcal{N}}$, we show that there exists V^Δ such that (18) holds, i.e.,

$$\sum_{i\in\mathcal{N}} \mu_i\left(Y(V^\Delta, V_L), V^\Delta, V_L\right) = k - 1. \tag{24}$$

First, consider the case where V^Δ approaches 0. For each $i \in \mathcal{N}$, when $p_i = \frac{\tilde{f}_i(0)}{Y}$, we have that

$$\begin{aligned}&\lim_{V^\Delta\searrow 0} \mu_i\left(Y(V^\Delta, V_L), V^\Delta, V_L\right)\\ =&\lim_{V^\Delta\searrow 0} \frac{1}{V^\Delta}\text{med}\left\{0, V^\Delta\left(1 - \frac{\tilde{f}_i(0)}{Y(V^\Delta, V_L)}\right), \frac{V^\Delta}{\delta_i} - \frac{\tilde{f}_i(0)}{Y(V^\Delta, V_L)}(V_L + V^\Delta)\right\} = 0,\end{aligned}$$

where the second equality follows from the fact that $\frac{V^\Delta}{\delta_i} - \frac{\tilde{f}_i(0)}{Y(V^\Delta,V_L)}(V_L + V^\Delta) \leq 0 \leq V^\Delta\big(1 - \frac{\tilde{f}_i(0)}{Y(V^\Delta,V_L)}\big)$ as V^Δ approaches 0.

When $p_i > \frac{\tilde{f}_i(0)}{Y}$, by (17), $\mu_i\left(Y(V^\Delta, V_L), V^\Delta, V_L\right) = 0$ for sufficiently small V^Δ. Therefore, we have that

$$\lim_{V^{\Delta}\searrow 0}\sum_{i\in\mathcal{N}}\mu_i\left(Y(V^{\Delta},V_L),V^{\Delta},V_L\right)=0.$$

Next, consider the case where V^{Δ} approaches infinity. For each $i\in\mathcal{N}$, we have that

$$\begin{aligned}0\leq & V^{\Delta}\left[1-p_i\left(Y(V^{\Delta},V_L),V^{\Delta},V_L\right)\right]\\ \leq & \frac{V^{\Delta}}{\delta_i}-p_i\left(Y(V^{\Delta},V_L),V^{\Delta},V_L\right)(V_L+V^{\Delta})+c_i\left(\tilde{f}_i^{-1}\left(Y(V^{\Delta},V_L)p_i\left(Y(V^{\Delta},V_L),V^{\Delta},V_L\right)\right)\right);\end{aligned}$$

together with (17), we can obtain that

$$\mu_i\left(Y(V^{\Delta},V_L),V^{\Delta},V_L\right)=1-p_i\left(Y(V^{\Delta},V_L),V^{\Delta},V_L\right),\text{ as }V^{\Delta}\to+\infty.$$

Therefore,

$$\lim_{V^{\Delta}\to+\infty}\sum_{i\in\mathcal{N}}\mu_i\left(Y(V^{\Delta},V_L),V^{\Delta},V_L\right)=\lim_{V^{\Delta}\to+\infty}\sum_{i\in\mathcal{N}}\left[1-p_i\left(Y(V^{\Delta},V_L),V^{\Delta},V_L\right)\right]=n-1.$$

Note that $\mu_i(Y,V^{\Delta},V_L)$ and $Y(V^{\Delta},V_L)$ are continuous for all $i\in\mathcal{N}$, and $0\leq k-1\leq n-1$. Therefore, there exists $V^{\Delta}\geq 0$ to satisfy (24). In what follows, we denote the largest V^{Δ} determined by V_L by $V^{\Delta}(V_L)$.

Step IV We show that there exists $V_L\in[0,1]$ to satisfy (19), i.e.,

$$\sum_{i\in\mathcal{N}_1}\frac{\delta_i}{1-\delta_i}\left[p_iV_L-c_i\left(\tilde{f}_i^{-1}(Yp_i)\right)\right]+\left(k-|\mathcal{N}_1|\right)V^{\Delta}+V_L=1,\tag{25}$$

where $V^{\Delta}=V^{\Delta}(V_L)$, $Y=Y(V^{\Delta},V_L)$, and $p_i=p_i(Y,V^{\Delta},V_L)$ for $i\in\mathcal{N}$, as defined above.

The left-hand side of (25) is non-negative; moreover, it is no less than 1 when $V_L=1$. To conclude the proof, it suffices to show that $\lim_{V_L\searrow 0}V^{\Delta}(V_L)=0$, from which we can conclude that the left-hand side of (25) approaches 0 as $V_L\searrow 0$.

Suppose, to the contrary, that $\lim_{V_L\searrow 0}V^{\Delta}(V_L)>0$. Then, as $V_L\searrow 0$, we have that

$$0\leq V^{\Delta}(1-p_i)<\frac{V^{\Delta}}{\delta_i}-p_i(V_L+V^{\Delta})+c_i\left(\tilde{f}_i^{-1}(Yp_i)\right),\ \forall\, i\in\mathcal{N}.$$

Recall that $\mathcal{N}_2$ is nonempty by definition. That is, there exists some agent $j\in\mathcal{N}_2$. By (17), we have

$$V^{\Delta}(1-p_j)\geq\frac{V^{\Delta}}{\delta_j}-p_j(V_L+V^{\Delta})+c_j\left(\tilde{f}_j^{-1}(Yp_j)\right).$$

A contradiction. ■

Proof of Theorem 2

Proof. We first show that the optimum can be achieved by setting $k = 1$. It suffices to show that for each $(\boldsymbol{\alpha}, \boldsymbol{\beta}, k)$ and a resulting equilibrium, there exists $(\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}})$ such that $(\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}, 1)$ induces the same equilibrium effort profile $\boldsymbol{x}$ and recognition probabilities $\boldsymbol{p}$.

By (1), (2) and (9), we have

$$p_i = \frac{\alpha_i f_i(x_i) + \beta_i}{\sum_{j\in\mathcal{N}}\left[\alpha_j f_j(x_j) + \beta_j\right]},$$

and

$$c_i'(x_i)\frac{\alpha_i f_i(x_i) + \beta_i}{\alpha_i f_i'(x_i)} \geq p_i(1-p_i)\left(V_L + \frac{(1-p_i-\mu_i)V^\Delta}{1-p_i}\right), \tag{26}$$

with equality holding if $x_i > 0$.

We construct $(\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}})$ as follows. For $x_i = 0$, we set $(\hat{\alpha}_i, \hat{\beta}_i) = (0, p_i)$. For $x_i > 0$, note by (13), we have that

$$1 = \sum_{i\in\mathcal{N}_1}(\delta_i v_i) + (k - |\mathcal{N}_1|)V^\Delta + V_L \geq V^\Delta + V_L, \tag{27}$$

where the inequality follows from $v_i \geq 0$ and $|\mathcal{N}_1| \leq k-1$. Combining (26) and (27) yields

$$\frac{c_i'(x_i)f_i(x_i)}{f_i'(x_i)} \leq c_i'(x_i)\frac{\alpha_i f_i(x_i) + \beta_i}{\alpha_i f_i'(x_i)} = p_i(1-p_i)\left(V_L + \frac{(1-p_i-\mu_i)V^\Delta}{1-p_i}\right) \leq p_i(1-p_i).$$

Define $\hat{\theta}_i := p_i(1-p_i)f_i'(x_i)/c_i'(x_i) - f_i(x_i)$. The above inequality indicates $\hat{\theta}_i \geq 0$. Set

$$\left(\hat{\alpha}_i, \hat{\beta}_i\right) := \left(\frac{p_i}{f_i(x_i) + \hat{\theta}_i}, \hat{\alpha}_i\hat{\theta}_i\right). \tag{28}$$

It remains to verify that $(\boldsymbol{x}, \boldsymbol{p})$ is the unique equilibrium effort profile and recognition probabilities under $(\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}, 1)$. When $k = 1$, the game degenerates to a standard contest with prize value of unity. It suffices to show that

$$p_i = \frac{\hat{\alpha}_i f_i(x_i) + \hat{\beta}_i}{\sum_{j\in\mathcal{N}}\left[\hat{\alpha}_j f_j(x_j) + \hat{\beta}_j\right]}, \tag{29}$$

and x_i solves

$$\max_{x_i\geq 0} \frac{\hat{\alpha}_i f_i(x_i) + \hat{\beta}_i}{\sum_{j\in\mathcal{N}}\left[\hat{\alpha}_j f_j(x_j) + \hat{\beta}_j\right]} - c_i(x_i). \tag{30}$$

Note that $p_i = \hat{\alpha}_i f_i(x_i) + \hat{\beta}_i$ for all $i \in \mathcal{N}$ by construction (see, e.g., (28)). Therefore, $\sum_{j\in\mathcal{N}}(\hat{\alpha}_j f_j(x_j) + \hat{\beta}_j) = \sum_{j\in\mathcal{N}} p_j = 1$, which implies (29).

Next, we verify that x_i solves the maximization problem (30). For $i \in \mathcal{N}$ with $x_i = 0$, it is evident that choosing $x_i = 0$ dominates $x_i > 0$ under $(\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\beta}}, 1)$ because $\hat{\alpha}_i = 0$. For $i \in \mathcal{N}$ with $x_i > 0$, by (28), we have

$$c_i'(x_i)\frac{\hat{\alpha}_i f_i(x_i) + \hat{\beta}_i}{\hat{\alpha}_i f_i'(x_i)} = c_i'(x_i)\frac{f_i(x_i) + \hat{\theta}_i}{f_i'(x_i)} = p_i(1 - p_i),$$

which is exactly the first-order condition for (30).

The above analysis shows that the optimum can be achieved by $k = 1$, in which case the game reduces to a standard static contest. By Theorem 2 in Fu and Wu (2020), the optimum can be achieved by choosing multiplicative biases $\boldsymbol{\alpha}$ only and setting headstart $\boldsymbol{\beta}$ to zero under Assumption 2. ■

Derivation for Equilibria in Example 1

First, consider the case of $k = 1$. The game reduces to a static Tullock contest. Let $Y := x_1 + x_2 + x_3$. The equilibrium conditions can be derived as

$$Yc_i = 1 - p_i,$$

from which we can solve for the aggregate effort Y and the equilibrium winning probabilities $\boldsymbol{p} := (p_1, p_2, p_3)$, and equilibrium efforts $\boldsymbol{x} := (x_1, x_2, x_3)$ as follows:

$$Y = \frac{2}{\sum_{i \in \mathcal{N}} c_i} = \frac{5}{6},$$

$$\boldsymbol{p} = 1 - Y\boldsymbol{c} = \left(\frac{23}{72}, \frac{24}{72}, \frac{25}{72}\right),$$

and

$$\boldsymbol{x} = Y\boldsymbol{p} = \frac{5}{6}\left(\frac{23}{72}, \frac{24}{72}, \frac{25}{72}\right),$$

Next, consider the case of $k = 2$. The equilibrium conditions in the proof of Theorem 1—i.e., conditions (14), (15), (16), (17), (18), and (19)—for this example can be written as follows:

$$Yp_i = x_i,$$

$$\sum_{i \in \mathcal{N}} p_i = 1,$$

$$Yc_i = (1 - p_i)(V_L + V^{\Delta}) - \mu_i V^{\Delta},$$

$$\mu_i = \frac{1}{\delta_i} - p_i - \frac{p_i V_L - c_i x_i}{V^{\Delta}},$$

$$\sum_{i \in \mathcal{N}} \mu_i = 1,$$

$$V_L + 2V^{\Delta} = 1.$$

It can be verified that $\boldsymbol{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $\boldsymbol{x} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, $\boldsymbol{\mu} = (\frac{7}{48}, \frac{16}{48}, \frac{25}{48})$, $V_L = \frac{13}{15}$, and $V^{\Delta} = \frac{1}{15}$ constitute an SSPE of the game. The equilibrium of the case of $k = 3$ can be similarly verified.

Derivation for Optimal Recognition Mechanism in Example 2

Next, we prove the optimality of $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ in Example 2. When the designer sufficiently concerns the profile of agents' recognition probabilities—i.e., when $\lambda \gg 1/c$—the optimal equilibrium winning probability profile must be $\boldsymbol{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ and the designer's payoff at $\boldsymbol{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is $\Lambda = x_1 + x_2 + x_3$. When c is sufficiently small, agent 3 is excessively strong and the designer's payoff is mainly determined by x_3. Therefore, it suffices to show that $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ maximizes x_3 among all rules $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ that induce $\boldsymbol{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Fix $\boldsymbol{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. We first rewrite the equilibrium conditions in the proof of Theorem 1—i.e., conditions (14)-(19). Evidently, condition (15) is satisfied and condition (14) becomes:

$$\alpha_i^* x_i + \beta_i^* = \frac{Y}{3}, \ \forall\, i \in \{1, 2, 3\}. \tag{31}$$

Next, consider condition (16). The condition holds with equality for $x_i > 0$. Further, if $x_i = 0$ for some $i \in \mathcal{N}$ and the strict inequality holds, we can increase α_i until the equality holds and at the same time keep the equilibrium effort profile $\boldsymbol{x}$ and recognition probabilities $\boldsymbol{p}$ unchanged. Therefore, we can assume that equality holds for all agents and the condition becomes

$$\frac{Yc_i}{\alpha_i^*} = \frac{2(V_L + V^{\Delta})}{3} - \mu_i V^{\Delta}, \ \forall\, i \in \{1, 2, 3\}. \tag{32}$$

Substituting (32) into (31) yields

$$3c_i x_i \leq \frac{2(V_L + V^{\Delta})}{3} - \mu_i V^{\Delta}, \ \forall\, i \in \{1, 2, 3\}, \tag{33}$$

with equality holding if $\beta_i^* = 0$. To establish the optimality of headstarts, it suffices to show that the inequality is strict for at least one agent.

Conditions (17), (18), and (19) are

$$\mu_i = \begin{cases} \frac{2}{3} \leq \frac{1}{\delta_i} - \frac{1}{3} - \frac{V_L}{3V^{\Delta}} + \frac{c_i x_i}{V^{\Delta}}, & i \in \mathcal{N}_1, \\ \frac{1}{\delta_i} - \frac{1}{3} - \frac{V_L}{3V^{\Delta}} + \frac{c_i x_i}{V^{\Delta}} \in \left[0, \frac{2}{3}\right], & i \in \mathcal{N}_2, \\ 0 \geq \frac{1}{\delta_i} - \frac{1}{3} - \frac{V_L}{3V^{\Delta}} + \frac{c_i x_i}{V^{\Delta}}, & i \in \mathcal{N}_3, \end{cases} \tag{34}$$

$$\mu_1 + \mu_2 + \mu_3 = 1, \tag{35}$$

and

$$\sum_{i \in \mathcal{N}_1} \frac{\delta_i}{1 - \delta_i} \left(\frac{V_L}{3} - c_i x_i \right) + (2 - |\mathcal{N}_1|) V^{\Delta} + V_L = 1. \tag{36}$$

Substituting (34) into (33) yields that

$$c_i x_i \leq \begin{cases} \frac{2}{9} V_L, & i \in \mathcal{N}_1, \\ \frac{1}{4}\left[V_L - (\frac{1}{\delta_i} - 1)V^\Delta\right], & i \in \mathcal{N}_2, \\ \frac{2}{9}(V_L + V^\Delta), & i \in \mathcal{N}_3, \end{cases} \tag{37}$$

from which we can conclude $c_i x_i \leq \frac{2V_L}{9}$ for $i \in \mathcal{N}_1$; together with (36), we can obtain that

$$\sum_{i \in \mathcal{N}_1} \frac{\delta_i}{1-\delta_i} \times \frac{V_L}{9} + \big(2 - |\mathcal{N}_1|\big) V^\Delta + V_L \leq 1. \tag{38}$$

In what follows, we will show that $c_3 x_3 \leq \frac{30}{144-\delta_3}$, and the equality holds if and only if $\boldsymbol{\alpha}^* = (\frac{62Y}{35}, \frac{62Y}{37}, \frac{62Yc}{39})$ and $\boldsymbol{\beta}^* = (0, \frac{17Y}{222}, 0)$. Consider the following three cases.

Case I: $\mathbf{3} \in \boldsymbol{\mathcal{N}_1}$. Note that $|\mathcal{N}_1| \leq k - 1 = 1$, we have that $\mathcal{N}_1 = \{3\}$. By (38), we can obtain that

$$\left[1 + \frac{\delta_3}{9(1-\delta_3)}\right] V_L + V^\Delta \leq 1;$$

together with (36), we can obtain that

$$c_3 x_3 \leq \frac{2V_L}{9} \leq \frac{2(1-\delta_3)}{9 - 8\delta_3} < \frac{30}{144 - \delta_3}.$$

Case II: $\mathbf{3} \in \boldsymbol{\mathcal{N}_2}$. By (34) and (37), we have that

$$0 \leq \frac{1}{\delta_3} - \frac{1}{3} - \frac{V_L}{3V^\Delta} + \frac{c_3 x_3}{V^\Delta} \leq \frac{1}{\delta_3} - \frac{1}{3} - \frac{V_L}{3V^\Delta} + \frac{V_L - (\frac{1}{\delta_3} - 1)V^\Delta}{4V^\Delta}.$$

Carrying out the algebra, we can obtain that

$$V_L \leq \left(\frac{9}{\delta_3} - 1\right) V^\Delta = \frac{35}{4} V^\Delta. \tag{39}$$

Further, $3 \notin \mathcal{N}_1$ implies that $\mathcal{N}_1 \in \big\{\{1\}, \{2\}, \emptyset\big\}$, and thus (38) becomes

$$1 \geq \left\{ \begin{array}{ll} \frac{16}{15} V_L + V^\Delta, & \text{if } \mathcal{N}_1 = \{1\} \\ \frac{10}{9} V_L + V^\Delta, & \text{if } \mathcal{N}_1 = \{2\} \\ V_L + 2V^\Delta, & \text{if } \mathcal{N}_1 = \emptyset \end{array} \right\} \geq \frac{16}{15} V_L + V^\Delta, \tag{40}$$

where the last inequality follows from (39).

Combining (37), (39) and (40), we have that

$$c_3 x_3 \le \frac{1}{4}\left[V_L - (\frac{1}{\delta_3} - 1)V^{\Delta}\right] \le \frac{30}{144-\delta_3} = \frac{13}{62}.$$

Note that equality holds in condition (37) if and only if $\beta_3^* = 0$. Further, equality holds in condition (39) only if $\mu_3 = 0$. Last, equality holds in condition (40) if and only if $\mathcal{N}_1 = \{1\}$ and $\beta_1^* = 0$.

Because $\mathcal{N}_1 = \{1\}$ and $\mu_3 = 0$, we have that $\mu_1 = \frac{2}{3}$ from (34); together with (35), we have $\mu_2 = \frac{1}{3}$. Moreover, by (34), we can conclude $2 \in \mathcal{N}_2$, which implies that $\mathcal{N}_2 = \{2, 3\}$ and $\mathcal{N}_3 = \emptyset$.

Combining (39) and (40) (recall that equality holds in these conditions), we can obtain $V_L = \frac{105}{124}$ and $V^{\Delta} = \frac{3}{31}$; together with (37), we have $x_1 = \frac{2V_L}{9} = \frac{35}{186}$. Substituting $\mu_2 = \frac{1}{3}$, $V_L = \frac{105}{124}$ and $V^{\Delta} = \frac{3}{31}$ into (34), we can obtain that $x_2 = \frac{V_L - 4V^{\Delta}}{3} = \frac{19}{124}$.

Last, we solve for $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$. Recall that $\beta_i^* = 0$ for $i \in \{1, 3\}$. Therefore, $\alpha_i^* = \frac{Y}{3x_i}$ from (31). For $i = 2$, we have $x_2 = \frac{19}{124}$. Further, by (32), we have $\frac{Y}{\alpha_2^*} = \frac{2V_L + V^{\Delta}}{3} = \frac{37}{62}$, which implies that $\alpha_2^* = \frac{62Y}{37}$; together with (31), we can conclude $\beta_2^* = \frac{Y}{3} - \alpha_2^* x_2 = \frac{17Y}{222}$.

In summary, the equality holds in $c_3 x_3 \le \frac{30}{144-\delta_3}$ if and only if $\boldsymbol{\alpha}^* = (\frac{62Y}{35}, \frac{62Y}{37}, \frac{62Yc}{39})$ and $\boldsymbol{\beta}^* = (0, \frac{17Y}{222}, 0)$, in which the equilibrium is $\boldsymbol{x} = (\frac{35}{186}, \frac{19}{124}, \frac{39}{186c})$, $\boldsymbol{p} = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $\boldsymbol{\mu} = (\frac{2}{3}, \frac{1}{3}, 0)$, $V_L = \frac{105}{124}$, and $V^{\Delta} = \frac{3}{31}$.

Case III: $3 \in \boldsymbol{\mathcal{N}_3}$. Condition (34), together with the postulated $3 \in \mathcal{N}_3$, implies that $\mu_3 = 0$. Analogous to derivation of (39), we can obtain that

$$V_L > \left(\frac{9}{\delta_3} - 1\right) V^{\Delta} = \frac{35}{4} V^{\Delta}. \tag{41}$$

Suppose $\mathcal{N}_1 \neq \emptyset$. By (40), we have that

$$1 \ge \frac{16V_L}{15} + V^{\Delta}. \tag{42}$$

Combining (33), (41), and (42) yields that

$$c_3 x_3 \le \frac{2(V_L + V^{\Delta})}{9} < \frac{30}{144 - \delta_3}.$$

Next, suppose $\mathcal{N}_1 = \emptyset$; together with $3 \in \mathcal{N}_3$ and $k = 2$, we can conclude $\mathcal{N}_2 = \{1, 2\}$. It follows from (36) that

$$V_L + 2V^{\Delta} = 1. \tag{43}$$

Recall $\mu_3 = 0$. Combining (34), (35), and (37), we can obtain that

$$1 = \mu_1 + \mu_2 = \frac{1}{\delta_1} + \frac{1}{\delta_2} - \frac{2}{3} - \frac{2V_L}{3V^\Delta} + \frac{x_1 + x_2}{V^\Delta} \leq 4 - \frac{2V_L}{3V^\Delta} + \frac{V_L}{2V^\Delta} - \frac{2}{3},$$

which in turn implies that

$$V_L \leq 14V^\Delta. \tag{44}$$

Therefore,

$$c_3x_3 \leq \frac{2(V_L + V^\Delta)}{9} \leq \frac{5}{24} < \frac{30}{144 - \delta_3},$$

where the first inequality follows from (33), the second inequality from (43) and (44).