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On equilibrium existence in generalized multi-prize nested lottery contests ☆

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Abstract

The generalized multiple-prize nested lottery contest framework has been broadly applied to model noisy competitions that award prizes to multiple recipients. Assuming homogeneous and risk-neutral players, previous studies have typically solved for the symmetric strategy profile that satisfies the first-order condition as the equilibrium solution to the game. The literature has yet to formally establish equilibrium existence because of a technical challenge caused by the presence of multiple prizes. The associated payoff structure dismisses the key property of contests as aggregative games and nullifies the usual approach for equilibrium analysis. We develop an alternative approach to ascertain the property of players' payoff functions without assuming homogeneous and/or risk-neutral players, which enables us to establish equilibrium existence. We then consider a setting that allows for incomplete information and develop an indirect approach to establish equilibrium existence of the Bayesian contest game.

JEL classification: C72; D72

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1. Introduction

Contest-like situations are common in the modern socioeconomic landscape: Contenders expend costly and nonrefundable efforts to vie for a limited number of prizes and they are rewarded based on their relative performance instead of on absolute output metrics. A wide variety of competitive activities resemble a contest, such as R&D races (Loury, 1979; Lee and Wilde, 1980), electoral campaigns (Snyder, 1989), influence politics (Che and Gale, 1998), and internal labor markets inside firms (Lazear and Rosen, 1981).

The game theoretical frameworks that model players' strategic behavior in contests can largely be classified into two broad categories: perfectly discriminatory contests and imperfectly discriminatory contests.[1] The former allows the player with the highest effort or bid to prevail with *certainty*, which refers to an all-pay auction (Hillman and Riley, 1989; Baye et al., 1993, 1996). The latter abstracts the situation whereby one's win or loss depends not only on his effort but also on *noises* and *randomness*. In particular, the generalized lottery contest is predominantly adopted to model noisy winner-selection mechanisms: For a given effort profile $\boldsymbol{x} \equiv (x_1, \dots, x_n)$, one wins with a probability

$$p_i(\boldsymbol{x}) = \frac{f_i(x_i)}{\sum_{j=1}^{n} f_j(x_j)}, \tag{1}$$

where the function $f_i(\cdot)$, conventionally called the impact function, converts a player's effort x_i into his effective output in the contest.[2] The two modeling frameworks provide drastically different predictions for equilibrium behavior and yield contrasting implications for contest design.[3]

A natural extension is to allow prizes to be awarded to multiple players.[4] Multiple prizes can readily be accommodated in an all-pay auction, as players are precisely ranked by their bids and each is awarded a prize according to his respective rank (see, e.g., Clark and Riis, 1998a; Barut and Kovenock, 1998; Bulow and Levin, 2006; Siegel, 2009, 2010, 2014; Xiao, 2016; Fang et al., 2020). Embedding multiple prizes into a generalized lottery contest was less straightforward until Clark and Riis (1996) proposed the multi-prize nested lottery contest model. Despite its broad applications (see, e.g., Azmat and Möller, 2009; Brown, 2011; Fu and Lu, 2009, 2012b; Fu and Wu, 2021), the numerous studies based on the model have provided little in terms of formal ways to shed light on its game theoretic fundamentals. This paper provides a general analysis of equilibrium existence in generalized multi-prize nested lottery contest games to close this long-existing gap.

1 See Fu and Wu (2019) for a recent survey of theoretical studies of contests.

2 Skaperdas (1996) provides an axiomatic foundation for this ratio-form contest success function. Tullock contest provides the most salient special case of the generalized contest model, which assumes that the impact function takes the form of $f_i(x_i) = x_i^r$.

3 In terms of equilibrium behavior, only mixed-strategy equilibria exist in all-pay auctions, while a pure-strategy equilibrium could exist in a generalized lottery contest. In terms of contest design, Baye et al. (1993) demonstrate that an all-pay auction may generate a higher amount of total effort if the designer excludes the strongest player, while Fang (2002) shows that this does not hold in a Tullock contest.

4 See Sisak (2009) on a survey on multi-prize contests.

Generalized multi-prize nested lottery contest A generalized multi-prize nested lottery contest can be conveniently described as a sequential lottery process. Suppose that $l \geq 2$ prizes are available to $n \geq l$ players, with each eligible for at most one prize. Players commit to their effort entries $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$, and the winner of the first prize is determined by the ratio-form contest success function (CSF, henceforth) (1). This winner is immediately removed from the pool of players eligible for the second prize; the recipient of the second prize is drawn from the rest of the players, and the probability of winning it—conditional on not having won the first prize—is given by the ratio of his effective output to the sum of those who remain in the pool. This process is repeated until all l prize recipients are distributed. Despite the intuitive and convenient analogy to a sequential lottery process, Fu and Lu (2012a) demonstrate that the contest is uniquely underpinned by a (simultaneous) noisy ranking system à la (McFadden, 1973, 1974), with the probability of one's being ranked in the mth place equal to that of being selected for the mth draw in the sequential lottery process.[5]

Prior analysis has largely been incomplete. Assuming complete information and homogeneous, risk-neutral players, the majority of studies solve for the symmetric effort profile that satisfies first-order conditions and simply adopt the solution as the equilibrium prediction. The literature nevertheless has yet to verify whether this convenient solution fulfills the requirement of Nash equilibrium, or to establish equilibrium existence in general.[6]

Game theoretic nature of the problem The multi-prize nested lottery contest fundamentally differs from its single-prize variant in terms of the underlying game theoretic structure. Previous results on equilibrium existence in contests and discontinuous games in general do not carry over.

Recall the ratio-form CSF (1) and consider, for simplicity, a standard lottery contest with $f_i(x_i) = x_i$. Discontinuity in players' payoff functions arises at origin, i.e., $(0, \ldots, 0)$. The analysis of a contest typically relies on its properties as an aggregative game to verify equilibrium existence: For a given prize value and an effort cost function, one's payoff depends only on his individual output x_i and the total effort $\sum_{j=1}^{n} x_j$, which enables the powerful tool of backward-reply correspondence (see Selten, 1970; Novshek, 1985; and Acemoglu and Jensen, 2013) in equilibrium analysis and, subsequently, share correspondence (Cornes and Hartley, 2005) in contests. However, this approach hits a roadblock in a multi-prize contest, in which one's payoff depends not only on his effort and the total effort, but also on the distribution of the effort profile. To see this, note that the ex ante probability of a player i's winning the second prize, which we denote by $P_{2,i}(\boldsymbol{x})$, is given by

$$P_{2,i}(\boldsymbol{x}) = \sum\nolimits_{i' \in \mathcal{N} \setminus \{i\}} \left[\frac{x_{i'}}{\sum_{j=1}^{n} x_j} \times \frac{x_i}{\sum_{j' \in \mathcal{N} \setminus \{i'\}} x_{j'}} \right],$$

where $\mathcal{N} \equiv \{1, \ldots, n\}$ is the set of players involved in the contest: $x_i / \sum_{j' \in \mathcal{N} \setminus \{i'\}} x_{j'}$ is the probability of player i's being picked in the second draw conditional on another player i''s winning the first prize, while $x_{i'} / \sum_{j=1}^{n} x_j$ is the probability of player i''s being picked in the first draw. The deviation from aggregative games nullifies the conventional approach to establishing equilibrium existence in contests.

[5] Lu and Wang (2015) provide an axiomatic foundation for the model. Letina et al. (2020) consider an optimal contest design problem, in which the designer is able to choose both the prize allocation and the contest success function; they show that the optimum can be achieved through running a nested Tullock contest.

[6] Clark and Riis (1996) examine the local concavity of players' payoff functions under this hypothetical equilibrium profile.

Furthermore, the multi-prize generalization compounds the discontinuity in payoffs. Consider, for instance, a five-player contest that awards three positive prizes. In addition to the origin, discontinuity arises at an effort profile $(1, 1, 0, 0, 0)$ as well: Player 3 would expect a jump in his payoff if he exerts an infinitesimal effort, which assures him of a prize.

Reny (1999) develops the famous thesis that a pure-strategy Nash equilibrium would emerge in a discontinuous game if the game is *better-reply secure* with convex and compact strategy spaces and payoffs are *quasiconcave* in players' strategies. However, this theorem cannot directly be applied. Write a player's payoff function as

$$\pi_i(x_i, \boldsymbol{x}_{-i}) = \sum_{m=1}^{l} \left[P_{m,i}(\boldsymbol{x}) \times u_i \left(V_{m,i} - c_i(x_i) \right) \right], \tag{2}$$

where $P_{m,i}(\cdot)$ is the probability of his winning each prize m, $V_{m,i}$ is his valuation of prize m, $c_i(\cdot)$ is his effort cost, and $u_i(\cdot)$ is an increasing and (weakly) concave Bernoulli utility function that represents his risk preference. With risk-neutral players and a single prize, the winning probability as given by (1) is well behaved, as it is concave in one's own effort. In contrast, with two or more prizes, the probability $P_{m,i}$ is concave only for the first prize, i.e., $m = 1$. The property of the payoff function $\pi_i(x_i, \boldsymbol{x}_{-i})$—as an aggregation—is thus elusive.[7]

Snapshot of the analysis In this paper, we first establish the existence of equilibrium in a complete-information setting in which players differ in their prize valuations, utility functions, impact functions, and effort cost functions. We then proceed to an incomplete-information setting and establish equilibrium existence in a Bayesian contest game.

More specifically, in the complete-information setting, we borrow Reny's (1999) better-reply security condition to establish equilibrium existence. We first develop a novel approach to verifying the concavity of a player's payoff in this model: Instead of writing the expected payoff function (2) as a linear combination of one's probabilities of winning each prize as above, we rearrange it as a linear combination of the probabilities of *not receiving one of the first several prizes*. The rearrangement illuminates the nature of a player's payoff structure in this game, which allows us to circumvent the usual analytical challenge and establish payoff concavity.

Further, we identify a lucid and intuitive sufficient condition (i.e., Assumption 1 in Section 2) that ensures better-reply security in the game. The condition requires that the numbers of *effective prizes* for each player—i.e., the prizes that provide positive incentives to a player—should be equal among all. We demonstrate that this condition plays a critical role and that better-reply security may fail to hold when it is violated.

We then relax the assumptions of complete information to consider the contest as a discontinuous Bayesian game. It allows for a rich information structure, in which players' private types may involve their prize valuations, utility functions, impact functions, and effort cost functions. The results from the recent literature on equilibrium existence in incomplete-information discontinuous games—i.e., He and Yannelis (2015) and Carbonell-Nicolau and McLean (2018)—do not directly apply to our context. In particular, the multi-prize nested lottery contest fails to meet the requirement of *aggregate upper semicontinuity* in every ex post game, which ensures upper semicontinuity of the sum of payoffs on players' behavioral strategies. We develop an indirect approach that bridges He and Yannelis (2015) and Carbonell-Nicolau and McLean (2018) to our setting. Specifically, we first construct an alternative game that modifies the original contest

[7] Schweinzer and Segev (2012) demonstrate the complexity and nonregularity of the payoff function in this game.

game by manipulating its "tie-breaking" rule. The constructed game is shown to meet the requirement of He and Yannelis (2015) and Carbonell-Nicolau and McLean (2018), and thus an equilibrium exists. We then demonstrate that the equilibrium of the constructed game is also that of the original game, which closes the loop.

Link to the literature Our paper belongs to the literature that explores the equilibrium fundamentals of imperfectly discriminatory contests. A large number of studies have been devoted to verifying the existence and/or uniqueness of Nash equilibrium in complete-information single-prize contests with risk-neutral players. These studies include Pérez-Castrillo and Verdier (1992), Szidarovszky and Okuguchi (1997), Cornes and Hartley (2005), Alcalde and Dahm (2010), Ewerhart (2015), and Feng and Lu (2017). Esteban and Ray (1999) and Brookins and Ryvkin (2016) establish equilibrium existence and uniqueness in group contests, while Franke and Öztürk (2015) and Xu et al. (2020) verify this in network contests. Lagerlöf (2020) applies the classic result of Reny (1999) to study a hybrid contest in which each player takes two actions, with one to be costly regardless of the outcome (all-pay) and the other to incur expenses only to the winner (winner-pay). While all the aforementioned studies assume risk-neutral players, Skaperdas and Gan (1995), Cornes and Hartley (2003, 2012), Yamazaki (2009), and Treich (2010) proceed to settings with risk aversion.

This research stream has extended to contests with incomplete information. Hurley and Shogren (1998b), Schoonbeek and Winkel (2006), Wärneryd (2003), Rentschler (2009), Zhang and Zhou (2016), and Denter et al. (2020) allow for one-sided asymmetric information. Hurley and Shogren (1998a), Malueg and Yates (2004), Schoonbeek and Winkel (2006), Fey (2008), and Serena (2021) consider two-sided incomplete information in bilateral contests.[8] Wasser (2013a, 2013b), Ewerhart (2014), Einy et al. (2015), and Ewerhart and Quartieri (2020) allow for private information with three or more players. Einy et al. (2017) establish equilibrium existence and uniqueness in a Tullock contest in which symmetrically informed players are uncertain about their common prize value and common effort cost function. In particular, Einy et al. (2015) and Ewerhart and Quartieri (2020) assume rich information structures and allow each player's payoff to depend on his private type in multiple dimensions, e.g., prize valuations, impact functions, and effort cost functions. The former establishes equilibrium existence, while the latter also verifies the uniqueness.[9]

All the aforementioned papers assume a single prize. To the best of our knowledge, our paper is the first to explore equilibrium existence in noisy multi-prize contests, and the analysis contributes notably to the contest literature. First, as previously mentioned, the shift from a single-prize contest to a multi-prize one fundamentally changes the nature of the game, dismissing the regularity inherent in aggregative games. As a result, the conventional results and approaches for single-prize contests do not extend. Second, our analysis illuminates players' payoff functions in multi-prize contests. Our approach and the identified regular properties may well be useful for future equilibrium analysis of this game. Third, our analysis encompasses a broad spectrum of modeling variations based on the framework of multi-prize nested lottery contest. For instance, the Bayesian game setting considered in Section 4 accommodates an enriched information structure and allows players to be privately informed.

[8] Carbonell-Nicolau and McLean (2018) provide an application of their Theorem 2 in an incomplete-information single-prize contest.

[9] Ewerhart and Quartieri (2020) further allow for budget constraints for players.

Our study also complements the recent literature on equilibrium existence in discontinuous games with incomplete information. Three papers relate closely to our work. First, our analysis of incomplete-information contests builds on He and Yannelis (2015) and Carbonell-Nicolau and McLean (2018), who notably extend Reny's (1999) equilibrium existence result to Bayesian games. As mentioned above, our contest game deviates from their settings because it lacks the key property of aggregate upper semicontinuity, which prevents direct application of their results. Our approach restores the relevance of their existence results in our context and broadens the scope of their applications. The technique may well be useful in future research for other forms of games that do not immediately match the requirements of He and Yannelis (2015) and Carbonell-Nicolau and McLean (2018). Second, our paper is also conceptually linked to Olszewski and Siegel (2020). They develop an alternative tool to verify equilibrium existence in discontinuous games with ties and relate its applications largely to contest games. Their study applies more to perfectly discriminatory contests, while ours focuses on imperfectly discriminatory contests.[10]

The rest of the paper is structured as follows. Section 2 describes the generalized multi-prize nested lottery contest. Section 3 establishes the main result regarding equilibrium existence in contests with complete information. Section 4 extends the model to accommodate incomplete information and establishes equilibrium existence, and Section 5 concludes. Appendix A collects the proofs that are not provided in the main text. Appendix B provides an example that demonstrates how a key property of the model relates to an assumption.

2. Generalized multi-prize nested lottery contests: complete information

There are $n \geq 2$ risk-neutral players competing for n prizes. The mth prize, with $m \in \{1, \dots, n\}$, has a value $V_{m,i} \geq 0$ for each player $i \in \mathcal{N} \equiv \{1, \dots, n\}$, which is common knowledge. We assume that $V_{1,i} \geq \cdots \geq V_{n,i}$, for all $i \in \mathcal{N}$, with strict inequality to hold for at least one.

Define $\ell_i := \max\left\{m = 1, \dots, n-1 \mid V_{m,i} > V_{m+1,i}\right\}$. Intuitively, ℓ_i can be interpreted as the actual number of prizes that could *effectively* incentivize player i's effort in the competition. Imagine a 7-person 7-prize contest and a series of prize valuations $(6, 6, 5, 4, 4, 2, 2)$ for a player i, which yields $\ell_i = 5$: Player i only strives to win one of the first five prizes, as he is assured of one of the two last prizes regardless of the outcome in the competition. The following assumption is imposed on the series of prize valuations.

Assumption 1 *(Equal number of effective prizes across players).* $\ell_1 = \cdots = \ell_n =: \ell$.

Assumption 1 is obviously satisfied in a winner-take-all contest, in which case $V_{m,i} = 0$, for all $m \in \{2, \dots, n\}$, and $i \in \mathcal{N}$. It is worth noting that a contest with $V_{1,i} > V_{m,i}$ and $V_{m,i} = V_{m',i} > 0$ for all $m, m' \in \{2, \dots, n\}$ and $i \in \mathcal{N}$ is strategically equivalent to a single-prize scheme: It also yields $\ell = 1$, and only the top prize lures players' efforts.

Assumption 1 is also satisfied when each prize carries a common monetary value, i.e., $V_{m,1} = \cdots = V_{m,n}$ for all $m \in \{1, \dots, n\}$. Our model allows players to value prizes differently—which implicitly alludes to the case of nonmonetary prizes—as long as the number of effective prizes that matter to each player is identical.

[10] See also Athey (2001), Jackson et al. (2002), Jackson and Swinkels (2005), and He and Yannelis (2016) for more results on equilibrium existence in discontinuous games of incomplete information.

Note that the collection of profiles $\boldsymbol{V}_i = (V_{1,i}, \ldots, V_{n,i}) \in \mathbb{R}^n_+$ with $V_{m,i} = V_{m',i}$ for some (m, m') pair has Hausdorff dimension $n-1$, and thus this collection of profiles has n-dimensional Lebesgue measure zero. Meanwhile, the set of all feasible prize profiles—i.e., $\{\boldsymbol{V}_i \mid V_{1,i} \geq \cdots \geq V_{n,i} \geq 0\}$—has Hausdorff dimension n and nonempty interior in $\mathbb{R}^n_+$. Altogether, these facts imply that if $\boldsymbol{V}_i$ is randomly "chosen" according to some probability measure that is absolutely continuous with respect to the Lebesgue measure, then $\ell_i = n-1$ for each $i \in \mathcal{N}$ and Assumption 1 is satisfied.[11]

We discuss the nuance of Assumption 1 in Section 3.2 and demonstrate that a pure-strategy equilibrium may fail to exist if this assumption is violated.

2.1. Winner-selection mechanism

Players simultaneously commit to their one-shot effort outlays x_is, which incur a cost $c_i(x_i)$, to vie for the prizes. We adopt the popularly studied multi-prize nested lottery contest (Clark and Riis, 1996, 1998b) to depict the winner-selection mechanism that allows for multiple prize recipients. Fixing an effort profile $\boldsymbol{x} := (x_1, \ldots, x_n)$, a player i wins the first prize with a probability

$$p_{1,i}(\boldsymbol{x}) := \begin{cases} \dfrac{f_i(x_i)}{\sum_{j\in\mathcal{N}} f_j(x_j)}, & \text{if } \sum_{j\in\mathcal{N}} f_j(x_j) > 0, \\ \dfrac{1}{n}, & \text{if } \sum_{j\in\mathcal{N}} f_j(x_j) = 0, \end{cases}$$

where the function $f_i(\cdot)$, labeled the impact function in the contest literature, converts one's effort into his effective output and satisfies $f_i(x_i) \geq 0$ and $f_i'(x_i) > 0$ for all $x_i \geq 0$. Each player is eligible for at most one prize: Once a player is picked as the recipient of the prize, he is immediately removed from the pool of players eligible for the second, and a similar lottery determines the recipient of the second prize from the remaining candidates. The process is repeated until all n prizes have been distributed.

To put this formally, let Ω_m, $m \in \{1, \ldots, n\}$, be the set of players who remain eligible for the mth draw—i.e., those who were not picked in the previous $m-1$ draws—with $\Omega_1 \equiv \mathcal{N}$. It is straightforward to see that $|\Omega_m| = n - m + 1$. The probability of a player i's receiving the mth prize *conditional* on his not having been picked in the previous $m-1$ draws (i.e., $i \in \Omega_m$) is given by

$$p_{m,i}(\boldsymbol{x}; \Omega_m) := \begin{cases} \dfrac{f_i(x_i)}{\sum_{j\in\Omega_m} f_j(x_j)} \times \mathbb{1}(i \in \Omega_m), & \text{if } \sum_{j\in\Omega_m} f_j(x_j) > 0, \\ \dfrac{1}{n-m+1} \times \mathbb{1}(i \in \Omega_m), & \text{if } \sum_{j\in\Omega_m} f_j(x_j) = 0, \end{cases} \tag{3}$$

where $\mathbb{1}(i \in \Omega_m)$ is an index function, with $\mathbb{1}(i \in \Omega_m) = 1$ if $i \in \Omega_m$ and $\mathbb{1}(i \in \Omega_m) = 0$ if $i \notin \Omega_m$.

We assume in (3) that players eligible for a prize win with equal probability when all of them exert zero effort. It is useful to point out that our result concerning equilibrium existence remains intact under an arbitrary tie-breaking rule. To be more specific, fixing Ω_m, $\frac{1}{n-m+1}$ in the above definition of $p_{m,i}(\boldsymbol{x}; \Omega_m)$ can be replaced by a positive constant $q_i > 0$ for each $i \in \Omega_m$, with $\sum_{i\in\Omega_m} q_i = 1$.

[11] We thank an anonymous referee for suggesting this justification for Assumption 1.

2.2. Players' preferences

Players' risk attitudes may play a nontrivial role in a contest game, given the gambling nature of contests and the resultant uncertainty inherent in such competitions.[12] Specifically, each player is assumed to have a (weakly) concave Bernoulli utility function $u_i(\cdot)$ that is twice-differentiable and satisfies $u_i'(w_i) > 0$, and $u_i''(w_i) \leq 0$ for all $w_i \in \mathbb{R}$. We further impose the following requirement on players' preferences.

Assumption 2 *(NIARA preferences).* Players' utility function exhibits nonincreasing absolute risk aversion (NIARA), i.e., $-u_i''(w_i)/u_i'(w_i)$ is nonincreasing in w_i for all $i \in N$.

This assumption was first proposed by Arrow (1970). The NIARA condition is satisfied by a broad spectrum of utility functions, such as the familiar constant absolute risk averse (CARA) and constant relative risk averse (CRRA) preferences. A plethora of experimental and empirical findings provide evidence for decreasing absolute risk aversion (see, e.g., Friend and Blume, 1975). Notably, NIARA implies that $u_i'''(w_i) \geq 0$, i.e., prudent players.[13]

2.3. Winning probabilities and expected payoffs

Fixing an effort profile $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$, denote by $P_{m,i}(\boldsymbol{x})$ a player i's ex ante probability of winning the mth prize. It can be verified that

$$P_{m,i}(\boldsymbol{x}) = \sum_{\Omega_m \in \binom{\mathcal{N}}{n-m+1}} \Big[\Pr(\Omega_m) \times p_{m,i}(\boldsymbol{x}; \Omega_m) \Big],$$

where $\binom{\mathcal{N}}{n-m+1}$ denotes the set of all subsets of $\mathcal{N}$ with cardinality $n-m+1$ and $\Pr(\Omega_m)$ is the probability that a particular set Ω_m of players are up for the mth draw.

Assuming nonseparable utility, a player i's expected utility is derived as

$$\pi_i(\boldsymbol{x}) := \sum_{m=1}^{n} \left[P_{m,i}(\boldsymbol{x}) \times u_i\left(V_{m,i} - c_i(x_i)\right) \right]. \tag{4}$$

We adopt nonseparable utility for two reasons. First, equilibrium existence is trivial if we assume that utility is separable in a player's income and effort cost, in which case the expected utility function takes the form of $\pi_i(\boldsymbol{x}) = \sum_{m=1}^{n} \left[P_{m,i}(\boldsymbol{x}) \times u_i(V_{m,i}) \right] - c_i(x_i)$. The contest game is then strategically equivalent to one in which players are risk neutral and prize valuation is reformulated as $\widehat{V}_{m,i} \equiv u_i\left(V_{m,i}\right)$. Second, multi-prize contests with nonseparable utility entail a substantially greater level of economic subtlety. The prevailing prize structure catalyzes an income effect that affects not only one's marginal benefit but also his marginal effort cost (Fu et al., 2021a).

[12] A growing body of literature explores contest games with risk-averse players, such as Cornes and Hartley (2003, 2012); March and Sahm (2018); Fu et al. (2021a); and Drugov and Ryvkin (2019) in imperfectly discriminatory contests, and Chen et al. (2017) and Klose and Schweinzer (2021) in all-pay auctions.

[13] To see this, note that $\frac{d}{dw_i}\left(-\frac{u_i''(w_i)}{u_i'(w_i)}\right) \leq 0$ is equivalent to $u_i'''(w_i) \geq [-u_i''(w_i)]^2/u_i'(w_i) \geq 0$.

2.4. Preliminary discussion

We now introduce an illustrative example to familiarize readers with the setup and to elucidate the game theoretical nuances of the model. For simplicity, we assume that $u_i(w_i) = w_i$ for all $i \in \mathcal{N}$; i.e., players are risk neutral.

Example 1. Consider a multi-prize contest with three risk-neutral players. Fixing $(x_2, x_3) > (0, 0)$, player 1's ex ante probability of winning the first prize is given by

$$P_{1,1}(\boldsymbol{x}) \equiv p_{1,1}(\boldsymbol{x}) = \frac{f_1(x_1)}{f_1(x_1) + f_2(x_2) + f_3(x_3)}.$$

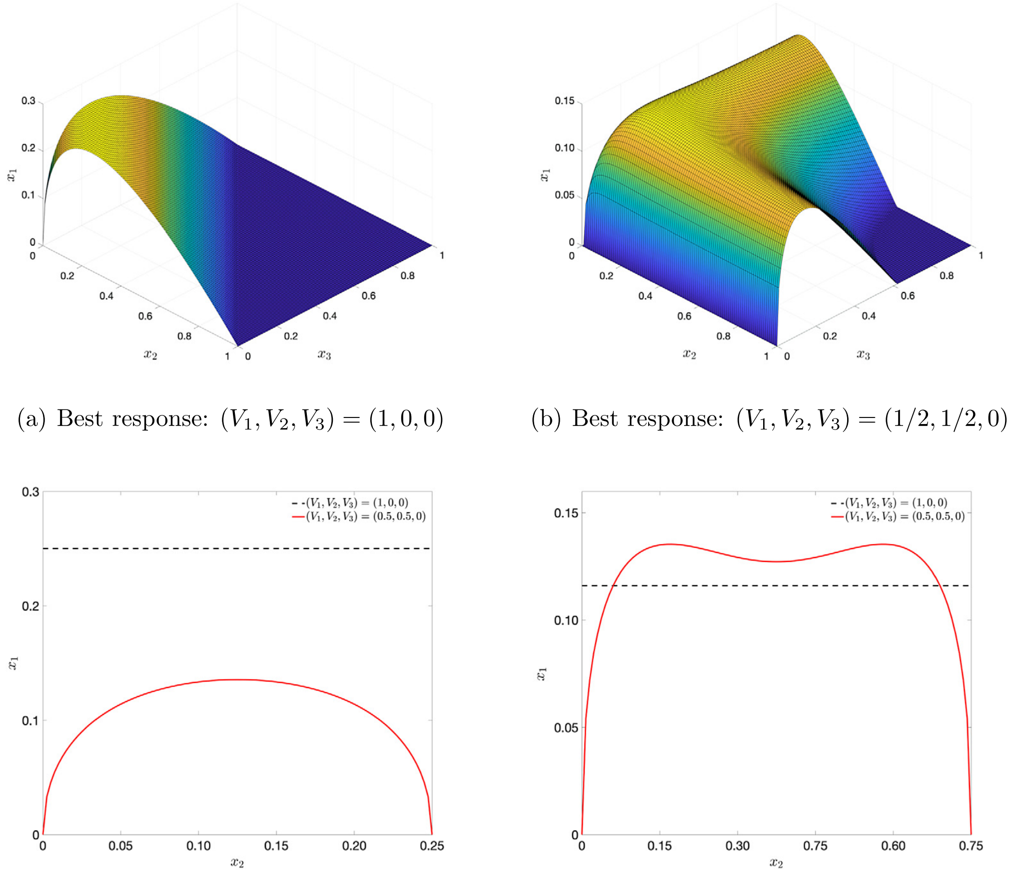
Moreover, his ex ante probability of winning the second prize is given by

$$P_{2,1}(\boldsymbol{x}) = \overbrace{\underbrace{\frac{f_2(x_2)}{f_1(x_1) + f_2(x_2) + f_3(x_3)}}_{\Pr(\{1,3\})} \times \underbrace{\frac{f_1(x_1)}{f_1(x_1) + f_3(x_3)}}_{p_{2,1}(\boldsymbol{x};\{1,3\})}}^{\text{Players 1 and 3 eligible for the second prize: } \Omega_2 = \{1,3\}.}$$
$$+ \overbrace{\underbrace{\frac{f_3(x_3)}{f_1(x_1) + f_2(x_2) + f_3(x_3)}}_{\Pr(\{1,2\})} \times \underbrace{\frac{f_1(x_1)}{f_1(x_1) + f_2(x_2)}}_{p_{2,1}(\boldsymbol{x};\{1,2\})}}^{\text{Players 1 and 2 eligible for the second prize: } \Omega_2 = \{1,2\}.}.$$

Two remarks are in order. First, it is straightforward to verify that player 1's probability of winning the first prize $P_{1,1}$—i.e., $f_1(x_1)/[f_1(x_1) + f_2(x_2) + f_3(x_3)]$—is concave in his effective output $f_1(x_1)$, as in a standard single-prize contest. However, his probability of winning the second prize $P_{2,1}$ is not. The property of his expected payoff function, $\pi_1(\boldsymbol{x}) = P_{1,1} \times V_{1,1} + P_{2,1} \times V_{2,1} + P_{3,1} \times V_{3,1} - c_1(x_1)$, remains elusive.

Second, with $\ell = 2$, the contest game differs fundamentally from its single-prize variant. With $\ell = 1$, player 1 wins the prize with a probability $P_{1,1} = f_1(x_1)/[f_1(x_1) + f_2(x_2) + f_3(x_3)]$: His payoff depends only on his own effective output $f_1(x_1)$, and the total output $\sum_{i=1}^{3} f_i(x_i)$, which renders the contest a classic aggregative game (Selten, 1970) if we change player i's decision variable from x_i to $f_i(x_i)$ for $i \in \{1, 2, 3\}$. The same does not hold when multiple effective prizes are in place. Player 1's probability of winning the second prize depends not only on the sum $\sum_{i=1}^{3} f_i(x_i)$ but also on the specific distribution of the opponents' output profile $\big(f_2(x_2), f_3(x_3)\big)$: For instance, the probability of player 1's winning the second prize conditional on player 2's being picked in the first draw is given by $f_1(x_1)/[f_1(x_1) + f_3(x_3)]$. As mentioned in Introduction, the departure from an aggregative game framework dismisses the application of the usual approaches based on backward-reply correspondence and share correspondence.

We construct a figure to visualize the subtlety. First, we assume a linear impact function, i.e., $f_i(x_i) = x_i$. Second, we consider a common-valued prize structure—i.e., $V_{m,1} = V_{m,2} = V_{m,3} =: V_m$ for all $m \in \{1, 2, 3\}$—and fix the total prize money: Each player has a prize value series $(1, 0, 0)$ in one scenario, which involves one effective prize ($\ell = 1$); each has a series $(0.5, 0.5, 0)$ in the other, which results in $\ell = 2$. In Fig. 1(a), the vertical axis traces player 1's best response in the former scenario, while Fig. 1(b) depicts the latter. Player 1's best response remains constant for a given sum $x_2 + x_3$, as in Fig. 1(a), which does not hold in the case represented by Fig. 1(b): He responds to (x_2, x_3) instead of $x_2 + x_3$. A similar observation can be obtained for Figs. 1(c)



(a) Best response: $(V_1, V_2, V_3) = (1, 0, 0)$ (b) Best response: $(V_1, V_2, V_3) = (1/2, 1/2, 0)$

(c) Best response: $x_2 + x_3 = 1/4$ (d) Best response: $x_2 + x_3 = 3/4$

Fig. 1. Player 1's best response: single prize vs. multiple prizes.

and 1(d). In each panel, the horizontal axis traces x_2 while keeping the sum $x_2 + x_3$ constant; the vertical axis depicts player 1's best response. The dashed horizontal line represents the case of a single-prize contest, which shows that the best response is invariant with the distribution of (x_2, x_3); the solid curve, however, shows that player 1's best response indeed varies when two prizes are available.

3. Analysis

Denote by $\boldsymbol{V}_i$ the vector of a player i's prize valuations $(V_{1,i}, \ldots, V_{n,i})$ for all $i \in \mathcal{N}$. Players' prize valuations $\{\boldsymbol{V}_i\}_{i=1}^n$, together with the set of utility functions $\{u_i(\cdot)\}_{i=1}^n$, impact functions $\{f_i(\cdot)\}_{i=1}^n$, and effort cost functions $\{c_i(\cdot)\}_{i=1}^n$, define a simultaneous-move generalized multi-prize nested lottery contest game, which we denote by

$$\left\langle \{\boldsymbol{V}_i, u_i(\cdot), f_i(\cdot), c_i(\cdot)\}_{i=1}^n \right\rangle.$$

Further define $y_i := f_i(x_i) \geq 0$—which denotes a player i's effective output—and $h_i(y_i) := c_i(f_i^{-1}(y_i))$. The following assumption is imposed throughout this section.

Assumption 3 *(Regular impact and effort cost functions).* $h_i(\cdot) \equiv c_i(f_i^{-1}(\cdot))$ is a twice-differentiable function, with $h_i(0) = 0$, $h_i'(y_i) > 0$, and $h_i''(y_i) \geq 0$ for all $y_i > 0$ and $i \in \mathcal{N}$.

The impact function $f_i(\cdot)$ and effort cost function $c_i(\cdot)$ are encapsulated into $h_i(\cdot)$. The conditions in the above definition automatically hold for an increasing and weakly concave impact function $f_i(\cdot)$ and an increasing and weakly convex effort cost function $c_i(\cdot)$, which are commonly assumed in the contest literature. However, it should be noted that the possibility of a convex impact function $f_i(\cdot)$ remains, as Assumption 3 can be satisfied when the effort cost function $c_i(\cdot)$ is sufficiently convex.

3.1. Equilibrium existence

The following theorem establishes the existence of a Nash equilibrium in pure strategy.

Theorem 1 *(Equilibrium existence in generalized multi-prize nested lottery contests). Suppose that Assumptions 1, 2, and 3 are satisfied. Then every generalized multi-prize nested lottery contest possesses a pure-strategy Nash equilibrium.*

It is evident that the multi-prize contest $\langle \{\boldsymbol{V}_i, u_i(\cdot), f_i(\cdot), c_i(\cdot)\}_{i=1}^n \rangle$ is strategically equivalent to one in which players with the same set of prize valuations $\{\boldsymbol{V}_i\}_{i=1}^n$ and utility functions $\{u_i(\cdot)\}_{i=1}^n$ choose $y_i \geq 0$ and have linear impact functions $\tilde{f}_i(y_i) = y_i$ and effort cost functions $h_i(y_i)$. It thus suffices to prove the equilibrium existence in the transformed multi-prize contest game $\langle \{\boldsymbol{V}_i, u_i(\cdot), \tilde{f}_i(\cdot), h_i(\cdot)\}_{i=1}^n \rangle$, and working on y_i instead of x_i proves convenient for the analysis.[14] In what follows, we call y_i player i's effort and $\boldsymbol{y} := (y_1, \ldots, y_n)$ the effort profile in the transformed contest game, and write $p_i(\cdot; \Omega_m)$, $P_{m,i}(\cdot)$, and $\pi_i(\cdot)$ as a function of $\boldsymbol{y} \equiv (y_1, \ldots, y_n)$ instead of $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$ with slight abuse of notation.

We make use of Reny (1999) to prove Theorem 1. In his seminal study on equilibrium existence in discontinuous games, Reny (1999) proposes the notion of "better-reply security," and shows in his Theorem 3.1 that a better-reply secure game possesses a pure-strategy Nash equilibrium when strategy spaces are convex and compact and payoffs are quasiconcave.

Note that bidding $y_i > h_i^{-1}(V_{1,i})$ is strictly dominated by $y_i = 0$ because for all $\boldsymbol{y}_{-i} \in \mathbb{R}_+^{n-1}$, we have that

$$\begin{aligned}\pi_i(y_i, \boldsymbol{y}_{-i}) &= \sum_{m=1}^{n} \left[P_{m,i}(y_i, \boldsymbol{y}_{-i}) \times u_i\left(V_{m,i} - h_i(y_i)\right) \right] \\ &\leq u_i\left(V_{1,i} - h_i(y_i)\right) < u_i(0) \leq \pi_i(0, \boldsymbol{y}_{-i}).\end{aligned}$$

Therefore, without loss of generality, we can restrict player i's effort choice to the interval $\mathcal{Y}_i := [0, h_i^{-1}(V_{1,i})]$, which ensures a convex and compact strategy space. It remains to verify (i) the quasiconcavity of each player's payoff function, and (ii) the better-reply security of the multi-prize contest. In particular, the condition of better-reply security states that if an effort profile

[14] A similar change of variables is invoked in Szidarovszky and Okuguchi (1997) and Cornes and Hartley (2005) to prove equilibrium existence and uniqueness in single-prize generalized lottery contests.

$\boldsymbol{y} \equiv (y_1, \ldots, y_n)$ is not an equilibrium, but the profile $(\boldsymbol{y}, \boldsymbol{\pi})$ of actions and payoffs belongs to the closure of the graph of the mapping from effort profiles to payoff profiles, then some player i has an effort entry y_i' that gives him a strictly higher payoff than π_i against the other players' effort profile in some open neighborhood of $\boldsymbol{y}_{-i}$.

We now lay out the proofs below that verify quasiconcave payoffs and better-reply security.

Payoff concavity We first prove that each player's payoff function is concave and thus quasiconcave in his own actions.

Lemma 1. *Suppose that Assumptions 2 and 3 are satisfied. Fixing $\boldsymbol{y}_{-i}$, player i's payoff $\pi_i(y_i, \boldsymbol{y}_{-i})$ in the multi-prize contest game $\langle \{\boldsymbol{V}_i, u_i(\cdot), \tilde{f}_i(\cdot), h_i(\cdot)\}_{i=1}^n \rangle$ is concave in y_i for $y_i \geq 0$.*

Proof. Define $\widetilde{P}_{m,i}(\boldsymbol{y})$ as

$$\widetilde{P}_{m,i}(\boldsymbol{y}) := \sum_{k=1}^{m} P_{k,i}(\boldsymbol{y}).$$

In words, $\widetilde{P}_{m,i}(\boldsymbol{y})$ is player i's probability of obtaining one of the first m prizes given the effort profile $\boldsymbol{y}$. It can be verified that

$$1 - \widetilde{P}_{m,i}(\boldsymbol{y}) = \sum_{\Omega_{m+1} \in \binom{\mathcal{N}}{n-m}} \left[\Pr(\Omega_{m+1}) \times \mathbb{1}(i \in \Omega_{m+1}) \right], \text{ for } m \in \{1, \ldots, n-1\}.$$

The above equation is intuitive: A player i does not receive one of the first m prizes if and only if he is eligible for the $(m+1)$th prize.

Case I: risk-neutral players Suppose that $u_i(w_i) = w_i$. A player i's expected payoff (4) can be rewritten as

$$\begin{aligned}
\pi_i(\boldsymbol{y}) &\equiv \sum_{m=1}^{n} \left[P_{m,i}(\boldsymbol{y}) \times V_{m,i} \right] - h_i(y_i) \\
&= V_{1,i} - \sum_{m=1}^{n-1} \left[\left[1 - \widetilde{P}_{m,i}(\boldsymbol{y}) \right] \times \left(V_{m,i} - V_{m+1,i} \right) \right] - h_i(y_i) \\
&= V_{1,i} - \sum_{m=1}^{n-1} \left\{ \left[\sum_{\Omega_{m+1} \in \binom{\mathcal{N}}{n-m}} \left[\Pr(\Omega_{m+1}) \times \mathbb{1}(i \in \Omega_{m+1}) \right] \right] \times (V_{m,i} - V_{m+1,i}) \right\} \\
&\quad - h_i(y_i).
\end{aligned} \tag{5}$$

Consider an arbitrary set of $m \in \{1, \ldots, n-1\}$ players that obtain the first m prizes. Let i_k indicate the index of the player that receives the kth prize. Denote by $\boldsymbol{I}_m$ the sequence of the index of players $(i_1, \ldots, i_m)$. In what follows, we write $i \in \boldsymbol{I}_m$ to indicate that i is an element of the sequence $\boldsymbol{I}_m$. Similarly, we write $i \notin \boldsymbol{I}_m$ to indicate that i is not an element of the sequence $\boldsymbol{I}_m$. Define $\boldsymbol{S}(m, i) := \{\boldsymbol{I}_m \mid i \notin \boldsymbol{I}_m\}$. In words, $\boldsymbol{S}(m, i)$ is the set of sequences of players that win the first m prizes and do not include player i. Further, define $Y := \sum_{j=1}^{n} y_j$. We can then obtain that

$$\sum_{\Omega_{m+1}\in\binom{\mathcal{N}}{n-m}}\left[\Pr(\Omega_{m+1})\times\mathbb{1}(i\in\Omega_{m+1})\right]=\sum_{\boldsymbol{I}_m\in\boldsymbol{S}(m,i)}\underbrace{\left[\prod_{k=1}^{m}\frac{y_{i_k}}{Y-\sum_{j=1}^{k-1}y_{i_j}}\right]}_{\Phi(y_i;\boldsymbol{I}_m)},\tag{6}$$

and

$$\ln\left(\Phi(y_i;\boldsymbol{I}_m)\right)=\sum_{k=1}^{m}\ln(y_{i_k})-\sum_{k=1}^{m}\ln\left(Y-\sum_{j=1}^{k-1}y_{i_j}\right).$$

The condition $\boldsymbol{I}_m\in\boldsymbol{S}(m,i)$ in (6) immediately implies $i_k\neq i$ for all $k\in\{1,\dots,m\}$, which allows us to conclude that $\ln\left(\Phi(y_i;\boldsymbol{I}_m)\right)$ is convex in y_i and thus $\Phi(y_i;\boldsymbol{I}_m)$ is convex in y_i. Combining (5) and (6), we learn that $\pi_i(y_i,\boldsymbol{y}_{-i})$ is concave in y_i for all $y_i>0$. Furthermore, it follows from (3) that $\lim_{y_i\searrow 0}\pi_i(y_i,\boldsymbol{y}_{-i})\geq\pi_i(0,\boldsymbol{y}_{-i})$. Therefore, a risk-neutral player's payoff $\pi_i(y_i,\boldsymbol{y}_{-i})$ is concave in y_i for $y_i\geq 0$.

Case II: risk-averse players Suppose that $u_i(\cdot)$ is increasing, (weakly) concave, and satisfies Assumption 2. Similar to (5), a player i's expected payoff (4) can be rearranged as

$$\begin{aligned}\pi_i(\boldsymbol{y})=&u_i\left(V_{1,i}-h_i(y_i)\right)\\&-\sum_{m=1}^{n-1}\left\{\left[1-\widetilde{P}_{m,i}(\boldsymbol{y})\right]\times\left[u_i\left(V_{m,i}-h_i(y_i)\right)-u_i\left(V_{m+1,i}-h_i(y_i)\right)\right]\right\}\\=&u_i\left(V_{1,i}-h_i(y_i)\right)\\&-\sum_{m=1}^{n-1}\sum_{\boldsymbol{I}_m\in\boldsymbol{S}(m,i)}\left\{\int_{V_{m+1,i}}^{V_{m,i}}\underbrace{\left[\overbrace{\left[\prod_{k=1}^{m}\frac{y_{i_k}}{Y-\sum_{j=1}^{k-1}y_{i_j}}\right]}^{\Phi(y_i;\boldsymbol{I}_m)}\times u_i'\left(\tau-h_i(y_i)\right)\right]}_{\Theta(y_i;\boldsymbol{I}_m)}d\tau\right\}.\end{aligned}\tag{7}$$

First, simple algebra would verify that

$$\frac{d^2u_i\left(V_{1,i}-h_i(y_i)\right)}{d(y_i)^2}=u_i''\left(V_{1,i}-h_i(y_i)\right)\left[h_i'(y_i)\right]^2-u_i'\left(V_{1,i}-h_i(y_i)\right)h_i''(y_i)\leq 0,$$

and thus $u_i\left(V_{1,i}-h_i(y_i)\right)$ is concave in y_i. Second, by Assumptions 2 and 3,

$$\frac{d\ln\left(u_i'\left(\tau-h_i(y_i)\right)\right)}{dy_i}=-\frac{u_i''\left(\tau-h_i(y_i)\right)}{u_i'\left(\tau-h_i(y_i)\right)}\times h_i'(y_i)$$

is non-decreasing in y_i, indicating that $u_i'\left(\tau-h_i(y_i)\right)$ is log-convex in y_i. Together with the fact that $\Phi(y_i;\boldsymbol{I}_m)$ is log-convex in y_i, we can conclude that $\Theta(y_i;\boldsymbol{I}_m)$ is log-convex and thus convex in y_i for all $y_i>0$. Last, note that $\lim_{y_i\searrow 0}\pi_i(y_i,\boldsymbol{y}_{-i})\geq\pi_i(0,\boldsymbol{y}_{-i})$. The above analysis implies immediately that $\pi_i(y_i,\boldsymbol{y}_{-i})$ is concave in y_i for $y_i\geq 0$. This concludes the proof. ■

It is noteworthy that Lemma 1 does not require Assumption 1. The key to the proof of payoff concavity is the rearrangement of expected payoff function. By Equation (4), a player i's expected payoff $\pi_i(y_i,\boldsymbol{y}_{-i})$ is expressed as a linear combination of the player's probability of receiving each prize, i.e., $P_{m,i}(\boldsymbol{y})$. However, as previously mentioned in Example 1, $P_{m,i}(\boldsymbol{y})$

is not concave in y_i in general. We circumvent this difficulty by rewriting $\pi_i(y_i, \boldsymbol{y}_{-i})$ as a linear combination of the probability of one's *not receiving one of the first several prizes*—i.e., $1 - \sum_{k=1}^{m} P_{k,i}(\boldsymbol{y})$—as Equation (7) in the proof illustrates.

We continue to use the three-player setting in Example 1 to illuminate the logic of the proof.

Example 1 *(Continued, proving payoff concavity).* Suppose that $V_{1,1} > V_{2,1} > V_{3,1} = 0$—i.e., there exist two effective prizes—and consider player 1. Recall that

$$P_{1,1}(\boldsymbol{y}) = \frac{y_1}{y_1 + y_2 + y_3},$$

and

$$P_{2,1}(\boldsymbol{y}) = \frac{y_1 y_2}{(y_1 + y_2 + y_3)(y_1 + y_3)} + \frac{y_1 y_3}{(y_1 + y_2 + y_3)(y_1 + y_2)}.$$

His expected payoff is given by $\pi_1(\boldsymbol{y}) = P_{1,1}(\boldsymbol{y})V_{1,1} + P_{2,1}(\boldsymbol{y})V_{2,1} - h_1(y_1)$, which can then be rewritten as

$$\pi_1(\boldsymbol{y}) = V_{1,1} - \left[1 - P_{1,1}(\boldsymbol{y})\right] \times (V_{1,1} - V_{2,1}) - \left[1 - P_{1,1}(\boldsymbol{y}) - P_{2,1}(\boldsymbol{y})\right] \times V_{2,1} - h_1(y_1),$$

where $\left[1 - P_{1,1}(\boldsymbol{y})\right]$ is the probability that he will not receive the first prize, while $[1 - P_{1,1}(\boldsymbol{y}) - P_{2,1}(\boldsymbol{y})]$ is the probability that player 1 will not receive either prize. Because $P_{1,1}(\boldsymbol{y})$ is concave and $h_1(y_1)$ is convex in y_1, it suffices to show that the term $[1 - P_{1,1}(\boldsymbol{y}) - P_{2,1}(\boldsymbol{y})]$ is convex in y_1. There are two possibilities for this event: (i) player 2 wins the first prize and player 3 wins the second prize; and (ii) player 3 wins the first prize and player 2 wins the second prize. Thus, we can obtain

$$\begin{aligned} &1 - P_{1,1}(\boldsymbol{y}) - P_{2,1}(\boldsymbol{y}) \\ &= \overbrace{\underbrace{\frac{y_2}{y_1 + y_2 + y_3} \times \frac{y_3}{y_1 + y_3}}_{\substack{\text{Player 2 wins the first prize,} \\ \text{player 3 wins the second prize.}}} + \underbrace{\frac{y_3}{y_1 + y_2 + y_3} \times \frac{y_2}{y_1 + y_2}}_{\substack{\text{Player 3 wins the first prize,} \\ \text{player 2 wins the second prize.}}}}^{\text{Player 1's being eligible for the third prize (i.e., no prizes): } \Omega_3 = \{1\}.}. \end{aligned}$$

It is evident that both terms in the above expression are log-convex and thus convex in y_1, which indicates that $[1 - P_{1,1}(\boldsymbol{y}) - P_{2,1}(\boldsymbol{y})]$ is convex in y_1.

Better-reply security We now verify the better-reply security of the contest game. The following definition is due to Reny (1999).

Definition 1 *(Better-reply security).* The multi-prize contest game $\left\langle \{\boldsymbol{V}_i, u_i(\cdot), \tilde{f}_i(\cdot), h_i(\cdot)\}_{i=1}^{n} \right\rangle$ is better-reply secure if, whenever $(\boldsymbol{y}^*, \boldsymbol{\pi}^*)$ is in the closure of the graph of its vector payoff function and $\boldsymbol{y}^*$ is not an equilibrium, there exists a player i, a strategy $\bar{y}_i \in \mathcal{Y}_i \equiv [0, h_i^{-1}(V_{1,i})]$, and a payoff $\alpha > \pi_i^*$ such that $\pi_i(\bar{y}_i, \boldsymbol{y}_{-i}) \geq \alpha$ for all $\boldsymbol{y}_{-i}$ in some open neighborhood of $\boldsymbol{y}_{-i}^*$.

Reny (1999) provides two useful conditions, referred to as *payoff security* and *reciprocal upper semicontinuity*, that lead to better-reply security. Intuitively, the former condition requires that for every effort profile $\boldsymbol{y}$, each player have a strategy that virtually guarantees the expected payoff he receives at $\boldsymbol{y}$ even if his opponents deviate slightly from $\boldsymbol{y}_{-i}$. The latter requires that some player's payoff jump up whenever some other's jumps down. It is noteworthy that payoff security is satisfied in our framework. However, reciprocal upper semicontinuity does not hold in general except for knife-edge cases, e.g., (i) there is only one prize available, i.e., $\ell = 1$, and (ii) all prizes bear the same values to all, i.e., $V_{m,i} = V_{m,j}$ for all $m \in \{1, \ldots, n\}$ and $i \neq j$.

We instead directly verify better-reply security by checking the condition stated in Definition 1.[15] The following can be obtained.

Lemma 2. *Suppose that Assumption 1 is satisfied.*[16] *Then the multi-prize contest game* $\langle \{\boldsymbol{V}_i, u_i(\cdot), \tilde{f}_i(\cdot), h_i(\cdot)\}_{i=1}^n \rangle$ *is better-reply secure.*

The proof is relegated to the Appendix A. But, again, we continue with the three-player setting in Example 1 to elaborate on its logic.

Example 1 *(Continued, proving better-reply security).* Let $h_i(y_i) = y_i$ and set $V_{1,i} = V_{2,i} = 1$ and $V_{3,i} = 0$ for $i \in \{1, 2, 3\}$, which yields a case of $\ell = 2$, i.e., two effective prizes. Suppose that $(\boldsymbol{y}^*, \boldsymbol{\pi}^*)$ is in the closure of the graph of its vector payoff function and the strategy profile $\boldsymbol{y}^* \equiv (y_1^*, y_2^*, y_3^*)$ does not constitute an equilibrium. If at least two active players exist—which corresponds to Case I in the proof of Lemma 2—then all players' payoff functions $\pi_i(\cdot)$ are continuous at $\boldsymbol{y}^*$. Better-reply security can then be implied from continuity and the definition of Nash equilibrium.

Next, imagine that the number of prizes exceeds that of active players. Specifically, suppose that there is only one active player, which corresponds to Case II in the proof of Lemma 2. Without loss of generality, we assume that $y_1^* > 0$ and $y_2^* = y_3^* = 0$. Consider a sequence of effort profiles $\{\boldsymbol{y}^k\}_{k=1}^{\infty}$ approaching $\boldsymbol{y}^*$ as $k \to \infty$ such that $\pi_i^* = \lim_{k\to\infty} \pi_i(\boldsymbol{y}^k)$. Clearly, player 1 must win the first prize with certainty at the limit. Further, it is impossible for both players 2 and 3 to win the second prize with certainty at the limit because their probabilities of obtaining the second prize must sum up to one. Suppose that player 2 wins the second prize with some probability $\kappa < 1$ at the limit. Then $\pi_2^* = \kappa$. In such a scenario, his probability of winning the second prize approaches 1 when he exerts a small amount of effort, given that other players deviate slightly from (y_1^*, y_3^*). To be more specific, suppose that player 2 deviates from zero effort and chooses $\bar{y}_2 = \delta$ for some small $\delta > 0$. Then player 2's probability of winning the second prize, given that (y_1, y_3) falls in the open ball with center (y_1^*, y_3^*) and radius δ^3, can be bounded from below by

$$\begin{aligned} P_{2,2}(y_1, \bar{y}_2, y_3) &= \frac{y_1}{y_1 + \bar{y}_2 + y_3} \times \frac{\bar{y}_2}{\bar{y}_2 + y_3} + \frac{y_3}{y_1 + \bar{y}_2 + y_3} \times \frac{\bar{y}_2}{y_1 + \bar{y}_2} \\ &\geq \frac{y_1}{y_1 + \bar{y}_2 + y_3} \times \frac{\bar{y}_2}{\bar{y}_2 + y_3} \\ &\geq \frac{y_1^* + \delta^3}{y_1^* + \delta^3 + \delta + \delta^3} \times \frac{\delta}{\delta + \delta^3}. \end{aligned}$$

Note that the last term in the above inequality approaches 1 as $\delta \to 0$, and player 2's expected payoff is no less than $P_{2,2}(y_1, \bar{y}_2, y_3)V_{2,2} - h_2(\bar{y}_2) = P_{2,2}(y_1, \bar{y}_2, y_3) - \delta$. This implies that

[15] Bagh and Jofre (2006) define the more general notion of weak reciprocal upper semicontinuity (wrusc) and show that wrusc and payoff security lead to better-reply security. It can be verified that wrusc is satisfied in our framework. Therefore, better-reply security can alternatively be established with payoff security and wrusc.

[16] The proof of the lemma makes use of the continuity and monotonicity of $u_i(\cdot)$ and $h_i(\cdot)$, as well as $h_i(0) = 0$. NIARA (and risk aversion) of $u_i(\cdot)$ in Assumption 2 and the convexity of $h_i(\cdot)$ in Assumption 3 are not necessary.

player 2 can secure a strictly larger expected payoff than κ by exerting an effort of a sufficiently small amount $\delta > 0$.

3.2. Further discussion: role of Assumption 1

We now elaborate on the role played by Assumption 1. Recall that by Reny (1999), every compact and quasiconcave deterministic game that satisfies better-reply security has a pure-strategy Nash equilibrium. As stated above, the payoff concavity established in Lemma 1 does not require Assumption 1. The absence of this condition, however, may cause the failure of better-reply security.

Consider a multi-prize contest with $n = 4$ players; assume $u_i(w_i) = w_i$ and $h_i(y_i) = y_i$ for all $i \in \{1, 2, 3, 4\}$. Set $\boldsymbol{V}_1 = \boldsymbol{V}_2 = (1, 0, 0, 0)$, $\boldsymbol{V}_3 = (1/4, 0, 0, 0)$, and $\boldsymbol{V}_4 = (1, 1, 1, 0)$, which yields $\ell_1 = \ell_2 = \ell_3 = 1$ and $\ell_4 = 3$ and thus violates Assumption 1. In words, players 1-3 do not care about the second and third prizes, and their effort decision is solely motivated by the first prize. In contrast, player 4 cares equally about the first three prizes.

3.2.1. Nonexistence of pure-strategy Nash equilibrium

Suppose to the contrary that there exists a pure-strategy equilibrium. Denote the effort profile and total effort by $\boldsymbol{y}^* = (y_1^*, y_2^*, y_3^*, y_4^*)$ and $Y^* := \sum_{i=1}^4 y_i^*$, respectively. We first show that $Y^* \geq 1/2$ in an equilibrium. The contest is essentially a single-prize contest to player $i = 1, 2$, and his optimization problem is

$$\max_{y_i \geq 0} \pi_i(y_i, \boldsymbol{y}_{-i}^*) = \frac{y_i}{y_i + \sum_{j \neq i} y_j^*} - y_i, \text{ for } i = 1, 2,$$

which further gives

$$y_i^* = \max\left\{0, \sqrt{Y^* - y_i^*} - (Y^* - y_i^*)\right\}.$$

If $y_i^* = 0$, then it must be the case that $\sqrt{Y^* - y_i^*} - (Y^* - y_i^*) \leq 0$, which implies $Y^* \geq 1 > 1/2$. If $y_i^* > 0$. We then have $y_i^* = \sqrt{Y^* - y_i^*} - (Y^* - y_i^*)$, from which we can solve for y_i^* as

$$y_i^* = Y^* - (Y^*)^2, \text{ for } i = 1, 2.$$

This gives

$$Y^* \equiv \sum_{i=1}^{4} y_i^* \geq y_1^* + y_2^* = 2Y^* - 2(Y^*)^2 \implies Y^* \geq \frac{1}{2}.$$

Given $Y^* \geq 1/2$, player 3 must stay inactive in the equilibrium, i.e., $y_3^* = 0$. He would otherwise end up with a negative expected payoff—i.e., $y_3^* \times [1/(4Y^*) - 1]$—which is clearly suboptimal.

Last, we examine player 4's effort incentive. Because $y_3^* = 0$, player 4 would win one of the first three prizes with probability one as long as he exerts a positive effort. However, the probability would drop to $1/2$ for $y_1^* = y_2^* > 0$ and to $3/4$ for $y_1^* = y_2^* = 0$ if player 4 stays inactive. A maximum of player 4's payoff function does not exist, which dissolves pure-strategy Nash equilibrium.

3.2.2. Failure of better-reply security and role of Assumption 1

The nonexistence of Nash equilibrium shown above indicates that better-reply security may not hold when condition of Assumption 1 fails. To see this, consider the sequence of effort profiles $\{\boldsymbol{y}^k\}_{k=1}^{\infty} = \{(1/4, 1/4, 0, 1/k)\}_{k=1}^{\infty}$ that approaches $\boldsymbol{y}^* = (1/4, 1/4, 0, 0)$. It is straightforward to verify that the sequence of payoff vectors approaches $\boldsymbol{\pi}^* = (3/4, 3/4, 0, 1)$ as $k \to \infty$. In other words, $(\boldsymbol{y}^*, \boldsymbol{\pi}^*)$ is in the closure of the graph of its vector payoff function. Because the contest game we construct in Section 3.2 has been proven to have no pure-strategy equilibrium, $\boldsymbol{y}^*$ is not an equilibrium. For player $i = 1, 2, 3$, there is no profitable deviation because y_i^* maximizes player i's expected payoff given the opponents' strategy profile. Therefore, player $i = 1, 2, 3$ cannot secure a payoff strictly higher than π_i^*. The same conclusion applies to player 4 because $\pi_4^* = 1$ is the highest payoff that player 4 can derive from the contest. The condition of better-reply security thus cannot be satisfied at $(\boldsymbol{y}^*, \boldsymbol{\pi}^*)$.

4. Bayesian generalized multi-prize nested lottery contests

In this section, we further generalize the model to allow for incomplete information. In what follows, we first set up the incomplete-information generalized multi-prize nested lottery contest model as a Bayesian game and present its preliminaries. We then explore equilibrium existence in the model.

4.1. Model setup and equilibrium concept

Let $(T_i, \mathcal{T}_i)$ be a measurable space for player $i \in \mathcal{N}$, where T_i is player i's nonempty type space. Let $T := \times_{i=1}^n T_i$ and $\mathcal{T} := \otimes_{i=1}^n \mathcal{T}_i$. Denote the common prior by λ, which is a probability measure on $(T, \mathcal{T})$. Further, let λ_i be the marginal probability measure induced by λ on T_i. We impose the following assumption on λ.

Assumption 4. λ is absolutely continuous with respect to $\lambda_1 \otimes \cdots \otimes \lambda_n$.

It is useful to point out that Assumption 4 is satisfied if the type space T is countable. We allow a player i's prize valuations $\boldsymbol{V}_i \equiv (V_1, \dots, V_n)$, utility function $u_i(\cdot)$, impact function $f_i(\cdot)$, and effort cost function $c_i(\cdot)$ to depend on the type profile $\boldsymbol{t} := (t_1, \dots, t_n)$. We assume that $\boldsymbol{V}_i(\boldsymbol{t})$ is $(\mathcal{T}, \mathcal{B}(\mathbb{R}^n))$-measurable and $\{u_i(w_i, \boldsymbol{t})\}_{i=1}^n$, $\{f_i(x_i, \boldsymbol{t})\}_{i=1}^n$, and $\{c_i(x_i, \boldsymbol{t})\}_{i=1}^n$ are $(\mathcal{B}(\mathbb{R}) \otimes \mathcal{T}, \mathcal{B}(\mathbb{R}))$-measurable, where $\mathcal{B}(\mathbb{R}^n)$ and $\mathcal{B}(\mathbb{R})$ denote the Borel σ-algebra on $\mathbb{R}^n$ and $\mathbb{R}$, respectively.

Define $\ell_i(\boldsymbol{t}) := \max\{m = 1, \dots, n-1 \mid V_{m,i}(\boldsymbol{t}) > V_{m+1,i}(\boldsymbol{t})\}$. The following assumption is imposed throughout this section.

Assumption 5. The following statements hold:

(i) The prize valuation $V_{m,i}(\boldsymbol{t}) < \bar{V} < \infty$ for all $m \in \{1, \dots, n\}$, $i \in \mathcal{N}$, and $\boldsymbol{t} \in T$, with $V_{1,i}(\boldsymbol{t}) \geq \cdots \geq V_{n,i}(\boldsymbol{t}) \geq 0$. Moreover, $\ell_1(\boldsymbol{t}) = \cdots = \ell_n(\boldsymbol{t}) =: \ell$ for all $\boldsymbol{t} \in T$.

(ii) Players' utility function $u_i(\cdot, \boldsymbol{t})$ is twice-differentiable and exhibits NIARA, with $\partial u_i(w_i, \boldsymbol{t})/\partial w_i > 0$ and $\partial^2 u_i(w_i, \boldsymbol{t})/\partial w_i^2 \leq 0$ for all $w_i \in \mathbb{R}$ and $\boldsymbol{t} \in T$. Moreover, $u_i(\bar{V}, \boldsymbol{t}) - u_i(V^\dagger, \boldsymbol{t}) < \bar{U} < \infty$ for all $\boldsymbol{t} \in T$, where $V^\dagger := -\sup_{\boldsymbol{t} \in T} c_i(\bar{X}, \boldsymbol{t})$ and $\bar{X}$ is defined in (iv) below.

(iii) The impact function $f_i(\cdot, \boldsymbol{t})$ is twice-differentiable, with $f_i(0, \boldsymbol{t}) = 0$, $\partial f_i(x_i, \boldsymbol{t})/\partial x_i > 0$, and $\partial^2 f_i(x_i, \boldsymbol{t})/\partial x_i^2 \leq 0$ for all $x_i \geq 0$, $i \in \mathcal{N}$, and $\boldsymbol{t} \in T$. Moreover, $\inf_{\boldsymbol{t}\in T} f_i(x_i, \boldsymbol{t}) > 0$ for all $x_i > 0$ and $i \in \mathcal{N}$.

(iv) The effort cost function $c_i(\cdot, \boldsymbol{t})$ is twice-differentiable, with $c_i(0, \boldsymbol{t}) = 0$, $\partial c_i(x_i, \boldsymbol{t})/\partial x_i > 0$, and $\partial^2 c_i(x_i, \boldsymbol{t})/\partial x_i^2 \geq 0$ for all $x_i > 0$, $i \in \mathcal{N}$, and $\boldsymbol{t} \in T$. Moreover, there exists $\bar{X} > 0$ such that $c_i(\bar{X}, \boldsymbol{t}) > \bar{V}$ for all $i \in \mathcal{N}$ and $\boldsymbol{t} \in T$.

(v) Fix $i \in \mathcal{N}$. For every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left|u_i(w_i, \boldsymbol{t}) - u_i(w_i', \boldsymbol{t})\right| < \epsilon, \forall \boldsymbol{t} \in T, w_i, w_i' \in \left[V^\dagger, \bar{V}\right], with \left|w_i - w_i'\right| < \delta,$$
$$\left|f_i(x_i, \boldsymbol{t}) - f_i(x_i', \boldsymbol{t})\right| < \epsilon, \forall \boldsymbol{t} \in T, x_i, x_i' \in \left[0, \bar{X}\right], with \left|x_i - x_i'\right| < \delta,$$

and

$$\left|c_i(x_i, \boldsymbol{t}) - c_i(x_i', \boldsymbol{t})\right| < \epsilon, \forall \boldsymbol{t} \in T, x_i, x_i' \in \left[0, \bar{X}\right], with \left|x_i - x_i'\right| < \delta.$$

Assumption 5 is intuitive. Part (i) naturally extends Assumption 1. Part (ii) reiterates Assumption 2. Parts (iii) and (iv) are essentially in line with Assumption 3. By part (iv), we can restrict a player i's action space to $\mathcal{X} := [0, \bar{X}]$, which is convex and compact.

Part (v) is a new condition and requires that the family of functions $(u_i(\cdot, \boldsymbol{t}))_{\boldsymbol{t}\in T}$, $(f_i(\cdot, \boldsymbol{t}))_{\boldsymbol{t}\in T}$, and $(c_i(\cdot, \boldsymbol{t}))_{\boldsymbol{t}\in T}$ be uniformly equicontinuous, a requirement that is clearly satisfied if the type space T is finite. The same uniform equicontinuity on payoff functions is imposed in Milgrom and Weber (1985) to prove distributional strategy Bayesian equilibrium in the setting of a continuum of types.[17,18]

Bayesian game and behavioral-strategy equilibrium A Bayesian generalized multi-prize nested lottery contest can then be denoted by

$$\boldsymbol{\Gamma} := \left\langle \{\boldsymbol{V}_i(\boldsymbol{t}), u_i(\cdot, \boldsymbol{t}), f_i(\cdot, \boldsymbol{t}), c_i(\cdot, \boldsymbol{t})\}_{i=1}^n, (T, \mathcal{T}, \lambda)\right\rangle,$$

and the ex post contest under a fixed type profile $\boldsymbol{t} \in T$ is denoted by

$$\boldsymbol{\Gamma_t} := \left\langle \{\boldsymbol{V}_i(\boldsymbol{t}), u_i(\cdot, \boldsymbol{t}), f_i(\cdot, \boldsymbol{t}), c_i(\cdot, \boldsymbol{t})\}_{i=1}^n\right\rangle.$$

A behavioral strategy for player $i \in \mathcal{N}$ in the ex ante Bayesian contest $\boldsymbol{\Gamma}$—which we denote by σ_i—is a transition probability from $(T_i, \mathcal{T}_i)$ to $\big(\mathcal{X}, \mathcal{B}(\mathcal{X})\big)$ such that $\sigma_i(\cdot|t_i)$ is a probability measure on $\big(\mathcal{X}, \mathcal{B}(\mathcal{X})\big)$ for all $t_i \in T_i$, and $\sigma_i(A|\cdot)$ is a $\mathcal{T}_i$-measurable function for every $A \in \mathcal{B}(\mathcal{X})$. Note that a behavior strategy $\sigma_i(\cdot|\cdot)$ degenerates to a pure one if for all $t_i \in T_i$, $\sigma_i(\cdot|t_i) = \delta_{x_i}(\cdot)$ holds for some $x_i \in \mathcal{X}$, where δ_{x_i} is the Dirac delta function at x_i. Further, let Σ_i represent the set of behavioral strategies for player $i \in \mathcal{N}$, and define $\boldsymbol{\Sigma} := \times_{i=1}^n \Sigma_i$.

[17] It is noteworthy that uniform equicontinuity on the modeling primitives—i.e., uniform equicontinuity on the family of functions $(u_i(\cdot, \boldsymbol{t}))_{\boldsymbol{t}\in T}$, $(f_i(\cdot, \boldsymbol{t}))_{\boldsymbol{t}\in T}$, and $(c_i(\cdot, \boldsymbol{t}))_{\boldsymbol{t}\in T}$—does not imply uniform equicontinuity on payoff functions in our setting due to discontinuity in players' payoff functions at origin. To see this, let $n = 2$ and set $\boldsymbol{V}_1(\boldsymbol{t}) = \boldsymbol{V}_2(\boldsymbol{t}) = (1, 0)$, $u_i(w_i, \boldsymbol{t}) = w_i$, and $c_i(x_i, \boldsymbol{t}) = f_i(x_i, \boldsymbol{t}) = x_i$ for all $\boldsymbol{t} \in T$. It follows immediately that $\lim_{x_i \searrow 0} \pi_i(x_i, 0, \boldsymbol{t}) = 1 \neq 0 = \lim_{x_{-i} \searrow 0} \pi_i(0, x_{-i}, \boldsymbol{t})$ for all $\boldsymbol{t} \in T$, i.e., player i's payoff is discontinuous at $(x_1, x_2) = (0, 0)$ for all $\boldsymbol{t} \in T$. Note that uniform equicontinuity implies that every member of the family is continuous, which suggests that the family of functions $(\pi_i(\cdot, \boldsymbol{t}))_{\boldsymbol{t}\in T}$ is not uniformly equicontinuous.

[18] Balder (1988) extends Milgrom and Weber's (1985) existence results to behavioral strategies as adopted in our paper. Balder's (1988) results do not require a uniform equicontinuity condition for payoff functions. Instead, he requires that players' payoff be continuous in the strategy profile for an arbitrary type profile—a condition that is not satisfied in our setting due to discontinuity in players' payoffs at origin.

Fixing a behavioral-strategy profile $\boldsymbol{\sigma} := (\sigma_1, \ldots, \sigma_n)$, player i's ex ante expected payoff in the Bayesian game $\boldsymbol{\Gamma}$ can be derived as

$$\Pi_i^{\boldsymbol{\Gamma}}(\boldsymbol{\sigma}) := \int_T \int_{\mathcal{X}} \cdots \int_{\mathcal{X}} \pi_i(\boldsymbol{x}, \boldsymbol{t})\sigma_1(dx_1|t_1)\ldots\sigma_n(dx_n|t_n)\lambda(d\boldsymbol{t}),$$

where $\pi_i(\boldsymbol{x}, \boldsymbol{t})$ is player i's expected payoff in the ex post contest $\boldsymbol{\Gamma_t}$ given $(\boldsymbol{x}, \boldsymbol{t})$.

We say that $\boldsymbol{\sigma}^* \equiv (\sigma_1^*, \ldots, \sigma_n^*)$ is a behavioral-strategy equilibrium of the Bayesian generalized multi-prize nested lottery contest $\boldsymbol{\Gamma}$ if for each player $i \in \mathcal{N}$,

$$\Pi_i^{\boldsymbol{\Gamma}}(\sigma_i^*, \boldsymbol{\sigma}_{-i}^*) \geq \Pi_i^{\boldsymbol{\Gamma}}(\sigma_i, \boldsymbol{\sigma}_{-i}^*), \forall \sigma_i \in \Sigma_i.$$

A pure-strategy equilibrium of the Bayesian contest $\boldsymbol{\Gamma}$ can be defined similarly.

4.2. Analysis

We now set out to establish equilibrium existence in the Bayesian contest $\boldsymbol{\Gamma}$ that we define in Section 4.1. Recall that choosing effort x_i is equivalent to choosing effective output $y_i \equiv f_i(x_i)$ for a player i in a contest with complete information, which enables an analysis in terms of y_i instead of x_i without loss of generality. This equivalence dissolves when the model involves incomplete information: Choosing effort is no longer equivalent to choosing effective output because the latter now depends on the type profile $\boldsymbol{t} \equiv (t_1, \ldots, t_n)$ and a player is uninformed of his opponents' types when making his effort decision. As a result, we perform the analysis in terms of the decision variable x_i directly.

We invoke Theorem 2 in He and Yannelis (2015) and Theorem 1 in Carbonell-Nicolau and McLean (2018), which extend Reny's (1999) equilibrium existence result to Bayesian games with discontinuous payoffs. As we mentioned in Section 3.1, Reny (1999) shows in his Proposition 3.2 that two conditions—namely, payoff security and reciprocal upper semicontinuity—imply better-reply security. He and Yannelis (2015) strengthen the notion of payoff security and introduce finite/finite* payoff security. With these stronger notions, they show that if the sum of payoffs in every ex post game is upper-semicontinuous in actions, then the ex ante Bayesian game is payoff-secure and reciprocal upper-semicontinuous. As a result, a pure-strategy equilibrium exists by Reny (1999), given that the ex ante Bayesian game is quasiconcave and the type space is countable. Similarly, Carbonell-Nicolau and McLean (2018) show that *uniform payoff security* in the ex post game—a notion proposed by Monteiro and Page (2007)—implies payoff security in the corresponding ex ante Bayesian games. In addition, they show that *aggregate upper semicontinuity* in every ex post game can guarantee the upper semicontinuity of the sum of payoffs defined on behavioral strategies. These results, together with those in Reny (1999), give rise to the existence of behavioral-strategy equilibrium in a Bayesian game.

It is thus natural to verify uniform payoff security and aggregate upper semicontinuity in order to extend our equilibrium existence result in Theorem 1 to the Bayesian contest game. Their formal definitions—due to Monteiro and Page (2007) and Carbonell-Nicolau and McLean (2018)—are restated below and adapted to our model.

Definition 2 *(Uniform payoff security).* A Bayesian game $\boldsymbol{\Gamma}$ is uniformly payoff-secure if for each $i \in \mathcal{N}$, $x_i \in \mathcal{X}$, and $\epsilon > 0$, there exists $\bar{x}_i \in \mathcal{X}$ such that for all $(\boldsymbol{t}, \boldsymbol{x}_{-i}) \in T \times \mathcal{X}^{n-1}$, there exists a neighborhood $U_{\boldsymbol{x}_{-i}}$ of $\boldsymbol{x}_{-i}$ such that

$$\pi_i(\bar{x}_i, \boldsymbol{x}'_{-i}, \boldsymbol{t}) \geq \pi_i(x_i, \boldsymbol{x}_{-i}, \boldsymbol{t}) - \epsilon, \forall \boldsymbol{x}'_{-i} \in U_{\boldsymbol{x}_{-i}}.$$

Definition 3 *(Aggregate upper semicontinuity).* An ex post contest $\boldsymbol{\Gamma_t}$ is aggregate upper-semicontinuous if the map $\sum_{i=1}^{n} \pi_i(\cdot, \boldsymbol{t}) : \mathcal{X}^n \to \mathbb{R}$ is upper-semicontinuous.

As will be clear later, uniform payoff security is satisfied given the assumption of uniform equicontinuity of $(u_i(\cdot, \boldsymbol{t}))_{\boldsymbol{t}\in T}$, $(f_i(\cdot, \boldsymbol{t}))_{\boldsymbol{t}\in T}$, and $(c_i(\cdot, \boldsymbol{t}))_{\boldsymbol{t}\in T}$ in Assumption 5. However, aggregate upper semicontinuity cannot be satisfied in our framework. To see this, consider a simple ex post two-player winner-take-all contest with $V_{1,i} > V_{2,i} = 0$, $u_i(w_i) = w_i$, and $f_i(x_i) = c_i(x_i) = x_i$ for $i \in \{1, 2\}$. Recall from Equation (3) that player i's winning probability is equal to $x_i/(x_1 + x_2)$ if $x_1 + x_2 > 0$ and is equal to $1/2$ if $x_1 + x_2 = 0$. It is straightforward to see that the sum of the players' payoff amounts to

$$\pi_1(x_1, x_2) + \pi_2(x_1, x_2) = \begin{cases} \dfrac{x_1}{x_1 + x_2} V_{1,1} + \dfrac{x_2}{x_1 + x_2} V_{1,2} - (x_1 + x_2), & \text{if } x_1 + x_2 > 0, \\ \dfrac{1}{2}\left(V_{1,1} + V_{1,2}\right), & \text{if } x_1 + x_2 = 0, \end{cases}$$

which is not upper-semicontinuous at $(x_1, x_2) = (0, 0)$ unless players are symmetric in their valuations, i.e., $V_{1,1} = V_{1,2}$. The results of Carbonell-Nicolau and McLean (2018) and He and Yannelis (2015) cannot be applied directly in our context. It is worth highlighting that an immediate detour is available to bypass this problem in the knife-edge case of a single-prize contest with risk-neutral players: We can normalize players' prize valuations to one and refine their effort cost functions to be $x_i/V_{1,i}$, which yields a strategically equivalent game and satisfies the symmetric-valuation condition. However, this transformation loses its bite when a general prize structure is in place. Imagine a case of three players competing for two effective prizes, i.e., $\ell = 2$: Normalizing their valuations for the top prize would not ensure symmetric valuations for the second.

4.2.1. Our approach: construction of alternative ex post contests

We take a detour to circumvent this difficulty. We first construct an alternative ex post contest for each type $\boldsymbol{t} \in T$—denoted by $\widehat{\boldsymbol{\Gamma}}_{\boldsymbol{t}}$—that satisfies aggregate upper semicontinuity. Specifically, we adjust players' conditional winning probability $p_{m,i}$ in Equation (3) for the case in which all players eligible for the mth prize exert zero effort. We then show that the corresponding Bayesian contest $\widehat{\boldsymbol{\Gamma}}$ has a behavioral-strategy equilibrium. Finally, we show that the equilibrium in the constructed Bayesian contest $\widehat{\boldsymbol{\Gamma}}$ is also an equilibrium of the original Bayesian contest $\boldsymbol{\Gamma}$.

Let us spell out more details regarding our construction of the ex post contest $\widehat{\boldsymbol{\Gamma}}_{\boldsymbol{t}}$. Fixing $\boldsymbol{t} \in T$ and an arbitrary effort profile $\boldsymbol{x}$, denote by $\mathcal{N}_+$ the set of active players. We modify the conditional winning probability $p_{m,i}(\boldsymbol{x}; \Omega_m)$ in Equation (3) for all Ω_m's, with $m \in \{1, \ldots, n-1\}$, such that $\Omega_m \cap \mathcal{N}_+ = \emptyset$. The procedure is summarized as follows.

- (i) We begin with the $(n-1)$-th draw: For all Ω_{n-1}'s such that $\Omega_{n-1} \cap \mathcal{N}_+ = \emptyset$, we change (3) in the case of $\sum_{j\in\Omega_{n-1}} f_j(x_j) = 0$ to allocate the $(n-1)$-th prize to the player with the highest payoff $u_i(V_{n-1,i})$ with probability one,[19] while leaving it unchanged for all Ω_{n-1}'s such that $\Omega_{n-1} \cap \mathcal{N}_+ \neq \emptyset$.
- (ii) We then proceed to the $(n-2)$-th draw. Recall the winner-selection mechanism in all possible Ω_{n-1}'s defined above. For all Ω_{n-2}'s such that $\Omega_{n-2} \cap \mathcal{N}_+ = \emptyset$, we change (3) in the case of $\sum_{j\in\Omega_{n-2}} f_j(x_j) = 0$ to allocate the $(n-2)$-th prize to maximize the total expected

[19] In the case where two or more such players exist, we pick the one with the smallest index i.

payoff of those in the set Ω_{n-2} under the winner-selection mechanism defined above. Again, we leave (3) unchanged for all Ω_{n-2}'s such that $\Omega_{n-2} \cap \mathcal{N}_+ \neq \emptyset$.

(iii) The process continues until we reach the first draw.

The following lemmas confirm that the constructed Bayesian contest $\widehat{\mathbf{\Gamma}}$ possesses the properties desirable for our purpose.

Lemma 3. *Suppose that Assumption 5 is satisfied. Then the constructed Bayesian contest $\widehat{\mathbf{\Gamma}}$ is uniformly payoff-secure.*

It should be noted that the uniform equicontinuity on $(u_i(\cdot, \boldsymbol{t}))_{t\in T}$, $(f_i(\cdot, \boldsymbol{t}))_{t\in T}$, and $(c_i(\cdot, \boldsymbol{t}))_{t\in T}$ required by Assumption 5(v) is key to establishing uniform payoff security of the constructed Bayesian contest $\widehat{\mathbf{\Gamma}}$. In Appendix B, we provide an example to demonstrate that Lemma 3 may not hold when the uniform equicontinuity assumption is violated.

Carbonell-Nicolau and McLean (2018) first furnish conditions on the primitives of a Bayesian game with discontinuous payoff functions that guarantee the existence of behavioral-strategy equilibrium. In particular, their Theorem 1 requires uniform payoff security. They illustrate their existence result in a common-value auction setting. Interestingly, uniform equicontinuity on modeling primitives is imposed to obtain uniform payoff security, as we do in Lemma 3.

Lemma 4. *Suppose that Assumption 5 is satisfied. Then the constructed ex post contest $\widehat{\mathbf{\Gamma}}_{\boldsymbol{t}}$ is aggregate upper-semicontinuous.*

Lemmas 3 and 4 enable us to apply Theorem 1 of Carbonell-Nicolau and McLean (2018) and Theorem 2 of He and Yannelis (2015) to establish the following result.

Lemma 5. *Suppose that Assumptions 4 and 5 are satisfied. Then a behavioral-strategy equilibrium exists in the Bayesian game $\widehat{\mathbf{\Gamma}}$. Moreover, if the type space T is countable, then a pure-strategy equilibrium exists in the Bayesian game $\widehat{\mathbf{\Gamma}}$.*

Finally, we obtain the following key result.

Lemma 6. *A behavioral-strategy (pure-strategy, respectively) equilibrium of the Bayesian contest $\widehat{\mathbf{\Gamma}}$ is also a behavioral-strategy (pure-strategy, respectively) equilibrium of the original Bayesian contest $\mathbf{\Gamma}$.*

The proof for Lemma 6 makes use of the fact that a player i's expected payoff $\pi_i(x_i, \boldsymbol{x}_{-i}, \boldsymbol{t})$ in the ex post contest $\mathbf{\Gamma}_{\boldsymbol{t}}$ is lower-semicontinuous at $x_i = 0$. The role of the lower semicontinuity of the expected payoff function will be elucidated in Section 4.2.2 below.

Although the key property of aggregate upper semicontinuity is missing in the original Bayesian contest $\mathbf{\Gamma}$, the constructed Bayesian game $\widehat{\mathbf{\Gamma}}$ and Lemma 6 bridge He and Yannelis (2015) and Carbonell-Nicolau and McLean (2018) to our environment and revive their results in this alternative context. Lemma 6 immediately yields equilibrium existence in the original Bayesian contest $\mathbf{\Gamma}$, which closes the loop.

Theorem 2 *(Equilibrium existence in Bayesian generalized multi-prize nested lottery contests). Suppose that Assumptions 4 and 5 are satisfied. Then every generalized multi-prize nested lottery* *contest with incomplete information* $\mathbf{\Gamma}$ *possesses a behavioral-strategy equilibrium. Furthermore, if the type space* T *is countable, then the Bayesian contest game* $\mathbf{\Gamma}$ *possesses a pure-strategy equilibrium.*

4.2.2. Discussion

It is useful to point out that the proof for Theorem 2—which critically relies on the construction of the modified Bayesian contest $\widehat{\mathbf{\Gamma}}$ and Lemma 6—can also shed light on the proof of Theorem 1 in complete-information settings. We close this section with an example in a complete-information setting to further illuminate the construction of $\widehat{\mathbf{\Gamma}}$, as well as the underlying logic of the proof for Lemma 6.

As in Example 1, we consider a simple three-player contest with $u_i(w_i) = w_i$ and $f_i(x_i) = c_i(x_i) = x_i$ for all $i \in \mathcal{N} \equiv \{1, 2, 3\}$, and set $\boldsymbol{V}_1 = \boldsymbol{V}_2 = (16, 8, 0)$ and $\boldsymbol{V}_3 = (2, 1, 0)$. There are two effective prizes, i.e., $\ell = 2$. Consider an effort profile $\boldsymbol{x} \equiv (x_1, x_2, x_3)$ such that $x_1 > 0$ and $x_2 = x_3 = 0$. It follows immediately that $\mathcal{N}_+ = \{1\}$ and $\Omega_2 = \{2, 3\}$. Because $\Omega_2 \cap \mathcal{N}_+ = \emptyset$ and $V_{2,2} = 8 > 1 = V_{2,3}$, we adjust the winner-selection mechanism in (3) by allocating the second prize to player 2 with certainty by our construction. For an arbitrary effort profile $\boldsymbol{x} \equiv (x_1, x_2, x_3)$, the conditional winning probability (3) can be adjusted in a similar manner. Specifically, in the modified contest $\widehat{\mathbf{\Gamma}}$, we always distribute a prize to a player with the smallest player index i whenever the effort entries for a draw are all zeros. It can be verified that $\boldsymbol{x}^* = (2, 2, 0)$ constitutes a pure-strategy equilibrium of the modified contest $\widehat{\mathbf{\Gamma}}$.

Next, we show that $\boldsymbol{x}^* = (2, 2, 0)$ is also an equilibrium of the original contest game $\mathbf{\Gamma}$. For notational convenience, fixing an effort profile $\boldsymbol{x}$, we denote by $\pi_i^{\mathbf{\Gamma}}(\boldsymbol{x})$ and $\pi_i^{\widehat{\mathbf{\Gamma}}}(\boldsymbol{x})$, respectively, player i's expected payoff, with $i \in \{1, 2, 3\}$, in the contests $\mathbf{\Gamma}$ and $\widehat{\mathbf{\Gamma}}$.

First, note that for player 3 we always have that $\pi_3^{\mathbf{\Gamma}}(2, 2, x_3) = \pi_3^{\widehat{\mathbf{\Gamma}}}(2, 2, x_3)$ for all $x_3 \geq 0$: By definition, player 3 has no incentive to deviate from $x_3 = 0$ in the modified contest $\widehat{\mathbf{\Gamma}}$. Therefore,

$$\pi_3^{\mathbf{\Gamma}}(2, 2, 0) = \pi_3^{\widehat{\mathbf{\Gamma}}}(2, 2, 0) \geq \pi_3^{\widehat{\mathbf{\Gamma}}}(2, 2, x_3) = \pi_3^{\mathbf{\Gamma}}(2, 2, x_3), \forall x_3 \geq 0,$$

which, in turn, implies that he has no incentive to deviate from $x_3 = 0$ in the original contest $\mathbf{\Gamma}$ given that $(x_1, x_2) = (2, 2)$.

Second, we examine player 1's incentive. Note that player 1 has no incentive to deviate to some other $x_1 > 0$ because $\pi_1^{\mathbf{\Gamma}}(x_1, 2, 0) = \pi_1^{\widehat{\mathbf{\Gamma}}}(x_1, 2, 0)$ for all $x_1 > 0$. It remains to show that player 1 cannot be strictly better off by choosing $x_1 = 0$. Note that $\pi_1^{\mathbf{\Gamma}}(x_1, 2, 0)$ is lower-semicontinuous at $x_1 = 0$. Therefore,

$$\pi_1^{\mathbf{\Gamma}}(0, 2, 0) \leq \liminf_{\epsilon \to 0} \pi_1^{\mathbf{\Gamma}}(\epsilon, 2, 0) = \liminf_{\epsilon \to 0} \pi_1^{\widehat{\mathbf{\Gamma}}}(\epsilon, 2, 0) \leq \pi_1^{\widehat{\mathbf{\Gamma}}}(2, 2, 0) = \pi_1^{\mathbf{\Gamma}}(2, 2, 0).$$

We then conclude that player 1 has no incentive to deviate to $x_1 = 0$. The analysis for player 2's incentive is exactly the same as that for player 1 due to symmetry. The above discussion implies that the effort profile $\boldsymbol{x}^* \equiv (2, 2, 0)$ constitutes a pure-strategy equilibrium of the original contest $\mathbf{\Gamma}$.

5. Conclusion

This paper comprehensively examines equilibrium existence in a generalized multi-prize nested lottery contest model. Despite its popular practice of modeling noisy contests that award multiple prizes, the literature has yet to formally explore its game-theoretic fundamentals. To the best of our knowledge, we are the first in the literature to establish equilibrium existence in such a contest game. The technique and results presented in this paper support the numerous studies that rely on the incomplete solutions obtained from this model, and pave the way for future research on noisy multi-prize contests, e.g., optimal design of multi-prize contests. For instance, Fu et al. (2021a) apply the results and approach of the current study to a multi-prize nested lottery contest with risk-averse players and Tullock contest specification $f_i(x_i) = x_i^r$ with $r \in (0, 1]$. They verify the uniqueness of a symmetric equilibrium, which enables the subsequent analysis of optimal prize allocation.

Our analysis of the incomplete-information setting also contributes to the current literature on discontinuous games with incomplete information. The technique we develop in the paper builds a linkage from He and Yannelis (2015) and Carbonell-Nicolau and McLean (2018) to our setting and revives the relevance of their results, despite the missing key property of aggregate upper semicontinuity. The indirect approach could open an avenue for future analysis of equilibrium existence in other forms of discontinuous games that do not immediately meet their requirement.

Significant space remains for future research along this stream. Our paper mainly focuses on equilibrium existence in the game. It would be interesting to identify useful conditions that ensure equilibrium uniqueness, especially when players are heterogeneous.[20] Further, the multi-prize nested lottery contest model has conventionally assumed simultaneous moves. A natural extension is to allow for sequential moves à la Hinnosaar (2021) in this framework.

Appendix A. Additional proofs

Proof of Lemma 2

Proof. We focus on the case of risk-neutral players, i.e., $u_i(w_i) = w_i$ for all $i \in \mathcal{N}$. The analysis for the case of risk-averse players is similar and omitted for brevity. Player i's expected payoff (4) can then be written as

$$\pi_i(\boldsymbol{y}) := \sum_{m=1}^{n} \left[P_{m,i}(\boldsymbol{y}) \times V_{m,i} \right] - h_i(y_i). \tag{8}$$

Suppose that $(\boldsymbol{y}^*, \boldsymbol{\pi}^*)$ is in the closure of the graph of its vector payoff function and $\boldsymbol{y}^*$ is not an equilibrium. By definition of Nash equilibrium, some player i has a profitable deviation given his opponents' effort profile $\boldsymbol{y}^*_{-i}$. Next, we show that there exists $\alpha > \pi_i^*$ and a strategy $\bar{y}_i \geq 0$ such that $\pi_i(\bar{y}_i, \boldsymbol{y}_{-i}) \geq \alpha$ for all $\boldsymbol{y}_{-i}$ in some open neighborhood of $\boldsymbol{y}^*_{-i}$.

Recall that $\ell \leq n-1$ is the number of actual prizes players care about. Fixing $\boldsymbol{y}^*$, denote the set of active players by $\mathcal{N}_+(\boldsymbol{y}^*)$, i.e, $\mathcal{N}_+(\boldsymbol{y}^*) := \{i = 1, \ldots, n \,|\, y_i^* > 0\}$. We consider the following two cases depending on ℓ relative to $|\mathcal{N}_+(\boldsymbol{y}^*)|$.

Case I: $\ell \leq |\mathcal{N}_+(\boldsymbol{y}^*)|$ In this case, the number of effective prizes is less than the number of active players. It follows immediately from Equation (3) that $\pi_i(\boldsymbol{y})$ is continuous at $\boldsymbol{y}^*$. Therefore, all sequences of strategies approaching $\boldsymbol{y}^*$ lead to the same payoff vector, and thus $\pi_i^* = \pi_i(\boldsymbol{y}^*)$ for all $i \in \mathcal{N}$, where $\pi_i(\cdot)$ is defined in Equation (8).

Because $\boldsymbol{y}^*$ is assumed not to be an equilibrium, there exists some player i and an effort level $\bar{y}_i$, that slightly deviates from y_i^*, such that $\pi_i(\bar{y}_i, \boldsymbol{y}^*_{-i}) > \pi_i(\boldsymbol{y}^*) = \pi_i^*$. The continuity of $\pi_i(\bar{y}_i, \cdot)$ implies that there exists an open neighborhood U of $\boldsymbol{y}^*_{-i}$ such that

[20] See Fu et al. (2021b) for further investigation in this direction.

$$\inf_{\boldsymbol{y}_{-i}\in U} \pi_i\left(\bar{y}_i, \boldsymbol{y}_{-i}\right) > \frac{1}{2} \times \left[\pi_i(\bar{y}_i, \boldsymbol{y}^*_{-i}) + \pi^*_i\right].$$

Let $\alpha = [\pi_i(\bar{y}_i, \boldsymbol{y}^*_{-i}) + \pi^*_i]/2$. It follows that $\alpha > \pi^*_i$ and $\pi_i(\bar{y}_i, \boldsymbol{y}_{-i}) > \alpha$ for all $\boldsymbol{y}_{-i} \in U$.

Case II: $\ell > |\mathcal{N}_+(\boldsymbol{y}^*)|$ Consider a sequence of strategies $\left\{\boldsymbol{y}^k\right\}_{k=1}^\infty$ approaching $\boldsymbol{y}^*$ as $k \to \infty$ such that $\pi^*_i = \lim_{k\to\infty} \pi_i(\boldsymbol{y}^k)$. Because $\left\{P_{m,i}(\boldsymbol{y}^k)\right\}_{k=1}^\infty$ is a bounded sequence, we can find a convergent subsequence of $\left\{P_{m,i}(\boldsymbol{y}^k)\right\}_{k=1}^\infty$. Without loss of generality, let us assume that $\left\{P_{m,i}(\boldsymbol{y}^k)\right\}_{k=1}^\infty$ is a convergent sequence for all $m \in \{1, \dots, n\}$ and $i \in \mathcal{N}$, and denote the limit by $P^*_{m,i}$. It follows from Equation (3) that

$$P^*_{m,i} \begin{cases} \geq 0 & \text{, if } i \in \mathcal{N}_+(\boldsymbol{y}^*), m \leq |\mathcal{N}_+(\boldsymbol{y}^*)|, \\ = 0 & \text{, if } i \notin \mathcal{N}_+(\boldsymbol{y}^*), m \leq |\mathcal{N}_+(\boldsymbol{y}^*)|, \\ = 0 & \text{, if } i \in \mathcal{N}_+(\boldsymbol{y}^*), m > |\mathcal{N}_+(\boldsymbol{y}^*)|, \\ \geq 0 & \text{, if } i \notin \mathcal{N}_+(\boldsymbol{y}^*), m > |\mathcal{N}_+(\boldsymbol{y}^*)|, \end{cases} \tag{9}$$

and thus

$$\sum_{i\notin\mathcal{N}_+(\boldsymbol{y}^*)} P^*_{m,i} = \mathbb{1}\left(m > \left|\mathcal{N}_+(\boldsymbol{y}^*)\right|\right), \; 1 < m \leq \ell.$$

The above equation implies that

$$\sum_{i\notin\mathcal{N}_+(\boldsymbol{y}^*)} \sum_{m=1}^{\ell} P^*_{m,i} = \sum_{m=1}^{\ell} \sum_{i\notin\mathcal{N}_+(\boldsymbol{y}^*)} P^*_{m,i} = \ell - \left|\mathcal{N}_+(\boldsymbol{y}^*)\right| < n - \left|\mathcal{N}_+(\boldsymbol{y}^*)\right|,$$

where the strict inequality follows from the fact that $\ell \leq n - 1$. Therefore, there exists some player $i \notin \mathcal{N}_+(\boldsymbol{y}^*)$ (i.e., $y^*_i = 0$) such that

$$\widetilde{P}^*_{\ell,i} := \sum_{m=1}^{\ell} P^*_{m,i} < 1.$$

Consider an effort strategy $\bar{y}_i := \delta$ for some $\delta > 0$ and an open ball with center $\boldsymbol{y}^*_{-i}$ and radius δ^3, which we denote by $B(\boldsymbol{y}^*_{-i}, \delta^3)$. Next, we show that player i can secure a payoff that is strictly greater than π^*_i for all $\boldsymbol{y}_{-i} \in B(\boldsymbol{y}^*_{-i}, \delta^3)$ by deviating from $y^*_i = 0$ to $\bar{y}_i$ when δ is sufficiently small.

Recall that $P_{m,i}$ is player i's ex ante probability of receiving the mth prize. It follows from Equation (3) and $\ell > |\mathcal{N}_+(\boldsymbol{y}^*)|$ that we can find a sufficiently small $\delta > 0$ to satisfy

$$\sum_{m=1}^{|\mathcal{N}_+(\boldsymbol{y}^*)|+1} P_{m,i}(\bar{y}_i, \boldsymbol{y}_{-i}) \geq 1 - \delta^2, \forall\, \boldsymbol{y}_{-i} \in B(\boldsymbol{y}^*_{-i}, \delta^3). \tag{10}$$

For a sufficiently small δ, player i's expected payoff of choosing effort $\bar{y}_i$ can then be bounded below by

$$\pi_i(\bar{y}_i, \boldsymbol{y}_{-i}) = \sum_{m=1}^{n} \left[P_{m,i}(\bar{y}_i, \boldsymbol{y}_{-i}) \times V_{m,i}\right] - h_i(\bar{y}_i)$$

$$
\begin{aligned}
&\geq \left[\sum_{m=1}^{|\mathcal{N}_+(\boldsymbol{y}^*)|+1} P_{m,i}(\bar{y}_i, \boldsymbol{y}'_{-i}) \right] \times V_{|\mathcal{N}_+(\boldsymbol{y}^*)|+1,i} - h_i(\bar{y}_i) \\
&\geq (1-\delta^2) \times V_{|\mathcal{N}_+(\boldsymbol{y}^*)|+1,i} - \delta h'_i(\delta) \\
&> \frac{1+\widetilde{P}^*_{\ell,i}}{2} \times V_{|\mathcal{N}_+(\boldsymbol{y}^*)|+1,i}, \forall\, \boldsymbol{y}_{-i} \in B(\boldsymbol{y}^*_{-i}, \delta^3),
\end{aligned} \tag{11}
$$

where the first inequality follows from $V_{1,i} \geq \cdots \geq V_{n,i}$; the second inequality follows from (10) and the convexity of $h_i(\cdot)$; and the third inequality holds when δ is sufficiently small.

Furthermore, π_i^* can be bounded from above by

$$
\begin{aligned}
\pi_i^* \equiv \sum_{m=1}^{n} \left[P^*_{m,i} \times V_{m,i}\right] - h_i(y_i^*) = & \sum_{m=|\mathcal{N}_+(\boldsymbol{y}^*)|+1}^{\ell} \left[P^*_{m,i} \times V_{m,i}\right] \\
\leq & \left[\sum_{m=|\mathcal{N}_+(\boldsymbol{y}^*)|+1}^{\ell} P^*_{m,i} \right] \times V_{|\mathcal{N}_+(\boldsymbol{y}^*)|+1,i} \\
\leq & \widetilde{P}^*_{\ell,i} \times V_{|\mathcal{N}_+(\boldsymbol{y}^*)|+1,i} \\
< & \frac{1+\widetilde{P}^*_{\ell,i}}{2} \times V_{|\mathcal{N}_+(\boldsymbol{y}^*)|+1,i},
\end{aligned} \tag{12}
$$

where the second equality follows from Equation (9), $y_i^* = 0$, and $h_i(0) = 0$; the first inequality follows from $V_{1,i} \geq \cdots \geq V_{n,i}$; and the last inequality follows from $\widetilde{P}^*_{\ell,i} < 1$.

Let $\alpha = \frac{1+\widetilde{P}^*_{\ell,i}}{2} \times V_{|\mathcal{N}_+(\boldsymbol{y}^*)|+1,i}$. It follows immediately from (11) and (12) that $\pi_i(\bar{y}_i, \boldsymbol{y}_{-i}) > \alpha > \pi_i^*$ for all $\boldsymbol{y}_{-i} \in B(\boldsymbol{y}^*_{-i}, \delta^3)$. This concludes the proof. ■

Proof of Lemma 3

Proof. Consider a player $i \in \mathcal{N}$ with type $t_i \in T_i$. Fix a strategy profile $\boldsymbol{x} \equiv (x_1, \ldots, x_n)$ and denote the set of active players by $\mathcal{N}_+$. In what follows, we use the superscript $\widehat{\boldsymbol{\Gamma}}$ to refer to the variables in the modified Bayesian contest $\widehat{\boldsymbol{\Gamma}}$ with slight abuse of notation. For instance, we denote player i's expected payoff and probability of winning the mth prize given an effort profile $\boldsymbol{x}$ in the ex post contest $\widehat{\boldsymbol{\Gamma}}_{\boldsymbol{t}}$ by $\pi_i^{\widehat{\boldsymbol{\Gamma}}}(\boldsymbol{x}, \boldsymbol{t})$ and $P^{\widehat{\boldsymbol{\Gamma}}}_{m,i}(\boldsymbol{x}, \boldsymbol{t})$, respectively. We consider the following two cases.

Case I: $i \in \mathcal{N}_+$ Then $x_i > 0$. Note that $(u_i(\cdot, \boldsymbol{t}))_{\boldsymbol{t}\in T}$, $(f_i(\cdot, \boldsymbol{t}))_{\boldsymbol{t}\in T}$, and $(c_i(\cdot, \boldsymbol{t}))_{\boldsymbol{t}\in T}$ are all uniformly equicontinuous by part (v) of Assumption 5. It can then be verified that for each $x_i > 0$, $\left(P^{\widehat{\boldsymbol{\Gamma}}}_{m,i}(x_i, \cdot, \boldsymbol{t})\right)_{\boldsymbol{t}\in T}$ is uniformly equicontinuous on $\mathcal{X}^{n-1}$. As a result, for each $x_i > 0$, $\left(\pi_i^{\widehat{\boldsymbol{\Gamma}}}(x_i, \cdot, \boldsymbol{t})\right)_{\boldsymbol{t}\in T}$ is uniformly equicontinuous on $\mathcal{X}^{n-1}$.

Set $\bar{x}_i = x_i$. It follows immediately that for each $x_i > 0$, $\epsilon > 0$, and all $(\boldsymbol{t}, \boldsymbol{x}_{-i}) \in T \times \mathcal{X}^{n-1}$, there exists a neighborhood $U_{\boldsymbol{x}_{-i}}$ of $\boldsymbol{x}_{-i}$ such that

$$
\pi_i^{\widehat{\boldsymbol{\Gamma}}}(\bar{x}_i, \boldsymbol{x}'_{-i}, \boldsymbol{t}) \geq \pi_i^{\widehat{\boldsymbol{\Gamma}}}(x_i, \boldsymbol{x}_{-i}, \boldsymbol{t}) - \epsilon, \forall\, \boldsymbol{x}'_{-i} \in U_{\boldsymbol{x}_{-i}}.
$$

Case II: $i \notin \mathcal{N}_+$ Then $x_i = 0$. Fix $\epsilon > 0$, the uniform equicontinuity of $u_i(\cdot,\cdot)$ implies that there exists $\delta_1 \in (0,1)$ such that

$$u_i\left(V_{m,i}(\boldsymbol{t}) - c, \boldsymbol{t}\right) > u_i\left(V_{m,i}(\boldsymbol{t}), \boldsymbol{t}\right) - \frac{\epsilon}{2}, \forall c \in (0,\delta_1), m \in \{1,\ldots,n\}, \boldsymbol{t} \in T. \tag{13}$$

By part (ii) of Assumption 5, there exists $\delta_2 > 0$ such that

$$\delta_2 \times \left[u_i\left(V_{1,i}(\boldsymbol{t}), \boldsymbol{t}\right) - u_i\left(V_{n,i}(\boldsymbol{t}), \boldsymbol{t}\right)\right] < \frac{\epsilon}{2}, \forall \boldsymbol{t} \in T. \tag{14}$$

Moreover, because $c_i(0,\boldsymbol{t}) = 0$ by part (iv) of Assumption 5 and $c_i(\cdot,\boldsymbol{t})$ is uniformly equicontinuous by part (v) of Assumption 5, there exists $\delta_3 > 0$ such that

$$c_i(\delta_3, \boldsymbol{t}) < \delta_1, \forall \boldsymbol{t} \in T. \tag{15}$$

Next, note that $\inf_{\boldsymbol{t}\in T} f_i(\delta_3,\boldsymbol{t}) > 0$ by part (iii) of Assumption 5 and $f_i(\cdot,\boldsymbol{t})$ is uniformly equicontinuous by part (v) of Assumption 5. Therefore, there exists $\delta_4 > 0$ such that

$$\frac{f_i\left(\delta_3,\boldsymbol{t}\right)}{f_i\left(\delta_3,\boldsymbol{t}\right) + \sum_{j\notin\mathcal{N}_+, j\neq i} f_j\left(\delta_4,\boldsymbol{t}\right)} \geq 1 - \delta_2, \forall \boldsymbol{t} \in T. \tag{16}$$

Fixing an arbitrary $\boldsymbol{x}_{-i} \in \mathcal{X}^{n-1}$, set $\bar{x}_i = \delta_3$ and $U_{\boldsymbol{x}_{-i}} = \{\boldsymbol{x}'_{-i} : \forall j \neq i, |x'_j - x_j| < \delta_4, x'_j \geq 0\}$. It suffices to show that for all $\boldsymbol{t} \in T$, we have that

$$\pi_i^{\widehat{\boldsymbol{\Gamma}}}\left(\bar{x}_i, \boldsymbol{x}'_{-i}, \boldsymbol{t}\right) \geq \pi_i^{\widehat{\boldsymbol{\Gamma}}}\left(x_i, \boldsymbol{x}_{-i}, \boldsymbol{t}\right) - \epsilon, \forall \boldsymbol{x}'_{-i} \in U_{\boldsymbol{x}_{-i}}.$$

Note that $i \notin \mathcal{N}_+$. Therefore, player i cannot win the first $|\mathcal{N}_+|$ prizes at $\boldsymbol{x}$, which in turn indicates that

$$\pi_i^{\widehat{\boldsymbol{\Gamma}}}\left(x_i, \boldsymbol{x}_{-i}, \boldsymbol{t}\right) \leq u_i\left(V_{|\mathcal{N}_+|+1,i}(\boldsymbol{t}), \boldsymbol{t}\right), \forall \boldsymbol{t} \in T. \tag{17}$$

Denote player i's probability of receiving the first m prizes given an effort profile $\boldsymbol{x}$ in the ex post contest $\widehat{\boldsymbol{\Gamma}}_{\boldsymbol{t}}$ by $\widetilde{P}^{\widehat{\boldsymbol{\Gamma}}}_{m,i}(\boldsymbol{x},\boldsymbol{t})$. It follows from (16) that

$$\widetilde{P}^{\widehat{\boldsymbol{\Gamma}}}_{|\mathcal{N}_+|+1,i}\left(\bar{x}_i, \boldsymbol{x}'_{-i}, \boldsymbol{t}\right) \geq \frac{f_i\left(\bar{x}_i,\boldsymbol{t}\right)}{f_i\left(\bar{x}_i,\boldsymbol{t}\right) + \sum_{j\neq i, j\notin\mathcal{N}_+} f_j\left(\delta_4,\boldsymbol{t}\right)} \geq 1 - \delta_2, \forall \boldsymbol{x}'_{-i} \in U_{\boldsymbol{x}_{-i}}, \boldsymbol{t} \in T. \tag{18}$$

Therefore, for all $\boldsymbol{t} \in T$, we can obtain that

$$\begin{aligned}
\pi_i^{\widehat{\boldsymbol{\Gamma}}}\left(\bar{x}_i, \boldsymbol{x}'_{-i}, \boldsymbol{t}\right) \geq{}& \widetilde{P}^{\widehat{\boldsymbol{\Gamma}}}_{|\mathcal{N}_+|+1,i}\left(\bar{x}_i, \boldsymbol{x}'_{-i}, \boldsymbol{t}\right) u_i\left(V_{|\mathcal{N}_+|+1,i}(\boldsymbol{t}) - c_i(\bar{x}_i,\boldsymbol{t}), \boldsymbol{t}\right) \\
&+ \left[1 - \widetilde{P}^{\widehat{\boldsymbol{\Gamma}}}_{|\mathcal{N}_+|+1,i}\left(\bar{x}_i, \boldsymbol{x}'_{-i}, \boldsymbol{t}\right)\right] u_i\left(V_{n,i}(\boldsymbol{t}) - c_i(\bar{x}_i,\boldsymbol{t}), \boldsymbol{t}\right) \\
\geq{}& (1-\delta_2) u_i\left(V_{|\mathcal{N}_+|+1,i}(\boldsymbol{t}) - c_i(\bar{x}_i,\boldsymbol{t}), \boldsymbol{t}\right) + \delta_2 u_i\left(V_{n,i}(\boldsymbol{t}) - c_i(\bar{x}_i,\boldsymbol{t}), \boldsymbol{t}\right) \\
\geq{}& (1-\delta_2) u_i\left(V_{|\mathcal{N}_+|+1,i}(\boldsymbol{t}) - \delta_1, \boldsymbol{t}\right) + \delta_2 u_i\left(V_{n,i}(\boldsymbol{t}) - \delta_1, \boldsymbol{t}\right) \\
\geq{}& (1-\delta_2) u_i\left(V_{|\mathcal{N}_+|+1,i}(\boldsymbol{t}), \boldsymbol{t}\right) + \delta_2 u_i\left(V_{n,i}(\boldsymbol{t}), \boldsymbol{t}\right) - \frac{\epsilon}{2} \\
={}& u_i\left(V_{|\mathcal{N}_+|+1,i}(\boldsymbol{t}), \boldsymbol{t}\right) - \delta_2 \times \left[u_i\left(V_{|\mathcal{N}_+|+1,i}(\boldsymbol{t}), \boldsymbol{t}\right) - u_i\left(V_{n,i}(\boldsymbol{t}), \boldsymbol{t}\right)\right] - \frac{\epsilon}{2} \\
\geq{}& u_i\left(V_{|\mathcal{N}_+|+1,i}(\boldsymbol{t}), \boldsymbol{t}\right) - \delta_2 \times \left[u_i\left(V_{1,i}(\boldsymbol{t}), \boldsymbol{t}\right) - u_i\left(V_{n,i}(\boldsymbol{t}), \boldsymbol{t}\right)\right] - \frac{\epsilon}{2} \\
\geq{}& u_i\left(V_{|\mathcal{N}_+|+1,i}(\boldsymbol{t}), \boldsymbol{t}\right) - \epsilon \\
\geq{}& \pi_i^{\widehat{\boldsymbol{\Gamma}}}\left(x_i, \boldsymbol{x}_{-i}, \boldsymbol{t}\right) - \epsilon, \forall \boldsymbol{x}'_{-i} \in U_{\boldsymbol{x}_{-i}},
\end{aligned}$$

where the first inequality follows from the definition of $\pi_i^{\widehat{\mathbf{\Gamma}}}(\boldsymbol{x},\boldsymbol{t})$ and the fact that $V_{1,i}(\boldsymbol{t}) \geq \cdots \geq V_{n,i}(\boldsymbol{t})$; the second inequality follows from (18); the third inequality follows from $\bar{x}_i = \delta_3$ and (15); the fourth inequality follows from (13); the fifth inequality follows from the fact that $V_{1,i}(\boldsymbol{t}) \geq V_{|\mathcal{N}_+|+1,i}(\boldsymbol{t})$; the sixth inequality follows from (14); and the last inequality follows from (17). This completes the proof. ■

Proof of Lemma 4

Proof. Fixing $\boldsymbol{t} \in T$ and $\boldsymbol{x} \in \mathcal{X}^n$, denote player i's expected payoff in the modified Bayesian contest $\widehat{\mathbf{\Gamma}}$ by $\pi_i^{\widehat{\mathbf{\Gamma}}}(\boldsymbol{x},\boldsymbol{t})$ for all $i \in \mathcal{N}$. We have that

$$\sum_{i\in\mathcal{N}} \pi_i^{\widehat{\mathbf{\Gamma}}}(\boldsymbol{x},\boldsymbol{t}) = \sum_{i\in\mathcal{N}_+} \pi_i^{\widehat{\mathbf{\Gamma}}}(\boldsymbol{x},\boldsymbol{t}) + \sum_{i\in\mathcal{N}\setminus\mathcal{N}_+} \pi_i^{\widehat{\mathbf{\Gamma}}}(\boldsymbol{x},\boldsymbol{t}).$$

Clearly, $\sum_{i\in\mathcal{N}_+} \pi_i^{\widehat{\mathbf{\Gamma}}}(\boldsymbol{x},\boldsymbol{t})$ is continuous at $\boldsymbol{x}$ because $x_i > 0$ for all $i \in \mathcal{N}_+$. Moreover, it is evident that $\sum_{i\in\mathcal{N}\setminus\mathcal{N}_+} \pi_i^{\widehat{\mathbf{\Gamma}}}(\boldsymbol{x},\boldsymbol{t})$ is upper-semicontinuous due to the construction of the Bayesian contest $\widehat{\mathbf{\Gamma}}$. Therefore, the expected aggregate payoff is upper-semicontinuous at $\boldsymbol{x}$ for all $\boldsymbol{t} \in T$ and $\boldsymbol{x} \in \mathcal{X}^n$. This concludes the proof. ■

Proof of Lemma 5

Proof. Note that uniform payoff security implies finite/finite* payoff security. Lemma 5 follows directly from Lemma 1, Lemma 3, Lemma 4, Theorem 1 in Carbonell-Nicolau and McLean (2018), and Theorem 2 in He and Yannelis (2015). This concludes the proof. ■

Proof of Lemma 6

Proof. We prove the lemma for behavioral-strategy equilibria. The analysis is similar if we consider pure-strategy equilibria. Denote the behavioral-strategy equilibrium in the Bayesian contest $\widehat{\mathbf{\Gamma}}$ by $\boldsymbol{\sigma}^*$. Let $\mathcal{A}^0_{\mathcal{N}_+} := \{\boldsymbol{x} : \text{The set of active players is } \mathcal{N}_+\}$ and $\mathcal{A}^0 := \cup_{|\mathcal{N}_+|<\ell}\mathcal{A}^0_{\mathcal{N}_+}$. The proof consists of two steps. In the first step, we show that in the equilibrium $\boldsymbol{\sigma}^*$, the probability that fewer than ℓ players remain active is zero, i.e., $\Pr(\mathcal{A}^0) = 0$. In the second step, we show that each behavioral-strategy equilibrium $\boldsymbol{\sigma}^*$ of $\widehat{\mathbf{\Gamma}}$ with $\Pr(\mathcal{A}^0) = 0$ is also a behavioral-strategy equilibrium of the original Bayesian contest $\mathbf{\Gamma}$.

Step I Suppose to the contrary that $\Pr(\mathcal{A}^0) > 0$. Then there exists $|\mathcal{N}_+| < \ell$ such that $\Pr(\mathcal{A}^0_{\mathcal{N}_+}) > 0$. Let $T^0 := \{\boldsymbol{t} \in T : \sigma_i^*(\{0\}|t_i) > 0, \forall i \notin \mathcal{N}_+\}$. Then $\Pr(\mathcal{A}^0) > 0$ implies that $\lambda(T^0) > 0$.

With slight abuse of notation, we denote player i's probability of winning the mth prize in the ex post contest $\widehat{\mathbf{\Gamma}}_{\boldsymbol{t}}$ by $P_{m,i}^{\widehat{\mathbf{\Gamma}}}(\boldsymbol{x},\boldsymbol{t})$. First, note that

$$\int\limits_{\boldsymbol{t}\in T^0} \int\limits_{x_j\geq 0, j\in\mathcal{N}_+} \int\limits_{x_i=0, i\notin\mathcal{N}_+} \sum_{s\notin\mathcal{N}_+} \sum_{m=|\mathcal{N}_+|}^{\ell} P_{m,s}^{\widehat{\mathbf{\Gamma}}}(\boldsymbol{x},\boldsymbol{t})\sigma_i^*(dx_i|t_i)\sigma_j^*(dx_j|t_j)\lambda(d\boldsymbol{t})$$

$$= (\ell - |\mathcal{N}_+|) \times \int\limits_{\boldsymbol{t}\in T^0} \int\limits_{x_j\geq 0, j\in\mathcal{N}_+} \int\limits_{x_i=0, i\notin\mathcal{N}_+} 1\sigma_i^*(dx_i|t_i)\sigma_j^*(dx_j|t_j)\lambda(d\boldsymbol{t})$$

$$< (n - |\mathcal{N}_+|) \times \int_{\boldsymbol{t}\in T^0} \int_{x_j \geq 0, j\in\mathcal{N}_+} \int_{x_i=0, i\notin\mathcal{N}_+} 1\sigma_i^*(dx_i|t_i)\sigma_j^*(dx_j|t_j)\lambda(d\boldsymbol{t}).$$

Therefore, there exists some player $s \notin \mathcal{N}_+$ such that

$$\int_{\boldsymbol{t}\in T^0} \int_{x_j \geq 0, j\in\mathcal{N}_+} \int_{x_i=0, i\notin\mathcal{N}_+} \sum_{m=|\mathcal{N}_+|+1}^{\ell} P_{m,s}^{\widehat{\boldsymbol{\Gamma}}}(\boldsymbol{x},\boldsymbol{t})\sigma_i^*(dx_i|t_i)\sigma_j^*(dx_j|t_j)\lambda(d\boldsymbol{t})$$
$$< \int_{\boldsymbol{t}\in T^0} \int_{x_j \geq 0, j\in\mathcal{N}_+} \int_{x_i=0, i\notin\mathcal{N}_+} 1\sigma_i^*(dx_i|t_i)\sigma_j^*(dx_j|t_j)\lambda(d\boldsymbol{t}).$$

Take $\epsilon > 0$ such that

$$\epsilon < \int_{\boldsymbol{t}\in T^0} \int_{x_j \geq 0, j\in\mathcal{N}_+} \int_{x_i=0, i\notin\mathcal{N}_+} \left\{ \begin{array}{c} \left[1 - \sum_{m=|\mathcal{N}_+|+1}^{\ell} P_{m,s}^{\widehat{\boldsymbol{\Gamma}}}(\boldsymbol{x},\boldsymbol{t})\right] \\ \times \left[u_s\left(V_{|\mathcal{N}_+|+1,s}(\boldsymbol{t}),\boldsymbol{t}\right) - u_s\left(V_{\ell+1,s}(\boldsymbol{t}),\boldsymbol{t}\right)\right] \end{array} \right\}$$
$$\times \sigma_i^*(dx_i|t_i)\sigma_j^*(dx_j|t_j)\lambda(d\boldsymbol{t}). \tag{19}$$

Because $(u_s(\cdot,\boldsymbol{t}))_{\boldsymbol{t}\in T}$ and $(c_s(\cdot,\boldsymbol{t}))_{\boldsymbol{t}\in T}$ are uniformly equicontinuous, there exists $\delta_5 > 0$ such that for all $\boldsymbol{t} \in T$ and $m \in \{1,\ldots,n\}$,

$$u_s\left(V_{m,s}(\boldsymbol{t}) - c_s(x_s,\boldsymbol{t}),\boldsymbol{t}\right) > u_s\left(V_{m,s}(\boldsymbol{t}),\boldsymbol{t}\right) - \frac{1}{4}\epsilon, \forall x_s \in [0,\delta_5]. \tag{20}$$

Define an alternative behavioral strategy σ_s' for player s as

$$\sigma_s'(A|t_i) = \begin{cases} \sigma_s^*(A|t_i), & \text{for } A\cap\{0,\delta_5\} = \emptyset \text{ or } \{0,\delta_5\}\subset A, \\ \sigma_s^*(A|t_i) - \sigma_s^*(\{0\}|t_i), & \text{for } A\cap\{0,\delta_5\} = \{0\}, \\ \sigma_s^*(A|t_i) + \sigma_s^*(\{0\}|t_i), & \text{for } A\cap\{0,\delta_5\} = \{\delta_5\}. \end{cases} \tag{21}$$

In words, $\sigma_s'(\cdot|t_i)$ corresponds to the equilibrium strategy $\sigma_s^*(\cdot|t_i)$ for all $t_i \in T_i$, except that player s chooses δ_5 instead of 0 with positive probability. Combining (20) and (21) yields

$$\int \pi_s^{\widehat{\boldsymbol{\Gamma}}}(x_s,\boldsymbol{x}_{-s},\boldsymbol{t})\sigma_s^*(dx_s|t_s) \leq \int \left(\pi_s^{\widehat{\boldsymbol{\Gamma}}}(x_s,\boldsymbol{x}_{-s},\boldsymbol{t}) + \frac{\epsilon}{4}\right)\sigma_s'(dx_s|t_s), \forall \boldsymbol{t}\in T, \boldsymbol{x}_{-s}\in\mathcal{X}^{n-1}.$$

The above inequality, together with (19), implies that

$$\int_{\boldsymbol{t}\in T_0} \int_{x_j \geq 0, j\in\mathcal{N}_+} \int_{x_i=0, i\notin\mathcal{N}_+, i\neq s} \int_{x_s=0} \pi_s^{\widehat{\boldsymbol{\Gamma}}}(x_s,\boldsymbol{x}_{-s},\boldsymbol{t})\sigma_s^*(dx_s|t_s)\sigma_i^*(dx_i|t_i)\sigma_j^*(dx_j|t_j)\lambda(d\boldsymbol{t})$$
$$\leq \int_{\boldsymbol{t}\in T_0} \int_{x_j \geq 0, j\in\mathcal{N}_+} \int_{x_i=0, i\notin\mathcal{N}_+, i\neq s} \int_{x_s=\delta_5} \pi_s^{\widehat{\boldsymbol{\Gamma}}}(x_s,\boldsymbol{x}_{-s},\boldsymbol{t})\sigma_s'(dx_s|t_s)\sigma_i^*(dx_i|t_i)\sigma_j^*(dx_j|t_j)\lambda(d\boldsymbol{t})$$
$$-\frac{3}{4}\epsilon. \tag{22}$$

Next, let us define

$$\Omega^* := \left\{(\boldsymbol{x},\boldsymbol{t}) : \boldsymbol{t}\in T_0; x_j \geq 0, j\in\mathcal{N}_+; x_i = 0, i\notin\mathcal{N}_+\right\},$$

and

$$\Omega' := \left\{(\boldsymbol{x},\boldsymbol{t}) : \boldsymbol{t} \in T_0; x_j \geq 0, j \in \mathcal{N}_+; x_i = 0, i \notin \mathcal{N}_+, i \neq s; x_s = \delta_5\right\}.$$

It follows from (20) and (22) that

$$\begin{aligned}
\Pi_s^{\widehat{\boldsymbol{\Gamma}}}(\sigma_s^*, \boldsymbol{\sigma}_{-s}^*) &= \int \pi_s^{\widehat{\boldsymbol{\Gamma}}}(\boldsymbol{x},\boldsymbol{t})\boldsymbol{\sigma}^*(d\boldsymbol{x}|\boldsymbol{t})\lambda(d\boldsymbol{t}) \\
&= \int_{\Omega^*} \pi_s^{\widehat{\boldsymbol{\Gamma}}}(\boldsymbol{x},\boldsymbol{t})\boldsymbol{\sigma}^*(d\boldsymbol{x}|\boldsymbol{t})\lambda(d\boldsymbol{t}) + \int_{\Omega^{*c}} \pi_s^{\widehat{\boldsymbol{\Gamma}}}(\boldsymbol{x},\boldsymbol{t})\boldsymbol{\sigma}^*(d\boldsymbol{x}|\boldsymbol{t})\lambda(d\boldsymbol{t}) \\
&\leq \int_{\Omega'} \pi_s^{\widehat{\boldsymbol{\Gamma}}}(\boldsymbol{x},\boldsymbol{t})\sigma_s'(dx_s|t_s)\boldsymbol{\sigma}_{-s}^*(d\boldsymbol{x}_{-s}|\boldsymbol{t}_{-s})\lambda(d\boldsymbol{t}) - \frac{3}{4}\epsilon \\
&\quad + \int_{\Omega'^c} \pi_s^{\widehat{\boldsymbol{\Gamma}}}(\boldsymbol{x},\boldsymbol{t})\sigma_s'(dx_s|t_s)\boldsymbol{\sigma}_{-s}^*(d\boldsymbol{x}_{-s}|\boldsymbol{t}_{-s})\lambda(d\boldsymbol{t}) + \frac{1}{4}\epsilon \\
&= \int \pi_s^{\widehat{\boldsymbol{\Gamma}}}(\boldsymbol{x},\boldsymbol{t})\sigma_s'(dx_s|\boldsymbol{t}_s)\boldsymbol{\sigma}_{-s}^*(d\boldsymbol{x}_{-s}|\boldsymbol{t}_{-s})\lambda(d\boldsymbol{t}) - \frac{1}{2}\epsilon \\
&< \int \pi_s^{\widehat{\boldsymbol{\Gamma}}}(\boldsymbol{x},\boldsymbol{t})\sigma_s'(dx_s|t_s)\boldsymbol{\sigma}_{-s}^*(d\boldsymbol{x}_{-s}|\boldsymbol{t}_{-s})\lambda(d\boldsymbol{t}) = \Pi_s^{\widehat{\boldsymbol{\Gamma}}}(\sigma_s', \boldsymbol{\sigma}_{-s}^*),
\end{aligned}$$

which contradicts the fact that $\boldsymbol{\sigma}^*$ is a behavioral-strategy equilibrium of the Bayesian contest $\widehat{\boldsymbol{\Gamma}}$. As a result, we must have that $\Pr(\mathcal{A}^0) = 0$.

Step II Next, we show that $\boldsymbol{\sigma}^*$ is an equilibrium of the original Bayesian contest $\boldsymbol{\Gamma}$. Fixing an arbitrary behavioral-strategy profile $\boldsymbol{\sigma}$, let us denote player i's expected payoff in the original Bayesian contest $\boldsymbol{\Gamma}$ and that in the modified contest $\widehat{\boldsymbol{\Gamma}}$ by $\Pi_i^{\boldsymbol{\Gamma}}(\boldsymbol{\sigma})$ and $\Pi_i^{\widehat{\boldsymbol{\Gamma}}}(\boldsymbol{\sigma})$, respectively.

First, note that player i's expected payoff in $\widehat{\boldsymbol{\Gamma}}$ is equal to that in $\boldsymbol{\Gamma}$ on $(\mathcal{A}^0)^c$, which occurs with probability 1. Therefore, we have that

$$\Pi_i^{\widehat{\boldsymbol{\Gamma}}}\left(\boldsymbol{\sigma}^*\right) = \Pi_i^{\boldsymbol{\Gamma}}\left(\boldsymbol{\sigma}^*\right), \forall i \in \mathcal{N}. \tag{23}$$

It remains to show that $\Pi_i^{\boldsymbol{\Gamma}}(\sigma_i', \boldsymbol{\sigma}_{-i}^*) \leq \Pi_i^{\boldsymbol{\Gamma}}(\sigma_i^*, \boldsymbol{\sigma}_{-i}^*)$ for all $i \in \mathcal{N}$ and $\sigma_i' \in \Sigma_i$. For each $\epsilon > 0$, let us define a strategy σ_i^ϵ such that

$$\sigma_i^\epsilon\left(A \,|\, t_i\right) = \sigma_i'\left(A - \epsilon \,|\, t_i\right),$$

where $A - \epsilon := \{x - \epsilon : x \in A\}$. In words, we shift player i's conditional bidding distribution for every type $t_i \in T_i$ to the right by ϵ. It follows that σ_i^ϵ converges weak-star to σ_i' and $\sigma_i^\epsilon(\{0\}|A) = 0$ for all $\epsilon > 0$ and $t_i \in T_i$. Note that $\pi_i(x_i, \boldsymbol{x}_{-i}, \boldsymbol{t})$ is lower-semicontinuous at $x_i = 0$. Therefore, we have that

$$\Pi_i^{\boldsymbol{\Gamma}}(\sigma_i', \boldsymbol{\sigma}_{-i}^*) \leq \liminf_{\epsilon \to 0} \Pi_i^{\boldsymbol{\Gamma}}(\sigma_i^\epsilon, \boldsymbol{\sigma}_{-i}^*) = \liminf_{\epsilon \to 0} \Pi_i^{\widehat{\boldsymbol{\Gamma}}}(\sigma_i^\epsilon, \boldsymbol{\sigma}_{-i}^*) \leq \Pi_i^{\widehat{\boldsymbol{\Gamma}}}(\boldsymbol{\sigma}^*) = \Pi_i^{\boldsymbol{\Gamma}}(\boldsymbol{\sigma}^*),$$

where the first inequality follows from weak* convergence of σ_i^ϵ and the lower semicontinuity of $\pi_i(x_i, \boldsymbol{x}_{-i}, \boldsymbol{t})$ at $x_i = 0$; the first equality follows from the fact that $\Pr(\mathcal{A}^0) = 0$ for all $(\sigma_i^\epsilon, \boldsymbol{\sigma}_{-i}^*)$; the second inequality follows from the fact that $\boldsymbol{\sigma}^*$ is an equilibrium of $\widehat{\boldsymbol{\Gamma}}$; and the last equality follows from (23). This implies that $\boldsymbol{\sigma}^*$ also constitutes an equilibrium of the original Bayesian contest $\boldsymbol{\Gamma}$ and concludes the proof. ■

Appendix B. Uniform equicontinuity on modeling primitives and uniform payoff security

The following example demonstrates that the uniform payoff security established in Lemma 3 may not hold when the uniform equicontinuity assumption required by Assumption 5(v) is violated.

Example 2. Consider a single-prize contest with two ex ante identical players. Let the type space be $T_i = \mathbb{R}_+$. Set $\boldsymbol{V}_1(\boldsymbol{t}) = \boldsymbol{V}_2(\boldsymbol{t}) = (1, 0)$, $u_i(w, \boldsymbol{t}) = w$, $f_i(x, \boldsymbol{t}) = x$, and $c_i(x, \boldsymbol{t}) = t_i x$. Players' types t_1 and t_2 are independent and identically distributed according to some unbounded continuous distribution function, e.g., the exponential distribution $H(t) = 1 - e^{-\beta t}$, with $\beta > 0$. According to the construction of the Bayesian contest $\widehat{\boldsymbol{\Gamma}}$ as described in Section 4.2.1, player 1 receives the first (unique) prize with probability one in the case of $(x_1, x_2) = (0, 0)$ for all type profiles, i.e., $p_{1,1}(0, 0, \boldsymbol{t}) = 1$ for all $\boldsymbol{t} \in T$.

It is straightforward to verify that parts (i) through (iv) of Assumption 5 are satisfied, whereas part (v) is violated. In particular, the family of functions $(c_i(\cdot, \boldsymbol{t}))_{\boldsymbol{t} \in T}$ is not uniformly equicontinuous.

Next, we show that the above two-player Bayesian contest $\widehat{\boldsymbol{\Gamma}}$ does not satisfy uniform payoff security. Suppose to the contrary that uniform payoff security holds. Consider player 1 and fix $x_1 = 0$. By Definition 2, for every $\epsilon > 0$, there exists $\bar{x}_1 \in \mathcal{X}$, which leads to the following: For all $(\boldsymbol{t}, x_2) \in \mathbb{R}_+^2 \times \mathcal{X}$, there exists a neighborhood U_{x_2} of x_2 such that

$$p_{1,1}(\bar{x}_1, x_2', \boldsymbol{t}) - t_1 \bar{x}_1 \geq p_{1,1}(0, x_2, \boldsymbol{t}) - \epsilon, \forall x_2' \in U_{x_2}. \tag{24}$$

The above condition indicates that

$$t_1 \bar{x}_1 \leq p_{1,1}(\bar{x}_1, x_2', \boldsymbol{t}) - p_{1,1}(0, x_2, \boldsymbol{t}) + \epsilon \leq 1 + \epsilon, \forall t_1 \in T_1.$$

Note that $T_1 = \mathbb{R}_+$ by assumption. As a result, we must have $\bar{x}_1 = 0$.

Next, fix a type profile (t_1, t_2), $\epsilon \in (0, 1)$, and $x_2 = 0$. Pick $x_2' > 0$ in the neighborhood U_{x_2} of x_2. It is obvious that $p_{1,1}(\bar{x}_1, x_2', \boldsymbol{t}) = p_{1,1}(0, x_2', \boldsymbol{t}) = 0$ and (24) can be simplified as

$$0 = p_{1,1}(\bar{x}_1, x_2', \boldsymbol{t}) - t_1 \bar{x}_1 \geq p_{1,1}(0, x_2, \boldsymbol{t}) - \epsilon = p_{1,1}(0, 0, \boldsymbol{t}) - \epsilon = 1 - \epsilon > 0,$$

which is a contradiction. Therefore, the constructed two-player Bayesian contest $\widehat{\boldsymbol{\Gamma}}$ is not uniformly payoff-secure.

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