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Multi-prize contests with risk-averse players ☆

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ABSTRACT

This paper studies a multi-prize imperfectly discriminatory contest with symmetric risk-averse contestants. Adopting a multi-winner nested Tullock contest model, we first establish the existence and uniqueness of a symmetric pure-strategy equilibrium under plausible conditions. We then investigate the optimal prize allocation in the contest. Our analysis provides a formal account of the incentive effects triggered by a variation in the prevailing prize structure when contestants are risk averse. We demonstrate that contestants' incentive subtly depends not only on the degree of a contestant's risk aversion but also that of his prudence. The former affects the marginal benefit of effort, while the latter affects the marginal cost. We derive sufficient conditions under which a single-(multi-)prize contest would be optimal when the contest designer aims to maximize total effort. We also discuss in depth the roles played by risk aversion and prudence in optimal prize allocation.

1. Introduction

Many competitive activities resemble a contest. Interest groups lobby to influence policies; litigants engage in legal battles for favorable court rulings; firms invest in R&D to secure market leadership; workers climb a firm's hierarchical ladder for promotion to higher rungs; and students vie for seats at elite colleges. All of these scenarios exemplify contests: Economic agents strive to get ahead of their peers, and scarcely supplied prizes are available only to top performers.

Investment in such competition is both rewarding and risky: On the one hand, effort improves one's relative standing in the competition and thus increases the odds of securing a prize; on the other hand, input is nonrecoverable regardless of the outcome of the competition. The tantalizing but limited rewards lure economic agents to pursue superiority relentlessly; this, amid the uncertainty inherent in the competition, could discourage upfront investment to avoid loss. The tension

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caused by the gambling nature of a contest game is particularly a concern for risk-averse participants who are sensitive to variance in terminal payoffs.

The prevailing prize structure of a contest may critically affect a risk-averse contestant's strategic choice of effort. Imagine a shift of prize money across prizes of different ranks: This alters one's wealth distribution across states—i.e., the outcomes of being ranked in different places—and causes nonlinear variations in his utility (and marginal utility) evaluated in these states, which ultimately affects his effort incentive. Two fundamental economics inquiries naturally ensue. First, how would risk-averse contestants respond to a variation in prize structure, and how does their risk attitude affect the cost-benefit trade-off in their equilibrium effort choice, compared with the risk-neutral counterpart? Second, how should a contest designer, with a fixed prize purse, fine-tune the prize structure of a contest to promote effort supply? Should she concentrate her entire prize purse on the top prize, or split it to lower the bar for reward?

This paper sets out to address these questions. Our analysis demonstrates that risk attitude fundamentally alters the nature of contestants' strategic trade-off and provides novel implications for the role played by the prevailing prize structure in contestants' incentives, which casts doubt on the conventional wisdom of the winner-take-all principle in our context. For this purpose, we set up a multi-prize contest model with risk-averse players and establish the existence and uniqueness of a symmetric pure-strategy bidding equilibrium under plausible conditions. This lays a foundation for a formal account of the incentive effect of prize structure under risk aversion, and further, allows us to identify relevant conditions under which a single-(multi-)prize contest emerges as an optimum.

Prize structure has long been recognized as an important structural element of a contest that could be manipulated to boost performance. The literature has conventionally espoused the high-power incentive provided by a single grand prize. Rosen (1986), for instance, proposed the celebrated thesis that prize money should be concentrated on a top final prize awarded to the grand winner.[1] The famous Netflix Prize, which sought algorithms of higher predictive power, provides one salient example: It awarded a US$1M grand prize to the BellKor's Pragmatic Chaos team. Competitions that award several prizes, however, are also widespread in practice. The most intuitive examples are seen in sporting events—e.g., the Olympic Games—that typically award three (gold, silver, and bronze) medals; in addition, athletes earn professional ranking points based on the stages of the tournament they manage to reach, even if they fail to win one of these medals. The various competitions hosted by the XPRIZE Foundation typically split the prize purse among a few winners: In the recent Water Abundance XPRIZE, for instance, the California-based Skysource/Skywater Alliance secured a grand prize of $1.5M for developing an easily deployable high-volume water generator; a second team, Hawaii-based JMCC WING, was awarded a $150K prize for its ingenious technological approach.[2] A growing strand of literature is devoted to the formal modeling of contests with multiple prizes and has developed competing theses to reconcile these diverging observations from various perspectives.[3,4] In this paper, we explore the implications of players' risk attitude for prize structure.

Incentive effect of prize structure under risk aversion We now briefly discuss how risk attitude reshapes contestants' response to a variation in prize structure. We adopt the popularly studied multi-winner nested Tullock contest (Clark and Riis, 1996) to model a mechanism that distributes multiple prizes. Following Konrad and Schlesinger (1997), Treich (2010), and Cornes and Hartley (2012), we assume that a contestant, indexed by $i \in \{1, \dots, N\}$, has a smooth concave function $u(\cdot)$ and an expected utility

$$\sum_{m=1}^{N} \left[P_m^i \times u(w + V_m - e^i) \right],$$

where V_m is the prize for the mth rank, P_m^i the probability of his achieving the mth rank, e^i his effort entry, and w the initial endowment of wealth. Begin with a winner-take-all contest, and imagine a hypothetical shift of a small amount of prize money from the single top prize to a prize for the runner-up. The shift of prize money triggers a three-way trade-off on a contestants' effort incentive, and the aggregate effect subtly depends not only on the degree of a contestant's risk aversion (i.e., the second-order property of utility function) but also that of his prudence (i.e., the third-order property of utility function).[5]

A negative effect is immediate: Setting a second prize softens the competition, which allows a contestant to be rewarded without outperforming all others, thereby diminishing the marginal benefit of effort and weakening his incentive to leapfrog.

[1] The winner-take-all principle is reaffirmed by Clark and Riis (1996, 1998b), Fu and Lu (2012a), and Schweinzer and Segev (2012) in imperfectly discriminatory contest settings.

[2] Google Code Jam, a renowned annual programming competition of algorithmic challenges, also selects multiple prize winners.

[3] Clark and Riis (1998b), Szymanski and Valletti (2005), Brown (2011), Fu and Lu (2009, 2012a), and Schweinzer and Segev (2012) adopt the multi-winner nested Tullock contest model (Clark and Riis, 1996) to model competitions that award several prizes. Glazer and Hassin (1988), Clark and Riis (1998a), Barut and Kovenock (1998), Moldovanu and Sela (2001), Bulow and Levin (2006), Moldovanu et al. (2007), Siegel (2009), and Fang et al. (2020) assume all-pay auctions. Further, Krishna and Morgan (1998), Akerlof and Holden (2012), and Drugov and Ryvkin (2020, 2019) consider tournament settings.

[4] See, e.g., Moldovanu and Sela (2001), Szymanski and Valletti (2005), and Drugov and Ryvkin (2020), for rationales that espouse multiple prizes. In addition, Hofstetter et al. (2018) provide empirical evidence that a multi-prize contest may outperform its single-prize counterpart.

[5] An economic agent is called "prudent" when the marginal utility function $u'(\cdot)$ is convex. Prudence is interpreted as a measure of the "sensitivity of the optimal choice of a decision variable to risk" (see Kimball, 1990). It is well known that a higher degree of prudence gives rise to a precautionary saving motive.

The negative effect underpins the usual rationale of the winner-take-all principle (see, for example, Fu and Lu, 2012a, and Schweinzer and Segev, 2012). Two additional competing forces, however, may loom large when contestants are risk averse and could counteract the negative incentive effect.

First, a risk-averse contestant, because of the concavity of his utility function, tends to discount the extra utility gain from a given wealth increase in more favorable states compared to his risk-neutral counterpart. With the shift of prize money, the contestant ends up with a smaller prize and, therefore, a utility loss in the event that he is the top performer, while he perceives extra utility in the state of being ranked in second place. The latter gain may more than offset the former loss when risk aversion is in place, which is impossible under risk neutrality. This effect, at least partly, offsets the above-mentioned negative effect. As a result, creating a second prize may incentivize risk-averse contestants more effectively than a single top prize. The ultimate effect on the marginal benefit of effort is ambiguous a priori and depends on the second-order property of the utility function.

Second, the hypothetical shift of prize money affects not only the marginal benefit of contestants' effort but also the marginal cost. Effort depletes contestants' wealth. Without risk aversion, one's marginal cost of effort is exogenously given and independent of the prevailing prize structure. With risk aversion, however, one's "marginal effort cost" boils down to the marginal disutility caused by wealth reduction aggregated in expectation over all possible states, i.e., at all possible ranks. The marginal effort cost curve is thus endogenously determined by the prize structure because the marginal disutility evaluated in each state depends on the associated prize. When the aforementioned hypothetical shift of prize money takes place, the contestant perceives an increase in his marginal disutility for the state of winning the top prize due to concavity, as well as a decrease for the state of obtaining the second rank. The aggregate effect thus depends on the second-order property of marginal utility, i.e., the third-order property of the utility function. When contestants are prudent, i.e., with a convex marginal utility, a contestant is more sensitive to downward risk, in which case the latter decrease more than offsets the former increase and reduces marginal effort cost. The second prize gives rise to a less polarized wealth distribution across states, which, analogous to precautionary saving, reduces the downward risk to the contestant and limits the overall disutility for effort. This would in turn encourage risky investment in the contest.

Snapshots of results Assuming homogeneous contestants, our study verifies that a unique symmetric pure-strategy equilibrium exists under a broad range of contest technologies when contestants exhibit nonincreasing absolute risk aversion (Theorem 1) or have quadratic preferences (Proposition 5). We do not impose specific restrictions on our setting to solve for the equilibrium in closed form. The equilibrium condition, however, suffices to shed light on the nature of contestants' effort incentive under risk aversion. The aforementioned three-way trade-off yields general insights for the implications of risk attitude for prize allocation.

Focusing on the symmetric equilibrium, we demonstrate that multiple prizes are more likely to be optimal when contestants are more risk averse and prudent. We formally derive sufficient conditions under which a single-/multi-prize contest is optimal. When contestants exhibit sufficiently mild risk aversion and relatively weak prudence, the two positive effects—which favor multiple prizes—are inadequate to counteract the negative effect. As a result, an optimal contest does not depart from the prediction obtained under risk neutrality, and again embraces the winner-take-all principle (Proposition 2). In contrast, when contestants are prudent and sufficiently risk averse, a multi-prize contest outperforms its single-prize counterpart (Proposition 3). In this case, higher-degree risk aversion strengthens the first positive effect—by which a second prize magnifies the marginal benefit of effort—while prudence catalyzes the second positive effect, in which case awarding a second prize reduces marginal cost. In particular, we demonstrate that positive prizes for lower ranks are likely in an optimum due to the cost-reducing effect of prudence, even if they provide direct negative incentive and reduce the marginal benefit of effort. The roles played by risk aversion and prudence are, respectively, discussed in detail in Sections 5.3.1 and 5.3.2.

We show that an optimum requires a strictly decreasing prize series, in that all positive prizes must be assigned in a descending order, with a larger prize awarded to a higher-ranked contestant (Proposition 1). The paper does not set out to provide a closed-form solution to optimal prize schedule, which is available only in specific setups. However, we derive an upper bound for the number of prizes in an optimum (Proposition 4). A few comparative statics are provided. In particular, we demonstrate that awarding a single prize is preferred over awarding multiple prizes under certain plausible conditions when contestants have large initial wealth holdings (Corollary 2). Further, we show that a larger prize purse facilitates multiple prizes; a similar effect can be seen in an increase in the number of contestants (Corollary 3).

This paper proceeds as follows. In Section 2, we elaborate on the paper's link to the literature. In Section 3, we set up the model. In Section 4, we characterize the unique symmetric pure-strategy equilibrium of the contest game. In Section 5, we lay out the main analysis and establish the conditions under which single (multiple) prize(s) would be optimal. We further discuss the role of contestants' risk aversion and prudence in determining the optimal prize schedule. In Section 6, we summarize our main findings and suggest directions for future research. Proofs of our main results are collected in the Appendix, while those for corollaries and additional results are relegated to an online appendix.

2. Relation to the literature

Our paper is related to three strands of the literature. First, it contributes to the literature that explores the fundamentals of equilibria in imperfectly discriminatory contest games. The existence and uniqueness of bidding equilibria have been thoroughly studied in winner-take-all Tullock contests with risk-neutral contestants.[6] Since the seminal contribution of Hillman and Katz (1984), an increasing number of studies introduce more general preferences into contest models. Skaperdas and Gan (1995) identify the conditions under which pure-strategy equilibria exist in two-player contests with a general contest success function when contestants exhibit constant absolute risk aversion (CARA). Cornes and Hartley (2003) allow for multiple heterogeneous contestants with CARA utility and verify that a unique equilibrium exists in a lottery contest. Assuming a concave impact function, Cornes and Hartley (2012) show that risk aversion may lead to multiple equilibria in a general lottery contest with homogeneous contestants. Yamazaki (2009) verifies that a unique pure-strategy equilibrium exists in a general lottery contest when contestants have nonincreasing absolute risk aversion. Jindapon and Yang (2017) further allow for a non-cash prize, risk-loving contestants, and sequential bidding. All of these studies assume a single prize. The framework of multi-winner nested Tullock contests (Clark and Riis, 1996, 1998b) has been popularly adopted to model contests that award several prizes. Somewhat surprisingly, the existence of equilibrium in the model was not formally established until the recent contribution of Fu et al. (2021). In this paper, we further identify the condition under which a unique symmetric pure-strategy equilibrium exists for homogeneous risk-averse contestants.

Second, our paper contributes to the literature that explores the strategic substance of contest games under risk aversion. Focusing on a symmetric contest with a general contest success function, Konrad and Schlesinger (1997) show that the impact of risk aversion on contestants' effort is ambiguous in a symmetric contest with general concave utility. In a similar setup, Treich (2010) concludes that risk aversion always leads to less effort when contestants are prudent.[7] Sahm (2017) shows in a general lottery contest that contestants are disadvantaged when they exhibit stronger aversion to downward risk, i.e., a higher degree of prudence. Schroyen and Treich (2016) explore how wealth endowment affects risk-averse contestants' effort incentives in a two-player contest. Chen et al. (2017) explore a complete-information all-pay auction when bidders have heterogeneous risk- or loss-averse utilities. Klose and Schweinzer (2021) examine an incomplete-information all-pay auction in which bidders have mean-variance preferences. To the best of our knowledge, our paper is the first to explore the impact of prize structure in Tullock contests with risk aversion.[8]

Third, our paper adds to the literature on optimal prize allocation in contests.[9] In a multi-winner nested Tullock contest model, Clark and Riis (1998b) show that a winner-take-all contest is optimal when homogeneous contestants are risk neutral and the cost function is linear. In a similar setup, Fu and Lu (2012a) consider a multi-stage sequential-elimination contest and establish a hierarchical winner-take-all principle, which requires that only a single grand prize be awarded in the finale of the contest when the winner-selection mechanism is sufficiently noisy. Schweinzer and Segev (2012) further extend Clark and Riis (1998b) by allowing for a nonlinear cost function and reaffirm the optimality of winner-take-all contests, provided that a symmetric pure-strategy equilibrium exists.[10] Szymanski and Valletti (2005) and Brown (2011) allow for heterogeneous contestants, and in a three-player case show that a second prize can incentivize contestants more effectively.

A few studies examine the optimal prize allocation in the setting of all-pay auctions. Glazer and Hassin (1988) pioneered in this stream of research. In an incomplete-information all-pay auction model, Moldovanu and Sela (2001) demonstrate that a winner-take-all contest can be suboptimal when effort costs are convex. Fang et al. (2020) consider a complete-information setting and find a demoralizing effect of a more unequal prize structure for homogeneous contestants with convex costs.[11] In a large-contest framework (Olszewski and Siegel, 2016), Olszewski and Siegel (2020) show that numerous heterogeneous prizes can be optimal when contestants' valuations for the prize are concave and effort costs are convex. Archak and Sundararajan (2009) model a crowdsourcing contest as an incomplete-information all-pay auction with a large number of risk-averse bidders; they search for the asymptotically optimal prize schedule when the sponsor cares about the quality of one or several top submissions.

Krishna and Morgan (1998), Akerlof and Holden (2012), and Drugov and Ryvkin (2020, 2019) explore optimal prize allocation in tournaments with additive noises. In particular, Krishna and Morgan establish a winner-take-all principle for optimal prize allocation in small tournaments. Akerlof and Holden highlight the contrasting roles played by rewards for top performers vis-à-vis punishment for bottom performers. Drugov and Ryvkin (2020) explore how the distribution of noise terms affects optimal prize allocation and demonstrate that the presence of heavy tails may compel the designer to split her prize purse into several uniform prizes. Drugov and Ryvkin (2019) further allow for risk aversion and characterize a sufficient condition under which winner-take-all principle can be retained.

[6] Szidarovszky and Okuguchi (1997) and Cornes and Hartley (2005), for instance, establish the existence and uniqueness of interior equilibrium for a sufficiently noisy contest. Baye et al. (1994), Alcalde and Dahm (2010), Ewerhart (2015, 2017a,b), and Feng and Lu (2017) venture into contests with large but finite discriminatory power.

[7] Liu et al. (2018b) consider a scenario in which only the winner of a contest pays for the resources used to compete.

[8] In contrast to this strand of literature that focuses on effort incentives, March and Sahm (2018) examine how risk aversion affects the selection efficiency of a contest in a two-player setting.

[9] See Sisak (2009) for a comprehensive survey of this topic.

[10] Liu and Treich (2019) study a multi-competition contest that allows a contestant to have multiple shots to win prizes when contestants are risk averse and prudent.

[11] Moldovanu and Sela (2006) allow the designer to decide on both the division of prize money and whether to embed a two-stage architecture in the contest, which eliminates a subset of contestants in a preliminary stage. Moldovanu et al. (2007) study the optimal distribution of status categories in an environment in which contestants value status as an intangible prize. Moldovanu et al. (2012), Thomas and Wang (2013), and Liu et al. (2018a) further expand the design space by allowing for negative prizes.

Among these studies, Archak and Sundararajan (2009), Krishna and Morgan (1998), Glazer and Hassin (1988), Akerlof and Holden (2012), and Drugov and Ryvkin (2019) allow for nonlinear valuation for prizes and hence include risk aversion. However, all these studies assume that the utility from prizes is additively separated from effort cost. In this case, a variation in prize structure does not affect each contestant's marginal cost of effort, which nullifies the effect of prudence. Our results differ from those obtained under separable utility, but our analysis also sheds light on that setting. In particular, nonseparable utility unleashes the incentive effect of prudence, which leads to more even prize allocation profiles. We show that positive prizes for lower ranks are likely under nonseparable utility even if they provide direct negative incentive, which is impossible under separable utility. This nuance is discussed in more detail in Section 5.3.3.

3. The model

A contest involves $N \geq 3$ homogeneous contestants, indexed by $i \in \mathcal{N} \equiv \{1, \cdots, N\}$.[12] Each is endowed with an initial income $w > 0$. A total of N prizes are to be given away in the contest based on contestants' ranks; they are ordered in a decreasing prize series $V_1 \geq \cdots \geq V_N \geq 0$. Contestants simultaneously commit to their costly effort e^is to vie for these prizes, and each contestant is eligible for at most one. The model boils down to a winner-take-all competition when $V_2 = 0$.

3.1. Winner-selection mechanism

We adopt the popularly studied multi-winner nested Tullock contest (Clark and Riis, 1996, 1998b) to depict the winner-selection mechanism that allows for multiple prize recipients.

The multi-winner nested contest can conveniently be described as a sequential lottery process. For a given effort profile $\boldsymbol{e} := (e^1, \ldots, e^N)$, a contestant i is picked as the recipient of the first prize, V_1, with a probability

$$p_1^i(\boldsymbol{e}) := \begin{cases} \frac{(e^i)^r}{\sum_{j\in\mathcal{N}}(e^j)^r}, & \text{if } \boldsymbol{e} \neq \boldsymbol{0}, \\ \frac{1}{N}, & \text{if } \boldsymbol{e} = \boldsymbol{0}, \end{cases}$$

which is equivalent to a standard winner-take-all Tullock contest. The parameter r indicates the discriminatory power of the contest technology. Recall that a contestant is eligible for only one prize. The recipient of the first prize is removed immediately from the pool of contestants eligible for the rest of the prizes, and a similar lottery picks the recipient of the second prize from the remaining candidates. The process is repeated until all prizes have been distributed.

To put this formally, let Ω^m, $m \in \{1, \ldots, N\}$, be the set of contestants who remain eligible for the mth-draw—i.e., those who were not picked in the previous $m-1$ draws—with $\Omega^1 \equiv \mathcal{N}$. Further denote by $\boldsymbol{e}^{\Omega^m}$ the effort profile of all contestants in the set Ω^m, with $\boldsymbol{e}^{\Omega^1} \equiv \boldsymbol{e}$. The probability of a contestant i's receiving the mth prize *conditional* on his not having been picked in the previous $m-1$ draws is given by

$$p_m^i\left(\boldsymbol{e}^{\Omega^m}; \Omega^m\right) := \begin{cases} \frac{(e^i)^r}{\sum_{j\in\Omega^m}(e^j)^r}, & \text{if } \boldsymbol{e}^{\Omega^m} \neq \boldsymbol{0}, \\ \frac{1}{N-m+1}, & \text{if } \boldsymbol{e}^{\Omega^m} = \boldsymbol{0}. \end{cases}$$

Despite the literal resemblance to a sequential lottery process, the multi-winner nested contest model is uniquely underpinned by a simultaneous noisy ranking system à la McFadden's (1973, 1974) discrete-choice model.[13]

3.2. Contestants' preference

Each contestant has an initial wealth $w \geq 0$. Contestants are assumed to be risk averse with an identical (weakly) concave Bernoulli utility function $u(\cdot)$ that satisfies the following conditions.

Assumption 1. ***(Risk-averse contestants)*** *Contestants' utility function $u(\cdot)$ is thrice continuously differentiable and satisfies $u'(c) > 0$, and $u''(c) \leq 0$ for all $c \in \mathbb{R}$.*

We assume that effort is costly and reduces a contestant's wealth at a unitary price. Therefore, a contestant ends up with a wealth of $w + V_m - e^i$ if he wins the mth prize. A contestant's wealth $w + V_m - e^i$ may take a negative value in a state when his endogenously chosen effort e^i exceeds the sum of the prize he wins, V_m, and his initial wealth w.[14] Following

[12] We assume $N \geq 3$ for ease of exposition. Our analysis can easily be extended to the case of $N = 2$.

[13] See Fu and Lu (2012b) for a detailed discussion of the model's stochastic foundation from a noisy ranking perspective. In addition, Lu and Wang (2015) provide an axiomatic foundation for the model.

[14] This scenario reflects the gambling nature of a contest game: Investment is nonrefundable regardless of the outcome, and thus one may bear a loss in an unfavorable state. The model does not depart from a typical setting with risk-neutral contestants in this respect.

the tradition in the literature—e.g., Cornes and Hartley (2012)—in Assumption 1 we allow the domain of the utility function $u(\cdot)$ to encompass the real line.[15]

Fixing an effort profile $\boldsymbol{e} \equiv (e^1, \ldots, e^N)$, denote by $P_m^i(\boldsymbol{e})$ a contestant i's ex ante probability of winning the mth prize. A contestant i's expected utility under a prize series $\boldsymbol{V} := (V_1, \ldots, V_N)$ is given by[16]

$$\sum_{m=1}^{N} \left[P_m^i(\boldsymbol{e}) \times u(w + V_m - e^i) \right].$$

3.3. Contest design

Prior to the competition, a contest designer splits a fixed prize purse of $V > 0$ into N nonnegative prizes with $V_1 \geq \cdots \geq V_N \geq 0$ and $\sum_{m=1}^{N} V_m = V$. She announces the prize schedule $\boldsymbol{V} \equiv (V_1, \ldots, V_N)$ publicly, which is thus commonly known to contestants when they exert their efforts.

The designer aims to maximize the total effort of the contest, i.e., $\sum_{i=1}^{N} e^i$. Clearly, a prize allocation $V_1 = \cdots = V_N$ is suboptimal, because no effort can be elicited in the equilibrium. Throughout our analysis, we focus on decreasing prize schedules $V_1 \geq \cdots \geq V_N \geq 0$, with strict inequality holding for at least one.

4. Equilibrium analysis

In this section, we characterize the bidding equilibrium of the multi-prize contest game given an arbitrary prize schedule $\boldsymbol{V} \equiv (V_1, \ldots, V_N)$. Assuming symmetric risk-neutral contestants, Fu et al. (2021) show that a unique equilibrium exists in a generalized multi-winner nested lottery contest, in which contestants adopt a symmetric pure strategy. Accommodating risk aversion would substantially complicate the analysis and may lead to multiple equilibria, even in symmetric single-prize contests.[17] Our analysis focuses on symmetric interior pure-strategy equilibria in which all contestants exert the same amount of effort, because a symmetric equilibrium is often viewed as a compelling and plausible focal point for a symmetric game (e.g., Morrow, 1994; Dutta, 1999).

To search for the equilibrium, we use the symmetric opponents form approach (SOFA)[18] and assume that all players other than one indicative player place the same bid e. An effort e' allows the indicative contestant to win the mth prize with a probability

$$P_m\left(e', e\right) \equiv \frac{(N-1)!}{(N-m)!} \times \left(\prod_{j=1}^{m-1} \frac{(e)^r}{(N-j)(e)^r + (e')^r} \right) \times \frac{(e')^r}{(N-m)(e)^r + (e')^r},$$

where the term $\frac{(e')^r}{(N-m)(e)^r + (e')^r}$ is the probability of his being picked in the mth draw conditional on that he has not been selected for the previous $m-1$ prizes. It is straightforward to verify that $P_m(e, e) = 1/N$.

The indicative contestant chooses his effort e' for the following expected utility maximization problem:

$$\max_{e' \geq 0} \pi\left(e', e\right) := \sum_{m=1}^{N} \left[P_m\left(e', e\right) \times u\left(w + V_m - e'\right) \right]. \tag{1}$$

The first-order condition with respect to e' leads to

$$\sum_{m=1}^{N} \left[\frac{\partial P_m\left(e', e\right)}{\partial e'} \times u\left(w + V_m - e'\right) \right] = \sum_{m=1}^{N} \left[P_m\left(e', e\right) \times u'\left(w + V_m - e'\right) \right]. \tag{2}$$

A symmetric equilibrium requires $e' = e$, in which case

[15] A caveat arises under constant relative risk aversion (CRRA) utility, in which case the utility function is undefined for $c < 0$. However, our analysis does not lose its bite: Negative consumption does not emerge in equilibrium because of the Inada condition implied by the utility function. The logic naturally unfolds after contestants' trade-off is laid out in Section 4.

[16] Alternatively, we can assume that effort is nonmonetary, and contestant's utility is given by $\sum_{m=1}^{N} [P_m^i(\boldsymbol{e}) \times u(w + V_m)] - e^i$. This case corresponds to the ability contest model of Schroyen and Treich (2016) and is widely adopted in tournament models. See Section 5.3.3 for more discussions of the differences between these two model specifications.

[17] Mixed observations are obtained in the context of single-prize contests. Yamazaki (2009) shows that a unique equilibrium exists with heterogeneous risk-averse contestants under nonincreasing absolute risk aversion preferences (NIARA). In contrast, Cornes and Hartley (2012) remarkably demonstrate the possibility of asymmetric equilibria in a symmetric single-prize contest under quadratic utility, in which case NIARA does not hold.

[18] The SOFA approach has been broadly applied in game theoretic and IO literature, such as auction and oligopoly models. See, e.g., Hefti (2017) and Drugov and Ryvkin (2020).

$$\left.\frac{\partial P_m\left(e',e\right)}{\partial e'}\right|_{e'=e}=\frac{r}{Ne}\times\left(1-\sum_{g=0}^{m-1}\frac{1}{N-g}\right)=:\mu_m\times\frac{r}{Ne}, \tag{3}$$

where μ_m, with $m \in \{1, \ldots, N\}$, is defined as

$$\mu_m := 1-\sum_{g=0}^{m-1}\frac{1}{N-g}.$$

It is straightforward to verify that $\mu_1 > \mu_2 > \cdots > \mu_N$ and $\sum_{m=1}^{N}\mu_m = 0$. Therefore, the term $\partial P_m(e',e)/\partial e'\big|_{e'=e}$ strictly decreases with m. To put this intuitively, additional effort affords him a higher probability of obtaining a better prize, and equivalently, renders him less likely to fall behind and end up with a less lucrative prize. Combining (2) and (3), the following condition must hold for an interior symmetric pure-strategy equilibrium:

$$\frac{r}{Ne}\times\sum_{m=1}^{N}\left[\mu_m\times u(w+V_m-e)\right]=\frac{1}{N}\times\sum_{m=1}^{N}u'(w+V_m-e). \tag{4}$$

Two additional regularity conditions are imposed. First, we require that the discriminatory power of the contest—i.e., the parameter r—be bounded from above. Define $\bar{r}$ as[19]

$$\bar{r} := \min\left\{1\,,\,\frac{4}{\sum_{g=2}^{N}\frac{1}{g}}\right\}.$$

Assumption 2. ***(Relevant bounds for the discriminatory power)*** $0 < r \le \bar{r}$.

It is well known that an interior pure-strategy equilibrium arises in single-prize contests with sufficient noise, e.g., small positive r in Tullock contests.[20] Randomized bidding emerges in equilibrium when the contest is sufficiently discriminatory, while the properties of the resultant mixed-strategy equilibria largely remain elusive.[21] Previous contest modeling exercises have typically been limited to the case of noisy contest technology (e.g., $r \le 1$) to obtain a pure-strategy equilibrium (see, e.g., Epstein et al., 2011; Drugov and Ryvkin, 2017; Fu and Wu, 2020). Few formal studies have been devoted to exploring the game theoretical fundamentals of multi-winner nested lottery contests. Fu et al. (2021) establish the existence of a pure-strategy equilibrium when the contest is sufficiently noisy, which can be translated to $r \le 1$ in our context.[22] Assumption 2 ensures a well-behaved payoff function and enables a tractable equilibrium analysis, which will be interpreted in more detail below.

We further impose the following requirement on contestants' preferences.

Assumption 3. ***(NIARA preferences)*** *Contestants' utility function exhibits nonincreasing absolute risk aversion (NIARA), i.e., $-u''(c)/u'(c)$ is nonincreasing in c.*

The NIARA condition is first proposed by Arrow (1970) and satisfied by a broad spectrum of utility functions, such as the familiar constant absolute risk aversion (CARA, henceforth) and constant relative risk aversion (CRRA, henceforth) utility functions. A plethora of experimental and empirical findings provide evidence for the prevalence of decreasing absolute risk aversion (see, for instance, Friend and Blume, 1975). It is worth noting that NIARA implies $u''' \ge 0$, i.e., contestants are prudent.[23] The following result is obtained.

Theorem 1. ***(Symmetric equilibrium existence and uniqueness under NIARA preferences)*** *Suppose that Assumptions 1, 2, and 3 are satisfied. For a prize schedule $V_1 \ge \cdots \ge V_N \ge 0$ with at least one strict inequality holding, there exists a unique symmetric pure-strategy equilibrium of the contest game, in which each contestant's equilibrium effort is the solution to Equation (4).*

[19] It can be verified that $4/[\sum_{g=2}^{N} 1/g] > 1$ for $N \le 82$, implying that $\bar{r} = 1$ for $N \le 82$.

[20] See Szidarovszky and Okuguchi (1997) and Cornes and Hartley (2005) for the risk-neutral case, and Yamazaki (2009) and Cornes and Hartley (2012) for the risk-averse case.

[21] With $r \in (2, \infty)$, Baye et al. (1994) identify a symmetric mixed-strategy equilibrium in a two-player single-prize Tullock contest that completely dissipates the rent. Ewerhart (2015) shows that the symmetric equilibria sharply differ from the pure-strategy equilibrium in a standard lottery contest (i.e., $r = 1$) and the mixed-strategy equilibrium in an all-pay auction (i.e., $r = \infty$). Alcalde and Dahm (2010) demonstrate the existence of an all-pay auction equilibrium in probabilistic contests with large but finite discriminatory power. Ewerhart (2017a) demonstrates, in a general class of probabilistic contests, that all equilibria are payoff-equivalent and revenue-equivalent when the contest involves small noise, i.e., being "close to all-pay auctions"; payoff and revenue equivalence nevertheless does not hold when the contest's discriminatory power is at an intermediate level.

[22] Fu and Lu (2012a) adopt the multi-winner nested lottery contest model and mainly focus on the case of $r \le 1$, which yields a pure-strategy solution to the game. Fu and Lu (2009) allow for a more general impact function but require concavity.

[23] To see this, note that $\frac{d}{dc}\left(-\frac{u''(c)}{u'(c)}\right) \le 0$ is equivalent to $u'''(c) \ge [-u''(c)]^2/u'(c) \ge 0$.

Theorem 1 shows that the symmetric effort profile that solves Equation (4) constitutes the unique symmetric pure-strategy equilibrium of the game when the mild requirements of Assumptions 1-3 are satisfied. Note that Assumption 2 requires that r be bounded by both 1 and the cutoff $4/[\sum_{g=2}^{N}\frac{1}{g}]$. The assumption of NIARA, together with the requirement of $r \leq 1$, guarantees that contestants' expected payoff $\pi(e', e)$ is strictly concave in his effort entry e' given $e > 0$, and thus the first-order condition $\partial\pi(e', e)/\partial e' = 0$ is not only a necessary but also a sufficient condition to characterize a contestant's unique best response. This largely paves the way for the existence of a symmetric equilibrium. Furthermore, the requirement of $r \leq 4/[\sum_{g=2}^{N}\frac{1}{g}]$ ensures that Equation (4) has a unique solution under NIARA preference, which establishes equilibrium uniqueness in the symmetric class. It is worth noting that $r \leq 4/[\sum_{g=2}^{N}\frac{1}{g}]$ imposes a conservative sufficient condition for the proof of uniqueness, and the upper bound can be relaxed under a broad array of utility functions. Consider, for instance, the usual CARA preferences.[24]

Corollary 1. ***(Symmetric equilibrium existence and uniqueness under CARA preferences)*** *Suppose that contestants exhibit CARA. For a prize schedule $V_1 \geq \cdots \geq V_N \geq 0$ with at least one strict inequality holding, there exists a unique symmetric pure-strategy equilibrium of the contest game for all* $0 < r \leq 1$.

5. Contest design: prize allocation

The equilibrium analysis allows us to formally explore the optimal prize allocation problem. The contest designer chooses the prize schedule $\boldsymbol{V} \equiv (V_1, \ldots, V_N)$ prior to the competition, anticipating that contestants play the symmetric equilibrium characterized by (4).[25] The condition (4) reveals the fundamental trade-off faced by a contestant when choosing his effort strategy, which is critical to understanding how a variation in prize schedule could affect contestants' bidding incentives. Recall that the condition is written as

$$\underbrace{\frac{r}{Ne} \times \sum_{m=1}^{N} [\mu_m \times u(w + V_m - e)]}_{\text{marginal benefit of effort}} = \underbrace{\frac{1}{N} \times \sum_{m=1}^{N} u'(w + V_m - e)}_{\text{marginal cost}}.$$

The left-hand and right-hand sides of Equation (4), respectively, represent the marginal benefit and marginal cost of a contestant's effort in terms of his utility gain/loss evaluated in the symmetric equilibrium. A higher effort varies the probability distribution of all possible outcomes. For each possible outcome, i.e., being ranked in an arbitrary place m, the marginal impact of effort on his utility through this channel is given by $\partial P_m(e', e)/\partial e'\big|_{e'=e} \times u(w + V_m - e')$; summing up over all possible outcomes, the overall marginal effect boils down to the left-hand side of Equation (4) by the fact that $\partial P_m(e', e)/\partial e'\big|_{e'=e} = \mu_m \times \frac{r}{Ne}$.

An increase in effort, however, also consumes the contestant's wealth and generates disutility in all possible outcomes. The marginal effect is captured by the right-hand side of Equation (4). The marginal disutility, $u'(w + V_m - e)$, is state-dependent, and the overall marginal cost is obtained by summing up $u'(w + V_m - e)$ over all states.

A variation in the prize allocation profile compels a contestant to rebalance between the marginal benefit of effort and marginal cost. This could increase equilibrium effort if it (i) increases each contestant's marginal benefit of effort, and/or (ii) decreases the marginal cost. Contestants' risk attitude plays a critical role in contestants' trade-off.

Let us first consider a benchmark case of risk-neutral contestants. In this case, the left-hand side of Equation (4) is

$$\frac{r}{Ne} \times \sum_{m=1}^{N} [\mu_m \times u(w + V_m - e)] = \frac{r}{Ne} \times \left[\sum_{m=1}^{N} \mu_m \times V_m\right],$$

while the right-hand side simply boils down to one, which is a constant and independent of the prevailing prize structure. Varying prize allocation affects only the marginal benefit of effort but not the marginal cost.

With risk-neutral contestants, maximizing the marginal benefit of effort, i.e., maximizing the sum $\sum_{m=1}^{N} \mu_m \times V_m$, simply requires concentrating the entire prize purse on the top prize V_1. To see this, imagine that a small amount of prize money ϵ is shifted from V_m to V_{m+1}. With linear utility, the change to the marginal benefit boils down to $-\frac{r}{Ne}(\mu_m - \mu_{m+1})\epsilon$, which is strictly negative because μ_m is strictly decreasing with m: The top prize incentivizes contestants more than any lower-rank prize. This logic underpins the winner-take-all result of Fu and Lu (2012a) and Schweinzer and Segev (2012).

Risk aversion, however, triggers the two additional effects mentioned in Section 1. First, by Equation (4), when prize money is shifted between prizes, the change in utility is nonlinear. Concentrating the entire prize purse on V_1 does not

[24] Simulations show that the upper bound for r can be relaxed to one with CRRA preferences.

[25] As noted above, multiple equilibria may arise in a symmetric multi-prize contest with risk-averse players. However, the literature has yet to provide an account of the (possible) asymmetric equilibria and their properties, which precludes a ranking between the symmetric and (possible) asymmetric equilibria in terms of players' welfare or effort contributions. Our contest design exercise focuses on the symmetric interior pure-strategy equilibrium, which is often viewed as a compelling and plausible focal point for a symmetric game (e.g., Morrow, 1994; Dutta, 1999).

necessarily maximize the marginal benefit of effort, $\frac{r}{Ne} \times \sum_{m=1}^{N} [\mu_m \times u(w + V_m - e)]$, because of the nonlinearity of $u(\cdot)$, despite the strictly decreasing μ_m. Second, varying the prize schedule also affects the marginal cost: $\frac{1}{N} \times \sum_{m=1}^{N} u'(w + V_m - e)$, in general, depends on the particular prize structure.[26]

Imagine, again, a hypothetical shift of a small amount of prize money ϵ from V_1 to V_2. Consider first the impact on marginal benefit. A direct loss results because of decreasing μ_m. The loss in $u(w + V_1 - e)$, however, can be (at least partly) compensated by a larger gain in $u(w + V_2 - e)$ because of the contestant's concave utility function. The overall change in the marginal benefit amounts to $\frac{r}{Ne} \times \{\mu_2[u(w + V_2 + \epsilon - e) - u(w + V_2 - e)] - \mu_1[u(w + V_1 - e) - u(w + V_1 - \epsilon - e)]\}$, which boils down to $\frac{r}{Ne} \times [\mu_2 u'(w + V_2 - e) - \mu_1 u'(w + V_1 - e)]\epsilon$ for small ϵ. The direction of the change depends on the curvature of the utility function $u(\cdot)$, i.e., its second-order property.

We next consider its impact on marginal cost, i.e., $\frac{1}{N} \times \sum_{m=1}^{N} u'(w + V_m - e)$. Because of decreasing marginal utility, the shift of prize money leads to a hike of marginal disutility in the state of achieving the top rank, i.e., $u'(w + V_1 - e) < u'(w + V_1 - \epsilon - e)$; this, however, reduces the marginal disutility in the state of achieving the second rank, i.e., $u'(w + V_2 - e) > u'(w + V_2 + \epsilon - e)$. The direction of the overall change in the marginal cost, $\frac{1}{N} \times [|u''(w + V_1 - e)| - |u''(w + V_2 - e)|]\epsilon$, depends on the curvature of the marginal utility function $u'(\cdot)$, i.e., the third-order property of the utility function $u(\cdot)$. Prudent contestants—with a positive $u'''(\cdot)$—implies that $\left|u''(w + V_1 - e)\right| - \left|u''(w + V_2 - e)\right| < 0$: They are more sensitive to downward risk because $\left|u''(\cdot)\right|$ is decreasing, so they perceive more significant disutility for a given amount of forgone effort when they end up with (the small) V_2 than they do if they end up with (the large) V_1. The shift in prize money alleviates the pain and reduces marginal cost of effort.

The former effect alludes to the usual preference for a smoother consumption profile across different states caused by risk aversion, whereas the second is analogous to that underlying the precautionary saving motive (Kimball, 1990) and self-protective behavior (Dachraoui et al., 2004) caused by prudence. An intuitive account of optimal prize allocation follows below.

(i) Multiple prizes are more likely to emerge in an optimum when contestants exhibit a higher degree of risk aversion. The preference for a smoother consumption profile across different states implies that shifting prize money to lower-rank prizes may increase the marginal benefit of effort.
(ii) Multiple prizes tend to be more appealing when u' is convex, i.e., when contestants are *prudent*, with $u'''(\cdot) \geq 0$. A more dispersed prize allocation profile effectively reduces contestants' pain in less favorable states, which reduces the marginal cost of effort.

Our subsequent results are all interpreted in light of the above rationale. Before we proceed, let us lay out some preliminaries. Define

$$\tau := \max\left\{\sup\left\{-\frac{u'''(c)}{u''(c)}\,\middle|\, w - \frac{V}{N} < c < w + V\right\}, 0\right\}, \tag{5}$$

which is useful for our analysis. The term $-u'''(c)/u''(c)$ in expression (5) is referred to as the coefficient of absolute prudence in the economics literature (see Kimball, 1990). The parameter τ is the maximum absolute prudence level of a contestant in his relevant support of wealth.[27]

Further, the following property of an optimum can readily be obtained.

Proposition 1. ***(Consecutive and monotone prize series)*** *Suppose that Assumptions 1, 2, and 3 are satisfied, such that the contest game has a unique symmetric pure-strategy equilibrium. If $V_j > 0$ for some $j \in \{2, \ldots, N\}$ in an optimal contest, then $V_{j-1} > V_j$.*

Proposition 1 states that positive prizes are never equal in an optimum despite contestants' risk aversion, and a higher rank is always rewarded more. As shown in our proof, a properly set prize premium for a higher rank increases the marginal benefit of effort and incentivizes contestants.

5.1. Optimality of single-prize contests

We first provide a sufficient condition under which the usual winner-take-all contest is optimal.

Proposition 2. ***(Optimality of single-prize contests)*** *Suppose that Assumptions 1, 2, and 3 are satisfied, such that the contest game has a unique symmetric pure-strategy equilibrium. A single-prize contest generates a larger amount of total effort than any multi-prize contest if*

[26] One exception is the quadratic utility function with linear marginal cost u', or equivalently, $u''' = 0$. It is straightforward to verify that the marginal cost curve depends on the designer's whole budget V but not the prize profile $(V_1, \ldots, V_N)$ in this case.

[27] To see this more clearly, note that the equilibrium effort level cannot exceed $\frac{V}{N}$. Therefore, a representative contestant's income in any state, i.e., $w + V_m - e$, is bounded from above by $w + V$ and bounded below by $w - \frac{V}{N}$.

$$\frac{u'\left(w-\frac{V}{N}\right)}{u'\left(w+V\right)} \leq \frac{r\mu_1+\frac{V}{N}\tau}{r\mu_2+\frac{V}{N}\tau}.$$

Although the condition required in Proposition 2 is sufficient but not necessary for the optimality of single-prize contests, it yields useful economic implications. First, the condition is more likely to be satisfied when a low degree of risk aversion is in place: A moderate $u'(w-\frac{V}{N})/u'(w+V)$ requires that $u'(\cdot)$ decrease gradually, which precludes excessive preference for a smooth consumption profile. Second, the right-hand side of the condition, $(r\mu_1+\frac{V}{N}\tau)/(r\mu_2+\frac{V}{N}\tau)$, strictly decreases with the degree of prudence τ. Therefore, the condition is also more likely to be satisfied when contestants are not excessively prudent.

Clearly, the above condition automatically holds for linear utility functions—i.e., with risk-neutral contestants: The left-hand side of the condition boils down to one, while the right-hand side is always strictly greater than one. Clark and Riis (1996, 1998b), Schweinzer and Segev (2012), and Fu and Lu (2012a) establish the optimality of single-prize contests in the case of symmetric and risk-neutral contestants. Proposition 2 extends the boundary for these winner-take-all results, as it allows for weak risk aversion and/or weak prudence. We illustrate this notion by using the convenient CARA and CRRA utility functions.

Example 1. (CARA utility) Suppose that the utility function takes the form

$$u(c)=1-\exp(-\alpha c), \quad \text{with } \alpha>0.$$

It is straightforward to verify that $-u'''/u''=\alpha$ and hence $\tau=\alpha$. The condition $\frac{u'\left(w-\frac{V}{N}\right)}{u'(w+V)} \leq \frac{r\mu_1+\frac{V}{N}\tau}{r\mu_2+\frac{V}{N}\tau}$ can equivalently be written as

$$\exp\left(\frac{N+1}{N}\alpha V\right) \leq \frac{r\mu_1+\frac{V}{N}\alpha}{r\mu_2+\frac{V}{N}\alpha}.$$

Note that the left-hand side and the right-hand side of the above inequality, respectively, approach 1 and $\frac{\mu_1}{\mu_2}>1$, as $\alpha \searrow 0$, which in turn indicates that the above condition holds for sufficiently small α.

Example 2. (CRRA utility) Suppose that the utility function takes the form

$$u(c)=\begin{cases}(c^{1-\beta}-1)/(1-\beta), & \text{if } \beta>0, \text{and } \beta\neq 1,\\ \ln(c), & \text{if } \beta=1.\end{cases}$$

Simple algebra can verify that $\tau=(1+\beta)/(w-\frac{V}{N})$ for $w>\frac{V}{N}$, and that $\frac{u'\left(w-\frac{V}{N}\right)}{u'(w+V)} \leq \frac{r\mu_1+\frac{V}{N}\tau}{r\mu_2+\frac{V}{N}\tau}$ holds for sufficiently small β.

A closer look at Proposition 2 yields further insights in the case of decreasing absolute risk aversion (DARA) preferences. Contestants' initial wealth w plays a nontrivial role in predicting optimal prize structure, which is shown in the following comparative statics.[28]

Corollary 2. *Suppose that Assumptions 1 and 2 are satisfied, such that the contest game has a unique symmetric pure-strategy equilibrium when the contestants' utility exhibits decreasing absolute risk aversion (DARA). Then there exists a threshold of contestants' initial wealth $\bar{w}$, such that awarding a single prize is optimal to an effort-maximizing contest designer for $w>\bar{w}$, if the utility function satisfies* $\lim_{w\nearrow\infty} u'\left(w-\frac{V}{N}\right)/u'\left(w+V\right)=1$.

It is straightforward to verify that the condition $\lim_{w\nearrow\infty} u'\left(w-\frac{V}{N}\right)/u'\left(w+V\right)=1$ can be met under CRRA preferences, which satisfy DARA.

5.2. Optimality of multi-prize contests

We now derive the condition under which the winner-take-all principle fades away. For notational convenience, denote by e_s the equilibrium effort level in a single-prize contest, i.e., under a prize schedule $\boldsymbol{V}_s := (V, 0, \ldots, 0)$. A sufficient condition for the optimality of multi-prize contests is laid out below.

[28] It is evident that the optimal prize schedule is independent of the initial wealth for CARA preferences.

Proposition 3. ***(Optimality of multi-prize contests)*** *Suppose that Assumptions 1, 2, and 3 are satisfied, such that $u''' \geq 0$ and the contest game has a unique symmetric pure-strategy equilibrium. There exists a multi-prize contest that generates more total effort than the single-prize contest (i.e., $\boldsymbol{V} = \boldsymbol{V}_s$) if*

$$\frac{u'(w - e_s)}{u'(w + V - e_s)} > \frac{\mu_1}{\mu_2}.$$

Proposition 3 states that a multi-prize contest can be optimal with risk-averse and weakly prudent contestants. Obviously, the condition $\frac{u'(w-e_s)}{u'(w+V-e_s)} > \frac{\mu_1}{\mu_2}$ degenerates to $1 > \frac{\mu_1}{\mu_2}$ under risk neutrality and can never be satisfied.

Imagine a single-prize contest with $V_1 = V$. Suppose that the designer shifts the prize money and awards a small second prize of size $\epsilon > 0$. The conditions stated in Proposition 3 can again readily be interpreted in light of the rationale laid out at the beginning of Section 5. The condition $u'(w - e_s)/u'(w + V - e_s) > \mu_1/\mu_2$ implies a strong risk aversion, under which the hypothetical shift of prize money strictly increases the marginal benefit of effort. The condition of prudence implied by NIARA preferences (Assumption 3)—i.e., $u''' \geq 0$—suggests that the hypothetical shift reduces the marginal cost. This renders the winner-take-all contest suboptimal.

A closer look at Proposition 3 further yields the following.

Corollary 3. *Suppose that Assumptions 1, 2, and 3 are satisfied, such that $u''' \geq 0$ and a unique symmetric pure-strategy equilibrium exists. Awarding multiple prizes is optimal to an effort-maximizing contest designer if (i) V is sufficiently large and $u'(\infty) < \frac{\mu_2}{\mu_1 u'(w)}$ or (ii) N is sufficiently large.*

When V is large, a winner-take-all prize structure renders the contest riskier because of the more polarized wealth distribution across states. The spread between $u'(w - e_s)$ and $u'(w + V - e_s)$ enlarges for large V, which makes more likely the condition $\frac{u'(w-e_s)}{u'(w+V-e_s)} > \frac{\mu_1}{\mu_2}$: A concave utility function could severely discount the marginal gain from a large prize. When the contest involves a larger number of contenders, one expects a smaller likelihood of achieving the top rank. Lower-rank prizes gain in their appeal, as they provide contestants with additional avenues for reward, which can in turn increase the marginal benefit of effort.[29]

Proposition 3 and Corollary 3 provide sufficient conditions for multiple prizes to be optimal. Although the exact form of an optimal prize schedule can be obtained only in a specific setting, we are able to establish an upper bound for the optimal number of prizes. Recall τ defined in (5). Further define

$$N_p := \max\left\{ j = 1, \ldots, N \,\middle|\, r \times \mu_j + \frac{V}{N}\tau > 0 \right\}. \tag{6}$$

The following can be obtained.

Proposition 4. ***(Upper bound of optimal number of prizes)*** *Suppose that Assumptions 1, 2, and 3 are satisfied, such that a unique symmetric pure-strategy equilibrium exists. The number of positive prizes in an optimal contest is no greater than N_p.*

Again, the logic can be understood in light of the rationale outlined above. Suppose that the contest designer considers setting aside a small additional prize indexed by j. The move would affect both the marginal benefit and marginal cost of effort, as previously discussed. Recall that the marginal benefit of effort is given by $\frac{r}{Ne} \times \sum_{m=1}^{N} [\mu_m \times u(w + V_m - e)]$ from Equation (4). The impact of the jth prize on the marginal benefit is thus proportional to μ_j. Note that the effect can be negative, depending on the sign of μ_j: Introducing a lower-rank prize softens the competition and weakens contestants' incentives to vie for more favorable positions. Further, the marginal cost of effort is given by $\frac{1}{N} \times \sum_{m=1}^{N} u'(w + V_m - e)$. It can be reduced if they are prudent, while the magnitude of the reduction depends on the degree of prudence, i.e., τ. The term $r \times \mu_j + \frac{V}{N}\tau$ in expression (6) thus captures the *net* effect on effort incentive of the small additional prize j. Awarding such a prize is suboptimal if the net effect turns negative.

It is noteworthy that by this rationale, a low-rank prize is likely to emerge in an optimum even if it creates a direct negative incentive and reduces marginal benefit of effort, i.e., $\mu_j < 0$: It could alleviate the pain caused by effort, which makes up for the loss in marginal benefit.

With risk-neutral contestants, an optimum requires $V_N = 0$ because incentives are provided solely by prize differentials. This does not necessarily hold for risk-averse and prudent contestants. An optimal contest may even award a positive prize for the bottom-ranked contestant (i.e., $V_N > 0$), depending on the specific context. We continue with Examples 1 and 2 to demonstrate the nuanced implications caused by different preferences.

[29] Relatedly, March and Sahm (2018) show in a single-prize setting that the size of the prize purse affects the selection efficiency of a contest when players are asymmetric and risk averse.

Example 1. (Continued, CARA utility) Suppose that contestants exhibit CARA preferences and a symmetric equilibrium is played. Then $V_N = 0$ in an optimal contest.[30]

Example 2. (Continued, CRRA utility) Suppose that $N = 3$, $(V, w, r) = (3, 0.5, 0.5)$, and $u(c) = \ln(c)$ and a symmetric equilibrium is played. An optimal prize schedule involves three positive prizes with $(V_1^*, V_2^*, V_3^*) \approx (1.9821, 0.9583, 0.0596)$, which yields an equilibrium effort of $e^* = 0.1820$.

To see the logic, note that $u'(0) = \infty$ with a CRRA utility function. A contestant is strongly averse to a state of low consumption because of the unbounded $u'(c)$ for c close to zero, which forces him to refrain from supplying effort in order to avoid wiping out his endowed wealth in the case of winning no prize. A less polarized prize allocation insures contestants against an unfavorable ranking outcome, which effectively reduces effort cost and restores their incentives.

In contrast, one can formally prove that $V_N = 0$ must hold with CARA preferences in an optimum. The aforementioned force that drives a positive bottom prize under CRRA preference is absent: One's marginal utility remains bounded and is equal to α when c drops to zero.

5.3. Discussion

Risk aversion and prudence are two highly related concepts used to describe economic agents' risk attitude. When the prize structure varies, the former determines its effect on marginal benefit of effort, while the latter influences the marginal cost. Next, we discuss in further detail the role played by risk aversion and prudence in determining an optimum. Propositions 1 to 4 provide a partial characterization of the contest designer's optimal prize schedule. The specific form of optimal prize allocation depends sensitively on the property of contestants' utility function. To gain additional mileage, we impose more structures on the utility function. We first follow Cornes and Hartley (2012, Example 1) and consider a quadratic utility function with $u'''(\cdot) = 0$: The effect of the utility functions' third-order property is entirely muted, which allows us to abstract away the role played by prudence and focus on that of risk aversion. We then consider a CARA utility function, which highlights the role played by prudence. Finally, we consider a variation to our model. We consider utility functions that are separable in one's income and effort cost, and discuss the implications of the modeling nuance.

5.3.1. Role of risk aversion

To examine the impact of contestants' risk preferences on optimal prize structure, we consider the popularly studied quadratic (mean-variance) utility function (e.g., Cornes and Hartley, 2012; Klose and Schweinzer, 2021). Suppose that the utility function takes the form

$$u(c) = c - \frac{\gamma}{2}c^2, \text{ with } \gamma > 0 \text{ and } c \leq 1/\gamma. \tag{7}$$

The parameter γ measures the degree of contestants' risk aversion. Note that the third-order derivative of the utility reduces to zero, i.e., $u''' = 0$. Therefore, when prize money is being shifted between prizes of different ranks, contestants' marginal cost of effort remains constant. As a result, the change in effort incentive is solely determined by the change in marginal benefit. It should be noted that a quadratic utility does not satisfy the requirement of NIARA preference (i.e., Assumption 3). We thus present the following additional result.

Proposition 5. ***(Symmetric equilibrium existence and uniqueness under quadratic preferences)*** *Suppose that contestants have quadratic utility as given by (7) and $u'(w + V) \geq 0$. There exists a unique symmetric pure-strategy equilibrium of the contest game for all $0 < r \leq \frac{1}{2(N-1)^2+1}$.*

The upper bound $\frac{1}{2(N-1)^2+1}$ ensures that a contestant's expected payoff $\pi(e', e)$ is strictly concave in his effort entry e' given that $e > 0$. Although a formal proof is absent, simulations show that concavity is preserved as long as $r \leq 1$, indicating that the symmetric pure-strategy equilibrium remains unique more generally.

Recall that Propositions 2 and 3 provide sufficient but not necessary conditions for the optimality of single prize and multiple prizes, respectively. An optimum remains ambiguous in general when neither condition is satisfied, in which case, as discussed above, an optimum depends on the aforementioned three-way trade-off, and both risk aversion and prudence may play a role. For quadratic utility, the ambiguity fades away because a contestant's absolute prudence $-u'''(\cdot)/u''(\cdot)$ reduces to zero. We obtain the following result, by which a single-prize contest is optimal once the condition in Proposition 3—which establishes the optimality of multi-prize contests—fails to hold.

[30] The formal proof of this claim is provided in Online Appendix B.

Corollary 4. ***(Optimal prize allocation with quadratic utility)*** *Suppose that each contestant has a quadratic utility function as given by (7) with $u'(w+V) \geq 0$ and $0 < r \leq \frac{1}{2(N-1)^2+1}$, such that a unique symmetric pure-strategy equilibrium exists. An effort-maximizing contest sets multiple prizes if $\frac{u'(w-e_s)}{u'(w+V-e_s)} > \frac{\mu_1}{\mu_2}$ and a single prize if $\frac{u'(w-e_s)}{u'(w+V-e_s)} < \frac{\mu_1}{\mu_2}$.*

The specific setting of quadratic utility allows us to further examine two main questions: (i) How would the optimal number of prizes change as contestants become more risk averse? (ii) How would the distribution of prizes change as contestants become more risk averse? Recall that the parameter γ measures the degree of risk aversion. We have the following observations.

Example 3. (Comparative statics of optimal prize allocation in risk aversion) Suppose that $u(c) = c - \frac{\gamma}{2}c^2$, $N = 10$, $r = 1$, $w = 0$, and $V = 1$.[31] By expression (6), the maximum number of positive prizes is 6 and thus we must have $V_7^* = \cdots = V_{10}^* = 0$ in an optimal contest.[32] The equilibrium effort e^* and optimal prize series $(V_1^*, \ldots, V_6^*)$ for different values of γ are reported as follows:

γ	e^*	V_1^*	V_2^*	V_3^*	V_4^*	V_5^*	V_6^*
$\gamma_1 = 0.1$	0.0864	1.0000	0.0000	0.0000	0.0000	0.0000	0.0000
$\gamma_2 = 0.3$	0.0816	0.6918	0.3082	0.0000	0.0000	0.0000	0.0000
$\gamma_3 = 0.5$	0.0786	0.5812	0.3702	0.0486	0.0000	0.0000	0.0000
$\gamma_4 = 0.7$	0.0765	0.4997	0.3581	0.1422	0.0000	0.0000	0.0000
$\gamma_5 = 0.9$	0.0748	0.4544	0.3514	0.1942	0.0000	0.0000	0.0000

The first pattern to notice is that the contest designer would (weakly) increase the number of prizes as contestants become more risk averse, i.e., when γ increases. The observation is intuitive. Contestants prefer a smoother consumption profile as γ ascends. The contest designer would set aside more prizes accordingly, because this enlarges the marginal benefit of effort. The second pattern to notice is that the distribution of prizes in an optimum tends to be more even as γ increases. Specifically, as γ increases from 0.5 to 0.9, the optimal number of prizes remains the same; however, the contest designer reallocates the prize money by reducing the sizes of prizes for higher ranks and increasing those for lower ranks. Both observations demonstrate that a higher degree of risk aversion renders prize allocation more dispersed over different ranks.

5.3.2. Role of prudence

We now consider the case of CARA utility. That is, each contestant has a utility function

$$u(c) = 1 - \exp(-\alpha c), \text{ with } \alpha > 0.$$

An economic agent with CARA utility must also be strictly prudent, and has a constant level of absolute prudence α. With both risk aversion and prudence in place, a variation in prize allocation affects both the marginal benefit and the marginal cost of effort. Recall that by Proposition 3, a multi-prize contest outperforms a single-prize one if $u'(w-e_s)/u'(w+V-e_s) > \mu_1/\mu_2$ and $u'''(\cdot) \geq 0$. When prize money is shifted from a single top prize to a second prize, the former condition ensures that sufficient risk aversion counteracts the superior incentive provided by top prize and magnifies marginal benefit. (Weak) prudence ensures that the marginal cost of effort would not increase: As a result, the former condition is not only sufficient but also necessary for the optimality of multiple prizes when $u''' = 0$, as seen in Corollary 4. One may conjecture that with strictly prudent contestants, multiple prizes can be optimal even if the condition established in Proposition 3 is not met, in which case a second prize does not amplify the marginal benefit of effort, but strictly reduces marginal cost. With CARA utility, we obtain the following result.

Corollary 5. ***(Optimal prize allocation with CARA utility)*** *Suppose that contestants' utility exhibits constant absolute risk aversion of $\alpha > 0$. The effort-maximizing contest sets multiple prizes if $\frac{u'(w-e_s)}{u'(w+V-e_s)} > \frac{N\mu_1}{\mu_1[1-\exp(-\alpha V)]+\mu_2[N-1+\exp(-\alpha V)]}$.*[33]

It is straightforward to verify that the right-hand side of the above condition, $N\mu_1/\{\mu_1[1-\exp(-\alpha V)] + \mu_2[N-1+\exp(-\alpha V)]\}$, strictly falls below the ratio μ_1/μ_2. Multiple prizes emerge in an optimum even if a shift of prize money from a single top prize to the second prize reduces marginal benefit of effort. Prudence plays a nontrivial role by decreasing marginal cost of effort. This complements the effect of risk aversion in catalyzing an optimum with multiple prizes.

[31] As previously mentioned in the main text, simulation shows that contestants' expected utility is concave in his effort entry for $r \leq 1$, and thus a unique symmetric pure-strategy equilibrium exists.

[32] Note that $\mu_6 = 1 - \sum_{g=0}^{5} \frac{1}{10-g} \approx 0.1544 > 0$ and $\mu_7 = 1 - \sum_{g=0}^{6} \frac{1}{10-g} \approx -0.0956 < 0$. Therefore, $N_p = 6$.

[33] It is useful to emphasize that the condition stated in Corollary 5 is not only sufficient but also necessary for the optimality of a multi-prize contest in the case of $N = 3$. See Online Appendix B for the proof.

5.3.3. Nonseparable utility vs. separable utility

Glazer and Hassin (1988), Krishna and Morgan (1998), Akerlof and Holden (2012), and Drugov and Ryvkin (2019) assume that individual's utility is additive and separable in her income and cost of effort, and the utility function in expression (1) becomes

$$\sum_{m=1}^{N}\left[P_m(e',e)\times u(w+V_m)\right]-e'.$$

Schroyen and Treich (2016) interpret such a setting as an account of an "ability contest" in which contestants engage in nonmonetary efforts to compete for monetary prizes. With a separable utility function in place, the symmetric pure-strategy equilibrium is given by

$$\frac{r}{Ne}\times\sum_{m=1}^{N}[\mu_m\times u(w+V_m)]=1. \tag{8}$$

In contrast to Equation (4), a contestant's marginal cost of effort is independent of the prize schedule with additive and separable utility functions. In other words, a variation in the prize structure affects only the marginal benefit of effort, while the aforementioned effect triggered by contestants' prudence on marginal cost of effort would disappear.

Intuitively, an optimum no longer factors in the third-order property of the utility function, which nullifies the role played by prudence. An additional prize boosts the performance of the contest only if it enlarges the marginal benefit. Our analysis thus sheds light on this setting, and the results can readily be adapted. Proposition 1 is preserved under separable utility: It espouses a consecutive and monotone prize series for an optimum, as the monotonicity serves to improve marginal benefit of effort regardless. Results in parallel to Propositions 2 and 3 can also be established immediately by removing their requirement regarding prudence.

To better illustrate the implications of (non)separability (and prudence), we take a closer look at prize allocation under this alternative setup. Define N_p^s as

$$N_p^s := \max\left\{j=1,\ldots,N\,\middle|\,\mu_j>0\right\}. \tag{9}$$

The following result can be obtained:

Proposition 6. ***(Maximum number of prizes under separable utility)*** *The number of positive prizes in an optimal contest is no greater than* N_p^s.

Recall that the optimal number of prizes cannot exceed N_p in the case of nonseparable utility, where N_p is given by (6). It is straightforward to observe that the upper bound N_p degenerates to N_p^s when the measure of contestants' prudence—i.e., τ—reduces to zero. Under separable utility, the role of prudence fades away, and thus the condition would no longer involve τ.

Proposition 6, together with Proposition 4, unveils the implications of separability in contestants' utility function on prize allocation. Proposition 6 indicates that a larger number of prizes can be awarded in an optimum under nonseparable utility vis-à-vis under separable utility. In the latter case, a positive prize V_j can be desirable only if it contributes to contestants' marginal benefit of effort, i.e., $\mu_j>0$. This requirement, however, can be relaxed under nonseparable utility, as a prize for low rank may serve to reduce marginal effort cost, despite the direct negative incentive it creates. The following example demonstrates such a possibility.

Example 4. (Nonseparable utility vs. separable utility) Suppose that $u(c)=1-\exp(-\alpha c)$, $N=5$, $r=1$, $w=0$, and $V=1$. By expression (9), the maximum number of positive prizes $N_p^s=3$ independent of contestants' degree of risk aversion α. The optimal prize series $(V_1^*,\ldots,V_5^*)$ for different values of α and model specifications are reported as follows:

α	utility	N_p or N_p^s	number of prizes	V_1^*	V_2^*	V_3^*	V_4^*	V_5^*
$\alpha_1=1$	nonseparable	$N_p=3$	2	0.6568	0.3432	0.0000	0.0000	0.0000
$\alpha_1=1$	separable	$N_p^s=3$	2	0.6873	0.3127	0.0000	0.0000	0.0000
$\alpha_2=3$	nonseparable	$N_p=4$	3	0.4415	0.3584	0.2001	0.0000	0.0000
$\alpha_2=3$	separable	$N_p^s=3$	3	0.5201	0.3952	0.0847	0.0000	0.0000
$\alpha_3=5$	nonseparable	$N_p=4$	4	0.3795	0.3369	0.2607	0.0229	0.0000
$\alpha_3=5$	separable	$N_p^s=3$	3	0.4454	0.3705	0.1841	0.0000	0.0000

Recall that $\tau=\alpha$ with CARA preferences. In the case of $\alpha=5$, the optimal number of prizes reaches its maximum, N_p or N_p^s, irrespective of nonseparable or separable utility. A positive prize for the 4th rank yields a direct negative incentive because $\mu_4\approx-0.2833<0$: It is impossible under separable utility, whereas it could arise under nonseparable utility. The loss caused to marginal benefit of effort is compensated for by reduced marginal effort cost when contestants become sufficiently prudent.

6. Concluding remarks

Both winner-take-all and multi-prize contests are often observed in practice. A growing strand of literature has examined optimal prize allocation in various economic contexts (see Sisak, 2009). Clark and Riis (1998b) and Schweinzer and Segev (2012) establish the winner-take-all principle for risk-neutral players within the framework of multi-winner nested Tullock contests. In this paper, we introduce risk aversion into this framework. We demonstrate that in contrast to the case of risk-neutral contestants, a variation in prize allocation affects both the marginal benefit and marginal cost of effort, and contestants' risk attitude reshapes the trade-off. The analysis provides sufficient conditions under which a multi-prize or a single-prize contest emerges in an optimum, assuming that the unique symmetric equilibrium is played. It is shown that a multi-prize contest is more likely to outperform a single-prize one when contestants become more risk averse and more prudent. In particular, prudence plays a nontrivial role that allows additional prizes to reduce marginal effort cost, which makes a low-rank prize possible even if it creates direct negative incentive.

Our study leaves large room for future extension. In this paper, we abstract away contestants' ex ante participation decisions and focus on a setting in which all contestants participate with zero entry cost. It would be interesting to endogenize contestants' entry with the presence of risk aversion. Intuitively, multiple prizes provide insurance against a contestant's income shock upon entry, and a natural conjecture is that multiple prizes would be more appealing when contestants bear an entry cost and decide whether to enter the contest in the first place.

Further, our paper assumes homogeneous players, which enables a tractable analysis but limits the scope of the study. Contenders in real-world competitive activities typically differ in their abilities and/or preferences. Heterogeneity could sensitively affect strategic behavior (e.g., Chen et al., 2017) and, ultimately, the optimal choice of contest structure. We provide a useful benchmark analysis for noisy multi-prize contests. However, the setting does not address the various concerns that could be pertinent for a designer in an environment of heterogeneous contestants. For example, Archak and Sundararajan (2009) consider contest design that maximizes the highest efforts; Fu and Wu (2020) maximize the expected winner's effort; March and Sahm (2018) focus on the selection efficiency of the contest. An analysis of multi-prize contests with heterogeneous risk-averse contestants is analytically challenging and beyond the scope of this study, but merits attempting in future research.

Another possible avenue for future research is to extend our analysis in the static model to a dynamic setting, in which contestants must survive successive elimination for advancement toward the top, i.e., sequential-elimination contests (e.g., Rosen, 1986; Gradstein and Konrad, 1999; Fu and Lu, 2012a, among others). The hierarchical winner-take-all principle—which requires that the entire prize purse be allocated to a single grand prize—deserves to be reexamined in an environment with risk-averse contestants. It would also be intriguing to explore whether our main results would persist under alternative or more general static winner selection mechanisms—for instance, the multi-prize "reverse" nested lottery contest proposed by Fu et al. (2014). Finally, we model contestants' risk attitude by introducing risk aversion, while the impact of risk attitude can be examined from alternative perspectives, such as preferences with loss aversion (e.g., Chen et al., 2017). It is worthwhile to compare the implications of these two approaches—i.e., risk aversion vs. loss aversion—on optimal prize structures. We leave the exploration of these possibilities to future research.

Appendix A

Proof of Theorem 1

Proof. It is useful to state several intermediate results.

Lemma 1. ***(Fu et al., 2021)*** *Suppose that Assumptions 1 and 3 are satisfied and $e > 0$. Then $\frac{\partial^2 \pi(e', e)}{\partial e'^2} < 0$ for all $r \in (0, 1]$.*

Lemma 2. *Suppose that Assumptions 1 and 3 are satisfied. Then*

$$[u(x) - u(y)] \times \left[u''(x) - u''(y)\right] \geq \left[u'(y) - u'(x)\right]^2, \textit{ for all } x > y.$$

Proof. Because $-u''(c)/u'(c)$ is nonincreasing in c, we have that $\frac{d}{dc}\left(-\frac{u''(c)}{u'(c)}\right) \leq 0$, which is equivalent to

$$u'(t)u'''(t) \geq \left[-u''(t)\right]^2. \tag{10}$$

This implies instantly that $u'''(\cdot) \geq 0$. Moreover, fixing $x > y$, we have that

$$[u(x) - u(y)] \times \left[(u''(x) - u''(y)\right] = \left(\int_y^x u'(t)dt\right)\left(\int_y^x u'''(t)dt\right)$$

$$\geq \left(\int_y^x \sqrt{u'(t)u'''(t)}dt \right)^2$$

$$\geq \left(\int_y^x -u''(t)dt \right)^2 = \left[u'(y) - u'(x)\right]^2,$$

where the first inequality follows from Cauchy's inequality and the second inequality follows directly from (10). This completes the proof. □

Now we can prove Theorem 1. Clearly, $\boldsymbol{e} = (0, \ldots, 0)$ cannot constitute an equilibrium; together with Lemma 1 and the first-order condition (4), it suffices to show that for all $r \leq \bar{r}$, there exists a unique solution to

$$F(e; \boldsymbol{V}) := r \times \sum_{m=1}^{N} [\mu_m \times u(w + V_m - e)] - e \times \sum_{m=1}^{N} u'(w + V_m - e) = 0.$$

Existence of symmetric equilibrium We first show that $F(0; \boldsymbol{V}) > 0$. Note that μ_m is strictly decreasing in m and $\sum_{m=1}^{N} \mu_m = 0$. Define $\kappa := \max\{j = 1, \ldots, N \mid \mu_j \geq 0\}$. It can be verified that κ is well defined, unique, and $\kappa \leq N - 1$. Moreover, we have that

$$\begin{aligned} F(0; \boldsymbol{V}) = r \times \sum_{m=1}^{N} [\mu_m \times u(w + V_m)] &= r \times \left\{ \sum_{m=1}^{\kappa} [\mu_m \times u(w + V_m)] + \sum_{m=\kappa+1}^{N} [\mu_m \times u(w + V_m)] \right\} \\ &\geq r \times \left\{ \sum_{m=1}^{\kappa} [\mu_m \times u(w + V_\kappa)] + \sum_{m=\kappa+1}^{N} [\mu_m \times u(w + V_{\kappa+1})] \right\} \\ &= r \times \left(\sum_{m=1}^{\kappa} \mu_m \right) \times [u(w + V_\kappa) - u(w + V_{\kappa+1})] \geq 0. \end{aligned}$$

Note that the equal sign in the above two inequalities occurs simultaneously if and only if $V_1 = \cdots = V_N$, which is excluded by assumption. Therefore, we have that $F(0; \boldsymbol{V}) > 0$.

Next, we show that $F(e; \boldsymbol{V}) < 0$ for sufficiently large e. Note that $F(e; \boldsymbol{V})$ can be bounded above by

$$\begin{aligned} F(e; \boldsymbol{V}) :&= r \times \sum_{m=1}^{N} [\mu_m \times u(w + V_m - e)] - e \times \sum_{m=1}^{N} u'(w + V_m - e) \\ &= r \times \sum_{m=1}^{N-1} \left\{ \left(\sum_{j=1}^{m} \mu_j \right) \times [u(w + V_m - e) - u(w + V_{m+1} - e)] \right\} - e \times \sum_{m=1}^{N} u'(w + V_m - e) \\ &\leq r \times \sum_{m=1}^{N-1} \left\{ \left(\sum_{j=1}^{m} \mu_j \right) \times (V_m - V_{m+1}) \times u'(w + V_{m+1} - e) \right\} - e \times \sum_{m=1}^{N} u'(w + V_m - e) \\ &\leq \sum_{m=1}^{N-1} \left[rNV \times u'(w + V_{m+1} - e) \right] - e \times \sum_{m=1}^{N} u'(w + V_m - e) \\ &= -e \times u'(w + V_1 - e) + (rNV - e) \times \sum_{m=2}^{N} u'(w + V_m - e), \end{aligned}$$

where the second equality follows from $\sum_{m=1}^{N} \mu_m = 0$; the first inequality follows from the concavity of $u(\cdot)$ and the fact that $\sum_{j=1}^{m} \mu_j \geq 0$ for all $m \in \{1, \ldots, N-1\}$; the second inequality follows from $V_m - V_{m+1} \leq V$ and $\sum_{j=1}^{m} \mu_j \leq N$. It is clear that the last term is negative if $e > rNV$. Therefore, there exists at least one symmetric equilibrium of the contest game.

Uniqueness of symmetric equilibrium Next, we prove the uniqueness of symmetric equilibrium. It suffices to show that if $F(e_0; \boldsymbol{V}) = 0$ for some $e_0 > 0$, then we must have $\frac{\partial F(e_0; \boldsymbol{V})}{\partial e} < 0$ (see Treich, 2010). For notational convenience, let us denote $\sum_{j=m+1}^{n} \mu_j$ by $\widetilde{\mu}_m$ for all $m \in \{1, \ldots, N-1\}$. It is straightforward to verify that $\widetilde{\mu}_m < 0$. Then $F(e_0; \boldsymbol{V}) = 0$ can be rewritten as

$$\begin{aligned}&\sum_{m=1}^{N-1}(N-m)\left[u'(w+V_{m+1}-e_0)-u'(w+V_m-e_0)\right]+Nu'(w+V_1-e_0)\\&=\frac{r}{e_0}\times\sum_{m=1}^{N-1}\widetilde{\mu}_m\left[u(w+V_{m+1}-e_0)-u(w+V_m-e_0)\right].\end{aligned}\tag{11}$$

Moreover, we have that

$$\begin{aligned}\left.\frac{\partial F(e;\boldsymbol{V})}{\partial e}\right|_{e=e_0}&=-r\times\sum_{m=1}^{N-1}\widetilde{\mu}_m\left[u'(w+V_{m+1}-e_0)-u'(w+V_m-e_0)\right]\\&\quad+e_0\times\sum_{m=1}^{N-1}(N-m)\left[u''(w+V_{m+1}-e_0)-u''(w+V_m-e_0)\right]+Ne_0u''(w+V_1-e_0)\\&\quad-\left\{\sum_{m=1}^{N-1}(N-m)\left[u'(w+V_{m+1}-e_0)-u'(w+V_m-e_0)\right]+Nu'(w+V_1-e_0)\right\}\\&=-r\times\sum_{m=1}^{N-1}\widetilde{\mu}_m\left[u'(w+V_{m+1}-e_0)-u'(w+V_m-e_0)\right]\\&\quad+e_0\times\sum_{m=1}^{N-1}(N-m)\left[u''(w+V_{m+1}-e_0)-u''(w+V_m-e_0)\right]\\&\quad-\frac{r}{e_0}\times\sum_{m=1}^{N-1}\widetilde{\mu}_m\left[u(w+V_{m+1}-e_0)-u(w+V_m-e_0)\right]+Ne_0u''(w+V_1-e_0)\\&\le-\sum_{m=1}^{N-1}\left\{\begin{array}{c}-r\widetilde{\mu}_m\left[u'(w+V_m-e_0)-u'(w+V_{m+1}-e_0)\right]\\+e_0(N-m)\left[u''(w+V_m-e_0)-u''(w+V_{m+1}-e_0)\right]\\-\frac{r}{e_0}\widetilde{\mu}_m\left[u(w+V_m-e_0)-u(w+V_{m+1}-e_0)\right]\end{array}\right\},\end{aligned}$$

where the second equality follows directly from (11), and the inequality follows from $u''\le 0$. Therefore, it suffices to show that for all $m\in\{1,\dots,N-1\}$ and $r\le\bar{r}$, the following inequality holds:

$$\left[\begin{array}{c}e_0(N-m)\left[u''(w+V_m-e_0)-u''(w+V_{m+1}-e_0)\right]\\+\frac{r}{e_0}(-\widetilde{\mu}_m)\left[u(w+V_m-e_0)-u(w+V_{m+1}-e_0)\right]\end{array}\right]\ge r(-\widetilde{\mu}_m)\left[\begin{array}{c}u'(w+V_{m+1}-e_0)\\-u'(w+V_m-e_0)\end{array}\right].$$

Recall that $\widetilde{\mu}_m<0$. Therefore, we have that

$$\begin{aligned}&e_0(N-m)\left[u''(w+V_m-e_0)-u''(w+V_{m+1}-e_0)\right]+\frac{r}{e_0}(-\widetilde{\mu}_m)\left[u(w+V_m-e_0)-u(w+V_{m+1}-e_0)\right]\\&\ge 2\sqrt{r(N-m)(-\widetilde{\mu}_m)}\sqrt{\left[u''(w+V_m-e_0)-u''(w+V_{m+1}-e_0)\right]\times\left[u(w+V_m-e_0)-u(w+V_{m+1}-e_0)\right]}\\&\ge 2\sqrt{r(N-m)(-\widetilde{\mu}_m)}\left[u'(w+V_{m+1}-e_0)-u'(w+V_m-e_0)\right],\end{aligned}$$

where the first inequality follows from the AM-GM inequality, and the second inequality follows from Lemma 2. To complete the proof of uniqueness of symmetric equilibrium, it suffices to show that

$$2\sqrt{r(N-m)(-\widetilde{\mu}_m)}\ge r(-\widetilde{\mu}_m),$$

which is equivalent to

$$r(-\widetilde{\mu}_m)\le 4(N-m).$$

The above condition holds for all $m\in\{1,\dots,N-1\}$ under Assumption 2. To see this more clearly, recall that

$$r\le\bar{r}\equiv\min\left\{1,\frac{4}{\sum_{g=2}^{N}\frac{1}{g}}\right\}\le\frac{4}{\sum_{g=2}^{N}\frac{1}{g}}.$$

This in turn implies that

$$r(-\widetilde{\mu}_m) \le \frac{4}{\sum_{g=2}^{N}\frac{1}{g}}(-\widetilde{\mu}_m) \equiv -\frac{4}{\sum_{g=2}^{N}\frac{1}{g}}\sum_{j=m+1}^{N}\mu_j$$
$$\le -\frac{4}{\sum_{g=2}^{N}\frac{1}{g}}\sum_{j=m+1}^{N}\mu_N$$
$$= -\frac{4}{\sum_{g=2}^{N}\frac{1}{g}}(N-m)\mu_N = 4(N-m),$$

where the second inequality follows from the fact that $\mu_m \ge \mu_N$ for all $m \in \mathcal{N}$, and the last equality follows from $\mu_N \equiv 1-\sum_{g=0}^{N-1}\frac{1}{N-g} = -\sum_{g=2}^{N}\frac{1}{g}$. This concludes the proof. □

Proof of Proposition 1

Proof. Denote the optimal prize allocation by $\boldsymbol{V}^* := (V_1^*, \ldots, V_N^*)$ and the equilibrium effort level by e^*. Suppose to the contrary that there exists some $j \in \{2, \ldots, N\}$ such that $V_j^* = V_{j-1}^*$. Consider an alternative prize structure $\boldsymbol{V}^\dagger := (V_1^\dagger, \ldots, V_N^\dagger)$, with $V_m^\dagger = V_m^*$ for $m \in \mathcal{N}\setminus\{j-1, j\}$, $V_{j-1}^\dagger = V_{j-1}^* + \epsilon$, and $V_j^\dagger = V_j^* - \epsilon$. Next, we show that the equilibrium effort under $\boldsymbol{V}^\dagger$ is greater than that under $\boldsymbol{V}^*$ for an infinitesimal $\epsilon > 0$. It suffices to show that $H(\epsilon) := F(e^*; \boldsymbol{V}^\dagger) - F(e^*; \boldsymbol{V}^*) > 0$. Simple algebra yields that

$$H(\epsilon) = r \times \left\{\mu_{j-1}\left[u(w+V_j^*+\epsilon-e^*) - u(w+V_j^*-e^*)\right] + \mu_j\left[u(w+V_j^*-\epsilon-e^*) - u(w+V_j^*-e^*)\right]\right\}$$
$$- e^* \times \left\{\left[u'(w+V_j^*+\epsilon-e^*) - u'(w+V_j^*-e^*)\right] + \left[u'(w+V_j^*-\epsilon-e^*) - u'(w+V_j^*-e^*)\right]\right\}.$$

Clearly, $H(0) = 0$. Moreover, we have

$$H'(0) = r \times \left(\mu_{j-1} - \mu_j\right) \times u'(w+V_j^*-e^*) > 0.$$

Therefore, $H(\epsilon) > 0$ for sufficiently small $\epsilon > 0$. This completes the proof. □

Proof of Proposition 2

Proof. Recall that the optimal prize allocation and equilibrium effort level are denoted by $\boldsymbol{V}^* := (V_1^*, \ldots, V_N^*)$ and e^* in the proof of Proposition 1, respectively. Suppose to the contrary that a contest with multiple prizes is optimal. Then there exists a prize $j \in \{2, \ldots, N\}$ such that $V_j^* > 0$. Consider the following alternative prize allocation, denoted by $\widehat{\boldsymbol{V}}$, that decreases V_j^* by a small amount $\epsilon > 0$, and increases V_1^* by the same amount. Next, we show that the constructed prize allocation generates more revenue to the contest designer. By the same argument as in the proof of Proposition 1, it suffices to show that $\zeta(\epsilon) := F(e^*; \widehat{\boldsymbol{V}}) - F(e^*; \boldsymbol{V}^*) > 0$. Clearly, $\zeta(0) = 0$. Carrying out the algebra, we have that

$$\zeta(\epsilon) = r \times \left\{\mu_1\left[u(w+V_1^*+\epsilon-e^*) - u(w+V_1^*-e^*)\right] + \mu_j\left[u(w+V_j^*-\epsilon-e^*) - u(w+V_j^*-e^*)\right]\right\}$$
$$- e^* \times \left\{\left[u'(w+V_1^*+\epsilon-e^*) - u'(w+\widehat{V}_1-e^*)\right] + \left[u'(w+V_j^*-\epsilon-e^*) - u'(w+V_j^*-e^*)\right]\right\},$$

and thus

$$\zeta'(0) = r \times \left[\mu_1 u'(w+V_1^*-e^*) - \mu_j u'(w+V_j^*-e^*)\right]$$
$$- e^* \times \left[u''(w+V_1^*-e^*) - u''(w+V_j^*-e^*)\right] \tag{12}$$

By Proposition 1, we have that $V_1^* > V_j^*$; together with $-u'''/u'' \le \tau$, we have $u''' \le -\tau u''$ and thus

$$u''(w+V_1^*-e^*) - u''(w+V_j^*-e^*) = \int_{w+V_j^*-e^*}^{w+V_1^*-e^*} u'''(e)de$$
$$\le -\tau \int_{w+V_j^*-e^*}^{w+V_1^*-e^*} u''(e)de$$
$$= -\tau\left[u'(w+V_1^*-e^*) - u'(w+V_j^*-e^*)\right]. \tag{13}$$

Combining (12) and (13) yields that

$$\begin{aligned}
\zeta'(0) &\geq (r\mu_1 + e^*\tau) \times u'(w + V_1^* - e^*) - (r\mu_j + e^*\tau) \times u'(w + V_j^* - e^*) \\
&\geq (r\mu_1 + e^*\tau) \times u'(w + V_1^* - e^*) - (r\mu_2 + e^*\tau) \times u'(w + V_j^* - e^*) \\
&= (r\mu_2 + e^*\tau) \times u'(w + V_1^* - e^*) \times \left[\frac{r\mu_1 + e^*\tau}{r\mu_2 + e^*\tau} - \frac{u'(w + V_j^* - e^*)}{u'(w + V_1^* - e^*)} \right] \\
&> (r\mu_2 + e^*\tau) \times u'(w + V_1^* - e^*) \times \left[\frac{r\mu_1 + e^*\tau}{r\mu_2 + e^*\tau} - \frac{u'(w - e^*)}{u'(w + V - e^*)} \right] \\
&> (r\mu_2 + e^*\tau) \times u'(w + V_1^* - e^*) \times \left[\frac{r\mu_1 + \frac{V}{N}\tau}{r\mu_2 + \frac{V}{N}\tau} - \frac{u'(w - \frac{V}{N})}{u'(w + V)} \right] \geq 0,
\end{aligned}$$

where the second inequality follows from $\mu_2 \geq \mu_j$ for $j \geq 2$ and $u' > 0$; the third inequality follows from $u'' \leq 0$, $V_1^* < V$, and $V_j^* > 0$; the fourth inequality follows from $e^* \in (0, \frac{V}{N})$; and the last inequality follows from the condition assumed in Proposition 2. This completes the proof. □

Proof of Proposition 3

Proof. Note that when $N \geq 3$, we have $\mu_1 > \mu_2 > 0$. Now consider introducing a small second prize $\epsilon > 0$, and denote the corresponding prize structure by $\widetilde{\boldsymbol{V}} := (\widetilde{V}_1, \ldots, \widetilde{V}_N) = (V - \epsilon, \epsilon, 0, \ldots, 0)$. Following the same argument as in the proof of Proposition 1, it suffices to show that $G(\epsilon) := F(e_s; \widetilde{\boldsymbol{V}}) - F(e_s; \widetilde{\boldsymbol{V}}) > 0$. Carrying out the algebra, we have that

$$\begin{aligned}
G(\epsilon) =& r \times \{\mu_1 [u(w + V - \epsilon - e_s) - u(w + V - e_s)] + \mu_2 [u(w + \epsilon - e_s) - u(w - e_s)]\} \\
&- e_s \times \left\{ \left[u'(w + V - \epsilon - e_s) - u'(w + V - e_s)\right] + \left[u'(w + \epsilon - e_s) - u'(w - e_s)\right] \right\}.
\end{aligned}$$

It is straightforward to see that $G(0) = 0$. Moreover, we have

$$\begin{aligned}
G'(0) &= r \times \left[-\mu_1 u'(w + V - e_s) + \mu_2 u'(w - e_s)\right] + e_s \times \left[u''(w + V - e_s) - u''(w - e_s)\right] \\
&\geq r \times \left[-\mu_1 u'(w + V - e_s) + \mu_2 u'(w - e_s)\right] \\
&= r\mu_1 \times u'(w + V - e_s) \times \left[\frac{\mu_2}{\mu_1} \times \frac{u'(w - e_s)}{u'(w + V - e_s)} - 1 \right] > 0,
\end{aligned}$$

where the first inequality follows from $u''' \geq 0$, the second inequality follows from $\frac{u'(w-e_s)}{u'(w+V-e_s)} > \frac{\mu_1}{\mu_2}$. This completes the proof. □

Proof of Proposition 4

Proof. Suppose to the contrary that there exists a prize indexed by $k > N_p$ such that $V_k^* > 0$. By the definition of N_p, we must have

$$r \times \mu_k + \frac{V}{N}\tau \leq 0. \tag{14}$$

Now consider an alternative prize structure that decreases V_k^* and increases V_1^* by a small amount $\epsilon > 0$. Denote the constructed prize allocation by $\boldsymbol{V}^{\dagger\dagger}$. Next, we show that the equilibrium effort level under $\boldsymbol{V}^{\dagger\dagger}$ is greater than that under $\boldsymbol{V}^*$. Again, it suffices to show that $\chi(\epsilon) := F(e^*; \boldsymbol{V}^{\dagger\dagger}) - F(e^*; \boldsymbol{V}^*) > 0$ for sufficiently small $\epsilon > 0$. It can be verified that $\chi(0) = 0$. Carrying out the algebra, we have that

$$\begin{aligned}
\chi(\epsilon) =& r \times \left\{\mu_1 \left[u(w + V_1^* + \epsilon - e^*) - u(w + V_1^* - e^*)\right] + \mu_k \left[u(w + V_k^* - \epsilon - e^*) - u(w + V_k^* - e^*)\right]\right\} \\
&- e^* \times \left\{ \left[u'(w + V_1^* + \epsilon - e^*) - u'(w + V_1^* - e^*)\right] + \left[u'(w + V_k^* - \epsilon - e^*) - u'(w + V_k^* - e^*)\right] \right\},
\end{aligned}$$

and

$$\chi'(0) = r \times \left[\mu_1 u'(w + V_1^* - e^*) - \mu_k u'(w + V_k^* - e^*)\right] - e^* \times \left[u''(w + V_1^* - e^*) - u''(w + V_k^* - e^*)\right] \tag{15}$$

Note that $-\frac{u'''(c)}{u''(c)} \leq \tau$ implies

$$u''(w + V_1^* - e^*) - u''(w + V_k^* - e^*) \leq -\tau \left[u'(w + V_1^* - e^*) - u'(w + V_k^* - e^*)\right]. \tag{16}$$

Combining (15) and (16) yields that

$$\begin{aligned}\chi'(0) &\geq (r\mu_1 + e^*\tau) \times u'(w + V_1^* - e^*) - (r\mu_k + e^*\tau) \times u'(w + V_k^* - e^*)\\ &\geq (r\mu_1 + e^*\tau) \times u'(w + V_1^* - e^*) - (r\mu_k + \frac{V}{N}\tau) \times u'(w + V_k^* - e^*)\\ &\geq (r\mu_1 + e^*\tau) \times u'(w + V_1^* - e^*) > 0,\end{aligned}$$

where the second inequality follows from $e^* \leq \frac{V}{N}$, and the third inequality follows from (14) and $u' > 0$. This completes the proof. □

Proof of Proposition 5

Proof. For notational convenience, let us define $z := e^r$, $z' := (e')^r$, and

$$P_m^{\dagger}\left(z', z\right) := \frac{(N-1)!}{(N-m)!} \times \left(\prod_{j=1}^{m-1} \frac{z}{(N-j)z + z'}\right) \times \frac{z'}{(N-m)z + z'}.$$

It follows immediately that $P_m\left(e', e\right) = P_m^{\dagger}\left(z', z\right)$. Next, let us define

$$\widetilde{P}_m(e', e) := \sum_{j=1}^{m} P_j(e', e).$$

In words, $\widetilde{P}_m(e', e)$ is an indicative contestant's probability of obtaining the first m prizes when he exerts effort e' and all other contestants exert effort e. An indicative contestant's expected payoff [i.e., Equation (1)] can then be rewritten as

$$\pi(e', e) = \sum_{m=1}^{N-1} \widetilde{P}_m(e', e)\left[u(w + V_m - e') - u(w + V_{m+1} - e')\right] + u(w + V_N - e'). \tag{17}$$

Similarly, we can define $\widetilde{P}_m^{\dagger}(z', z)$ as

$$\widetilde{P}_m^{\dagger}(z', z) := \sum_{j=1}^{m} P_j^{\dagger}(z', z).$$

It can be verified that

$$1 - \widetilde{P}_m^{\dagger}(z', z) = \frac{N!}{(N-m)!} \times \left(\prod_{j=0}^{m-1} \frac{z}{z' + (N-1)z - jz}\right). \tag{18}$$

The following lemmata can be obtained:

Lemma 3. ***(Fu et al., 2021)*** $\frac{\partial^2 \widetilde{P}_m^{\dagger}(z', z)}{\partial z'^2} \leq 0$ *for all* $m \in \mathcal{N}$.

Lemma 4. *Suppose that* $r \leq 1$. *Then* $\frac{\partial^2 \widetilde{P}_m(e', e)}{\partial e'^2} \leq 0$ *for all* $m \in \mathcal{N}$.

Proof. Carrying out the algebra, we have that

$$\frac{\partial^2 \widetilde{P}_m(e', e)}{\partial e'^2} = \frac{\partial^2 \widetilde{P}_m^{\dagger}(z', z)}{\partial z'^2} r^2 (e')^{2r-2} + \frac{\partial \widetilde{P}_m^{\dagger}(z', z)}{\partial z'} r(r-1)(e')^{r-2} \leq 0,$$

where the inequality follows from Lemma 3 and $r \in (0, 1]$. This competes the proof. □

Now we can prove Proposition 5. Without any loss of generality, we can suppose that contestants' utility function is $u(c) = c - \frac{\gamma}{2}c^2$, with $\gamma > 0$. Note that $u'(w + V) \geq 0$ implies $w + V \leq 1/\gamma$. It is straightforward to verify that the first-order condition (4) has a unique positive solution with quadratic utility functions. From the proof of Theorem 1, it suffices to show that $\frac{\partial^2 \pi(e', e)}{\partial e'^2} \leq 0$ for all $r \leq \frac{1}{2(N-1)^2 + 1}$.

For notational convenience, let us define $d_m := V_m - V_{m+1} \geq 0$, with $m \in \{1, \ldots, N-1\}$. It follows from Equation (17) that

$$\begin{aligned}\frac{\partial^2 \pi(e',e)}{\partial e'^2} &= \sum_{m=1}^{N-1}\left\{d_m \frac{\partial^2 \widetilde{P}_m(e',e)}{\partial e'^2}\left[1-\frac{\gamma}{2}\left(2w+V_{m+1}+V_m-2e'\right)\right]+2\gamma\frac{\partial \widetilde{P}_m(e',e)}{\partial e'}d_m\right\}-\gamma \\ &\le \sum_{m=1}^{N-1}\left[d_m \frac{\partial^2 \widetilde{P}_m(e',e)}{\partial e'^2}\left\{1-\frac{\gamma}{2}\left[V_{m+1}+V_m-2\left(V_1-\frac{1}{\gamma}\right)\right]\right\}+2\gamma\frac{\partial \widetilde{P}_m(e',e)}{\partial e'}d_m\right]-\gamma \\ &= \frac{\gamma}{2}\times\left\{\sum_{m=1}^{N-1}\left[\frac{\partial^2 \widetilde{P}_m(e',e)}{\partial e'^2}d_m\left(2\sum_{j=1}^{m-1}d_j+d_m\right)+4\frac{\partial \widetilde{P}_m(e',e)}{\partial e'}d_m\right]-2\right\} \\ &\le \frac{\gamma}{2}\times\sum_{m=1}^{N-1}\left[\frac{\partial^2 \widetilde{P}_m(e',e)}{\partial e'^2}(d_m)^2+4\frac{\partial \widetilde{P}_m(e',e)}{\partial e'}d_m-\frac{2}{N-1}\right],\end{aligned}$$

where the first inequality follows from Lemma 4 and $w-e'\le\frac{1}{\gamma}-V_1$; and the second inequality follows again from Lemma 4 and $d_m\ge 0$. Therefore, it suffices to show that for all $m\in\{1,\ldots,N-1\}$,

$$\frac{\partial^2 \widetilde{P}_m(e',e)}{\partial e'^2}(d_m)^2+4\frac{\partial \widetilde{P}_m(e',e)}{\partial e'}d_m-\frac{2}{N-1}\le 0.$$

View the left-hand side of the above inequality as a function of d_m. Note that $\frac{\partial^2 P_m(e',e)}{\partial e'^2}\le 0$ by Lemma 4. Therefore, a sufficient condition for the above inequality to hold is

$$2\left[\frac{\partial \widetilde{P}_m(e',e)}{\partial e'}\right]^2+\frac{1}{N-1}\times\frac{\partial^2 \widetilde{P}_m(e',e)}{\partial e'^2}\le 0. \tag{19}$$

Note that

$$\frac{\partial \widetilde{P}_m(e',e)}{\partial e'}=\frac{\partial \widetilde{P}^{\dagger}_m(z',z)}{\partial z'}r(e')^{r-1}, \tag{20}$$

and

$$\begin{aligned}\frac{\partial^2 \widetilde{P}_m(e',e)}{\partial e'^2} &= \frac{\partial^2 \widetilde{P}^{\dagger}_m(z',z)}{\partial z'^2}r^2(e')^{2r-2}+\frac{\partial \widetilde{P}^{\dagger}_m(z',z)}{\partial z'}r(r-1)(e')^{r-2} \\ &\le -\frac{\partial \widetilde{P}^{\dagger}_m(z',z)}{\partial z'}r(1-r)(e')^{r-2},\end{aligned} \tag{21}$$

where the inequality follows from Lemma 3. Combining (19), (20), and (21), it suffices to show that

$$z'\frac{\partial \widetilde{P}^{\dagger}_m(z',z)}{\partial z'}\le\frac{1-r}{2r(N-1)}.$$

Next, note that the right-hand side of the above inequality can be bounded above by

$$\begin{aligned}z'\frac{\partial \widetilde{P}^{\dagger}_m(z',z)}{\partial z'} &= z'\widetilde{P}^{\dagger}_m(z',z)\frac{\partial \ln \widetilde{P}^{\dagger}_m(z',z)}{\partial z'} \\ &= z'\widetilde{P}^{\dagger}_m(z',z)\left(-\frac{\partial \ln\left(1-\widetilde{P}^{\dagger}_m(z',z)\right)}{\partial z'}\right) \\ &= z'\widetilde{P}^{\dagger}_m(z',z)\left[-\frac{\partial}{\partial z'}\left(\ln(N!)-\ln(N-m)!+m\ln z-\sum_{j=0}^{m-1}\ln\left(z'+(N-1)z-jz\right)\right)\right] \\ &= \widetilde{P}^{\dagger}_m(z',z)\sum_{j=0}^{m-1}\frac{z'}{z'+(N-1)z-jz}\le m\le N-1,\end{aligned}$$

where the third equality follows from (18); and the first inequality follows from the fact that $\widetilde{P}^{\dagger}_m(z',z)\le 1$ and $\frac{z'}{z'+(N-1)z-jz}\le 1$. The above inequality, together with $r\le\frac{1}{2(N-1)^2+1}$, implies instantly that

$$z'\frac{\partial \widetilde{P}^{\dagger}_m(z',z)}{\partial z'}\le N-1\le\frac{1-r}{2r(N-1)}.$$

This completes the proof. □

Proof of Proposition 6

Proof. With separable utility functions, denote the equilibrium effort under an arbitrary prize schedule $\boldsymbol{V} \equiv (V_1, \ldots, V_N)$ by $e^s(\boldsymbol{V})$. It follows from (8) that

$$e^s(\boldsymbol{V}) = \frac{r}{N} \times \sum_{m=1}^{N} [\mu_m \times u(w + V_m)].$$

Suppose there exists some $j \in \mathcal{N}/\{1\}$ such that $\mu_j \leq 0$ and $V_j > 0$. Consider an alternative prize structure $\widehat{\boldsymbol{V}}^s := (\widehat{V}_1^s, \ldots, \widehat{V}_N^s)$, with $\widehat{V}_m^s = V_m$ for $m \in \mathcal{N} \setminus \{1, j\}$, $\widehat{V}_1^s = V_1 + V_j$, and $\widehat{V}_j^s = 0$. It remains to show that the equilibrium effort under $\widehat{\boldsymbol{V}}^s$ is greater than that under $\boldsymbol{V}$. Carrying out the algebra, we have that

$$e^s(\widehat{\boldsymbol{V}}^s) - e^s(\boldsymbol{V}) = \frac{r}{N} \left\{\mu_1 \left[u(w + V_1 + V_j) - u(w + V_1)\right] + \mu_j \left[u(w) - u(w + V_j)\right]\right\} > 0,$$

where the strict inequality follows from $\mu_1 > 0 \geq \mu_j$, $V_1 + V_j > V_1 \geq V_j > 0$. This completes the proof. □

Appendix B. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.geb.2021.07.003.

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