Optimal Sorting in Team Tournaments*

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October 13, 2022

Abstract

This paper investigates the optimal formation of teams in a tournament. A manager sorts four workers—who differ in their productivity—into two teams. Workers on each team join force to produce team output, and one team wins a prize, e.g., bonus package. Two sorting patterns are possible: Positive sorting requires that each team consist of players of same caliber, while negative sorting does the opposite. We characterize the optimum with respect to relevant environmental factors, ranging from the degree of effort complementarity in the team production process and the curvature of effort cost functions, to the degree of worker heterogeneity. We further extend the model to allow the manager to set prize schedule for the workers on each team upon a win and allocate productive resources between teams.

Keywords: Team Tournaments; Tournament Design; Sorting; Heterogeneous Players; Prize Allocation; Resource Allocation.

JEL Classification Codes: C72; D72; M54; L23

*We thank Mikhail Drugov, Allen Vong, and participants at the University of Macau, the 21st annual SAET conference and the 2022 Conference on Mechanism and Institution Design for helpful discussions, suggestions, and comments. Fu thanks the Singapore Ministry of Education Tier-1 Academic Research Fund (R-313-000-139-115) for financial support. Wu thanks the National Natural Science Foundation of China (Nos. 72222002, 72173002, and 71803003), the Wu Jiapei Foundation of the China Information Economics Society (No. E21100383), and the seed fund of the School of Economics, Peking University, for financial support. Any errors are our own.

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1 Introduction

Teamwork is commonplace in modern firms (e.g., Che and Yoo, 2001; Kambhampati and Segura-Rodriguez, 2022). General Electric (GE) famously relied on an autonomous team at Durham to fulfill its colossal jet engine project; teamwork has since been promoted throughout the entire company, which was widely viewed as a key driver for GE's miraculous performance during Jack Welch's tenure as its CEO. Pixar Studio has vehemently promoted teamwork to foster collective creativity; Steve Jobs, for instance, deliberately remodeled the physical layout of the studio's office space to facilitate collaborations among peers of diverse expertise. Four Seasons is known to engage teamwork across departments to provide customers with integrated service solutions. As Deloitte's 2017 Global Human Capital Trend Survey indicates, more than thirty percent of the survey respondents operate primarily in teams.[1] Many firms strive to destruct hierarchical layers and functional lines, while reinventing themselves toward network of teams. The transformation has also been well underway even in the U.S. military, which has increasingly resorted to decentralized team operations since the Iraq War (see McChrystal, Collins, Silverman, and Fussell, 2015).

Although team productions have been broadly embraced as an efficient means to cultivate agility and productivity, management pundits and practitioners have endeavored to search for the approach to successful teamwork. Google, for instance, launched Project Aristotle in 2012 to study how to build "the perfect team." In this paper, we explore a natural question central to successful team management: How should a manager mix and match workers of diverse levels of productivity to boost the performance of an organization?

We investigate the optimal sorting of heterogeneous workers in a setting with two teams inside a firm to be ranked and rewarded by relative performance evaluation (RPE) scheme, which can intuitively be interpreted as a tournament. Internal competitions based on collective performance are prevalent. The Houston Independent School District (HISD) offered incentive pays to high schools based on their school-level value-added rank in each subject (see Imberman and Lovenheim, 2015). Birkinshaw (2001) documented a number of salient cases of internal competitions between units or divisions. In the early 1990s, Fuji Xerox, for instance, deliberately pitted two project teams against each other in developing next-generational copier technologies, and the steering committee selected one as the winner to be integrated into the new copier after one year. In Telstar, an IT infrastructure company, two teams developed competing middleware technology platforms in parallel, being aware that only one would be picked by the new business unit.

We construct a team tournament model to shed light on the question raised above. Analogous to Franco, Mitchell, and Vereshchagina (2011) and Kaya and Vereshchagina (2014),

[1] See https://www2.deloitte.com/content/dam/Deloitte/global/Documents/About-Deloitte/central-europe/ce-global-human-capital-trends.pdf.

production requires joint work of two workers, and four workers of different levels of productivity are to be sorted into two teams. Workers' ability or strength, measured by an effort cost parameter, can be either high or low, with two of each type. The game proceeds in two stages. A manager first forms the teams—i.e., sorting the workers into two teams—to maximize the aggregate output of the tournament. Two sorting patterns may arise. Under *positive sorting*, each team consists of homogeneous workers—i.e., two strong (low-cost) or two weak (high-cost) workers on each team; in contrast, under *negative sorting*, each team involves workers of different types. Teams compete in the second stage. Workers simultaneously contribute their efforts, and a CES production function converts individual efforts into team-level output. Workers on the winning team equally share the prize, e.g., the bonus packages.

The manager's choice is governed by a fundamental trade-off between (intra-team) production efficiency and (inter-team) competition. The former requires that the choice of sorting pattern address two typical issues that concern collective production: (i) efficient provision of incentives to workers; and (ii) efficient conversion of individual efforts into composite output. The conventional wisdom in economics tells that supermodular production function calls for positive sorting: Positive sorting—which pools workers of equal caliber into the same teams—effectively leverages effort complementarity. Sorting patterns generate diverging incentive effects in the case of strong effort complementary vis-à-vis that of weak complementarity. Positive sorting incentivizes the two strong (low-cost) workers more effectively when efforts are sufficiently complementary: Each expects a higher marginal return of his effort given the higher input contributed by the teammate. However, this effect tends to wane when the degree of complementarity declines: Negative sorting can gain its appeal when efforts are relatively less complementary (more substitutable), in which case a strong worker on each team can be disciplined from shirking when partnered with a weak teammate (negative sorting), as he is less able to freeride on the other.

The latter concern compels the manager to ensure sufficient competition under RPE. Positive sorting generates polarized distribution of talents and team output, causing a lopsided competition, with one strong team to confront a weak one. The unlevel playing field dilutes the competition, as the conventional wisdom of the tournament/contest literature would tell.

The optimal sorting pattern must strike a balance between the two aforementioned concerns, which depends complexly on the combination of the nature of production technology (the degree of effort complementarity), the convexity of effort cost functions, and the degree of worker heterogeneity. Our analysis fully characterizes the equilibrium under each sorting pattern, which enables us to compare resultant equilibrium total output and obtain the optimum. The main results can be summarized as follows.

First, we establish a unique threshold for the degree of effort complementarity, which is a function of the convexity of effort cost functions and the degree of workers' heterogeneity: Positive sorting emerges in the optimum when efforts are sufficiently complementary, i.e.,when the degree of effort complementarity exceeds that threshold; negative sorting arises otherwise, i.e., when efforts are relatively less complementary. This prediction evidences the fundamental trade-off delineated above.

Second, we show that positive sorting is more likely to emerge when effort cost functions become more convex. The convexity can intuitively be interpreted as a measure of the difficulty of the task. The result thus implies that a more difficult task tends to favor positive sorting.

Third, our framework allows us to generate lucid comparative statics with respect to workers' heterogeneity. Increasing worker heterogeneity affects both sides of the above-mentioned trade-off, which causes complications. We show that the comparative statics depend on both the nature of the production technology as well as the convexity of effort cost functions. Increasing heterogeneity favors positive sorting when efforts are sufficiently complementary and effort costs are moderately convex; in contrast, it renders negative sorting more likely when efforts are moderately complementary and efforts costs are strongly convex.

We explore two extended settings of practical relevance. The baseline model assumes that workers on each firm equally split the prize upon a win. The first extension allows the manager—with a fixed prize purse—to set a prize schedule that specifies the reward each worker on a team would receive when the team wins. It deserves to point out that this flexibility endogenizes not only the prevailing prize structure of the tournament, but also the organizational form of the team production process: She can dismantle the team structure by concentrating the prize purse on one worker, in which case only one worker remains active on a team. This scenario occurs when the degree of effort complementarity remains weak. The analysis shows that positive sorting is less likely to emerge in the optimum when the manager is allowed to set prize schedule endogenously.

Further, we assume that the manager is endowed with a fixed amount of productive resources and allocates the resources between teams. For instance, a pharmaceutical company may provide laboratory equipment or funding to research labs that race toward an innovative solution. The resources amplify a recipient team's productivity. The allocation, however, alters teams' relative competency and reshapes the competitive balance of the tournament. An imbalanced competition could arise when the manager unevenly splits the resources even if teams are ex ante symmetric, with each team consisting of heterogeneous workers, i.e., under negative sorting. The baseline model is a special case of the extension in which the manager evenly splits the resources. We demonstrate that positive sorting emerges in the optimum more often when the manager can allocate the resources endogenously.

Finally, we relax our assumption of two cost types and allow the four workers to be entirely heterogeneous. We demonstrate that our predictions do not lose their bites and the main results remain qualitatively robust.

The economic logic underlying these results will be delved in depth as the analysis unfolds. Our results not only provide theoretical insights into the sorting problem, but also generate useful implications for the organization of team production in practice. The managerial implications of our analysis will be laid out in Section 5.

Link to Literature There has been an extensive literature on optimal sorting of players with heterogeneous traits into teams and how the sorting outcome is subject to various frictions. For instance, Lamberson and Page (2012), Chade and Eeckhout (2018), and Kaya and Vereshchagina (2022) focus on players of different abilities to acquire information. Our paper is particularly related to one strand of this literature—e.g., Franco, Mitchell, and Vereshchagina (2011) and Kaya and Vereshchagina (2014, 2015)—which focuses on the sorting of players with different levels of productivity in the presence of moral hazard. Franco et al. (2011) focus on a principal-agent setting in which workers inside a firm are organized into teams. They demonstrate that with complementary technology, the optimal sorting pattern may turn out to be negative because of the cost to provide incentive under moral hazard. Kaya and Vereshchagina (2015) examine a setting of partnership. Kaya and Vereshchagina (2014) investigate how the different organizational forms—i.e., partnership versus corporations—would endogenously emerge, depending on the different roles played by moral hazard in different organizational environments. Kambhampati and Segura-Rodriguez (2022) allow for not only moral hazard but also adverse selection, such that workers possess private information about their own productivity. They find that nonassortative matching may arise when complementarity is sufficiently weak. Glover and Kim (2021), in a repeated team production setting, demonstrate that diverse teams—which generate productive complementarity—may facilitate tacit cooperation within teams. Imhof and Kräkel (2022) consider a setting in which workers on a team not only join force to produce composite output but also compete for career advance. They contend that more diverse distribution of competence may incentivize workers more effectively.[2]

In contrast to our paper, these studies do not consider the competition between teams under RPE. Our paper is thus closely related to Ryvkin (2011) and Brookins, Lightle, and Ryvkin (2015). Both studies examine optimal sorting of players with different abilities into group contests. Ryvkin (2011) assumes that efforts are perfect substitutes and demonstrate

[2]Li, Lim, and Chen (2020) and Bergeron, Bessone, Kabeya, Tourek, and Weigel (2022) examine the effect of team heterogeneity on team performance. Li et al. (2020) demonstrate that workers' effort responses to increasing heterogeneity within teams depend on the prevailing compensation schemes. Bergeron et al. (2022) espouse the merit of positive sorting and demonstrate that the benefit stems out of complementarity.

that the optimum depends on the curvature of cost functions. Brookins et al. (2015) assume a CES production and contend that technology plays a role. In contrast to our paper, both studies assume $n \times m$ players, with $n \geq 2$ groups and each consisting of $m \geq 2$ players. Because multi-player asymmetric contests, in general, do not yield closed-form solutions, they resort to quadratic approximation to the equilibrium efforts and limit their focus on the case of small degree of asymmetry among players' types. Our setting enables a complete characterization of the equilibrium under a full range of player heterogeneity, which allows for comprehensive comparative statics of the optimal sorting patterns with respect to the various environmental factors and lucid economic interpretation. Our paper is thus complementary to these studies.

The literature on contests/tournaments between groups dates back to Nitzan (1991a). The majority of these studies assume that there are no complementarities between players within a group (e.g., Nitzan, 1991a,b; Esteban and Ray, 1999, 2001, 2008; Ryvkin, 2011; Eliaz and Wu, 2018). Chowdhury, Lee, and Topolyan (2016) assume a "weakest link" group production function, in that the minimum of the contribution within the group determines the aggregate output. Chowdhury, Lee, and Sheremeta (2013) and Barbieri, Malueg, and Topolyan (2014) assume the opposite, such that aggregate output is given by the maximum, i.e., the "best shot."[3] Kolmar and Rommeswinkel (2013) and Choi, Chowdhury, and Kim (2016), as in Brookins, Lightle, and Ryvkin (2015), allow individual efforts to complement each other and are converted into their group's outlay through a CES production function. However, these studies do not consider endogenous formation of teams.

This strand of literature either lets players in the winning group equally share the prize or allows for a merit-based sharing scheme—i.e., a portion of the prize to be distributed among the winning players based on the relative performance (efforts) within the group.[4] Our paper assumes that efforts are noncontractible. In one of the aforementioned extensions, we allow the manager to set identity-dependent awards contingent on stochastic outcome (i.e., win or loss). This feature enriches the research streams of optimal sorting in group contests/tournaments. It also links our study to Franco et al. (2011) and Kaya and Vereshchagina (2015): They consider principal-agent settings, in which the principal sets optimal contract based on stochastic output under independent performance evaluation (IPE) scheme, while we consider RPE.

The other extension of our paper regarding resource allocation is related to Fu, Lu, and Lu (2012), Deng, Fu, and Wu (2021), and Gao, Fan, Huang, and Chen (2022), which

[3]Chowdhury and Topolyan (2016a,b) allow for a combination of weakest links and best shot: One group's output is the weakest link of individual efforts, while the other's is the best shot.

[4]Technically, an equal sharing rule is no different from assuming a public-good prize, e.g., Esteban and Ray (1999), Baik (2008), Ryvkin (2011), Kolmar and Rommeswinkel (2013), Barbieri, Malueg, and Topolyan (2014), Chowdhury and Topolyan (2016a,b), Chowdhury, Lee, and Topolyan (2016), and Eliaz and Wu (2018).

allow a principal to allocate productive resources between competing parties. They focus on contests/tournaments between individuals, where the problem of free-riding is absent. In contrast, we consider competition between teams.

2 Baseline Model

The production in a firm requires joint work of two workers. The firm employs four workers, and a manager of the firm splits the pool of workers into two teams; teams—with each consisting of two workers—independently execute the production tasks.

Teams are indexed by $i \in \{1, 2\}$, and workers on a team i are indexed by ik, with $k \in \{1, 2\}$. Workers simultaneously exert their efforts $e_{ik} \geq 0$. The efforts contributed by the workers on a team i are converted into a composite team output through a constant elasticity of substitution (CES) aggregation function

$$\mathcal{Y}_i(e_{i1}, e_{i2}) = \left(\frac{1}{2}e_{i1}^{\rho} + \frac{1}{2}e_{i2}^{\rho}\right)^{1/\rho}, \text{ with } \rho \leq 1.$$

The function is supermodular for $\rho \leq 1$, and the parameter ρ measures the degree of complementarity of the team production process. When $\rho \to 1$, workers' efforts are perfect substitutes. When $\rho \to 0$, the production technology degenerates to the Cobb-Douglas function. When $\rho \to -\infty$, the Leontief production function $\mathcal{Y}_i(e_{i1}, e_{i2}) = \min\{e_{i1}, e_{i2}\}$ ensues, and workers' efforts exhibit perfect complementarity.

A worker ik, when exerting an effort e_{ik}, bears a cost[5]

$$\mathcal{C}(e_{ik}; c_{ik}) = c_{ik}\frac{e_{ik}^{1+\gamma}}{1+\gamma}, \text{ with } \gamma > 0.$$

The parameter γ captures the convexity of a worker's effort cost function and can be interpreted as an indicator of the difficulty of the task. The parameter $c_{ik} > 0$ measures the worker's ability, which can take either of two values, c_H or c_L, with $c_H > c_L > 0$. Workers differ in their ability; there are two workers of each ability type and their types are publicly observable. Define $x := c_H/c_L \in (1, \infty)$, which measures the degree of heterogeneity in workers' abilities.

[5]The complete free-riding result of Baik (2008) emerges when $\gamma = 0$. This implies that with linear effort costs and linear production function, the optimum requires placing the n most competent workers into n teams, while the assignment of the other workers does not matter, since on each team, only the most competent will exert positive effort. Complete freeriding, however, rarely occurs in real life. Our analysis thus abstracts away this trivial case and assumes that workers' effort cost functions are strictly convex, i.e., $\gamma > 0$.

Winner-selection Mechanism and Workers' Payoffs The manager organizes a tournament between the two teams to incentivize workers. The winning team is awarded a team prize with a normalized value of two, and its workers equally share the prize. Teams compete by their composite output: With $\boldsymbol{e}_i := (e_{i1}, e_{i2})$ and an effort profile $(\boldsymbol{e}_1, \boldsymbol{e}_2)$, a team $i \in \{1, 2\}$ wins the tournament with a probability

$$p_i(\boldsymbol{e}_1, \boldsymbol{e}_2) = \begin{cases} \dfrac{\mathcal{Y}_i(e_{i1}, e_{i2})}{\mathcal{Y}_1(e_{11}, e_{12}) + \mathcal{Y}_2(e_{21}, e_{22})} & \text{if } \mathcal{Y}_1(e_{11}, e_{12}) + \mathcal{Y}_2(e_{21}, e_{22}) > 0, \\ \dfrac{1}{2} & \text{otherwise.} \end{cases} \tag{1}$$

This winning probability formulation is called a lottery tournament in the literature. Clark and Riis (1996) provide a microfoundation for this winning probability specification from a noisy ranking perspective related to the discrete choice model of McFadden (1973, 1974). Further, Baye and Hoppe (2003) show that the tournament is isomorphic to the research tournament model proposed by Fullerton and McAfee (1999) and the patent race model of Loury (1979) and Dasgupta and Stiglitz (1980).

Workers are risk-neutral. With an effort profile $(\boldsymbol{e}_1, \boldsymbol{e}_2)$, a worker ik receives an expected payoff

$$\pi_{ik} := p_i(\boldsymbol{e}_1, \boldsymbol{e}_2) - \mathcal{C}(e_{ik}; c_{ik}). \tag{2}$$

Manager's Sorting Decision Prior to the tournament, the manager decides how to sort the four workers into two teams. There are two sorting patterns, which we denote by $\theta \in \{N, P\}$, with N and P to indicate *positive sorting* and *negative sorting*, respectively. Under positive sorting, the manager sorts workers of like types into each team; negative sorting arises if she does the opposite.

The manager forms the teams to maximize the total output $\sum_{i=1}^{2} \mathcal{Y}_i(e_{i1}, e_{i2})$ of the firm. Worker assignment is revealed publicly before they sink efforts.

3 Analysis

In this section, we first characterize the equilibrium under each sorting pattern. We then compare the performances of the tournament under different sorting patterns to obtain the optimum. Finally, we present and discuss the comparative statics of the optimal sorting patterns with respect to the various environmental factors.

3.1 Equilibria under Different Sorting Patterns

A worker ik chooses his effort $e_{ik} \geq 0$ to maximize his expected payoff (2). Given the winning probability formulation (1) and zero marginal effort cost at $e_{ik} = 0$, each worker must exert a positive effort in the equilibrium. Let $\mathcal{Y} := \mathcal{Y}_1 + \mathcal{Y}_2$. The first-order condition with respect to e_{ik} gives

$$\frac{1}{2}e_{ik}^{\rho-1}\left(\frac{1}{2}e_{i1}^{\rho}+\frac{1}{2}e_{i2}^{\rho}\right)^{\frac{1}{\rho}-1}\frac{\mathcal{Y}-\mathcal{Y}_i}{(\mathcal{Y}_1+\mathcal{Y}_2)^2} = c_{ik}e_{ik}^{\gamma}, \forall i \in \{1,2\}, k \in \{1,2\}. \quad (3)$$

The above condition is not only necessary but also sufficient to determine workers' payoff-maximizing efforts, since the expected payoff π_{ik} is strictly concave in e_{ik} for all $\gamma > 0$ and $\rho \leq 1$.

Denote by e_H^{θ} and e_L^{θ}, respectively, the equilibrium individual efforts for the high- and low-cost types associated with a sorting pattern $\theta \in \{N, P\}$. Solving for the above system of equations (3) yields the following.

Lemma 1 ***(Equilibrium Effort Profile)*** *Fixing a sorting pattern $\theta \in \{N, P\}$, there exists a unique pure-strategy Nash equilibrium in the tournament game. The equilibrium individual efforts $(e_H^{\theta}, e_L^{\theta})$ are fully characterized as the following.*

(i) Under positive sorting, the equilibrium individual efforts of the high- and low-cost types are, respectively,

$$e_H^P = \frac{c_L^{\frac{1}{1+\gamma}}}{2^{\frac{1}{1+\gamma}}c_H^{\frac{\gamma}{(1+\gamma)^2}}c_L^{\frac{\gamma}{(1+\gamma)^2}}\left(c_H^{\frac{1}{1+\gamma}}+c_L^{\frac{1}{1+\gamma}}\right)^{\frac{2}{1+\gamma}}},$$

and

$$e_L^P = \frac{c_H^{\frac{1}{1+\gamma}}}{2^{\frac{1}{1+\gamma}}c_H^{\frac{\gamma}{(1+\gamma)^2}}c_L^{\frac{\gamma}{(1+\gamma)^2}}\left(c_H^{\frac{1}{1+\gamma}}+c_L^{\frac{1}{1+\gamma}}\right)^{\frac{2}{1+\gamma}}}.$$

(ii) Under negative sorting, the equilibrium efforts of the high-cost and the low-cost type are, respectively,

$$e_H^N = \frac{c_L^{\frac{1}{1+\gamma-\rho}}}{2^{\frac{2}{1+\gamma}}c_H^{\frac{1}{1+\gamma}}c_L^{\frac{1}{1+\gamma}}\left(c_H^{\frac{\rho}{1+\gamma-\rho}}+c_L^{\frac{\rho}{1+\gamma-\rho}}\right)^{\frac{1}{1+\gamma}}},$$

and

$$e_L^N = \frac{c_H^{\frac{1}{1+\gamma-\rho}}}{2^{\frac{2}{1+\gamma}}c_H^{\frac{1}{1+\gamma}}c_L^{\frac{1}{1+\gamma}}\left(c_H^{\frac{\rho}{1+\gamma-\rho}}+c_L^{\frac{\rho}{1+\gamma-\rho}}\right)^{\frac{1}{1+\gamma}}}.$$

Note that the equilibrium effort profile under positive sorting is independent of the degree of effort complementarity—i.e., the parameter ρ—in the team production process. This observation echoes the finding of Kolmar and Rommeswinkel (2013) that effort complementarity does not play an explicit role in determining equilibrium efforts when team members are homogeneous—i.e., under positive sorting in our context. Effort complementarity, however, affects equilibrium effort supply when workers on each team are heterogeneous—i.e., under negative sorting.

3.2 Positive versus Negative Sorting

Lemma 1 allows us to explore the optimal sorting pattern. Recall that the manager chooses $\theta \in \{N, P\}$ to maximize the equilibrium total output

$$\mathcal{Y} \equiv \sum_{i=1}^{2} \mathcal{Y}_i = \left(\frac{1}{2}e_{11}^{\rho} + \frac{1}{2}e_{12}^{\rho}\right)^{1/\rho} + \left(\frac{1}{2}e_{21}^{\rho} + \frac{1}{2}e_{22}^{\rho}\right)^{1/\rho}. \tag{4}$$

By Lemma 1, the equilibrium total output under positive and negative sorting, denoted respectively by $\mathcal{Y}^P$ and $\mathcal{Y}^N$, can be obtained as

$$\mathcal{Y}^P := \mathcal{Y}_1^P + \mathcal{Y}_2^P = \frac{\left(c_H^{\frac{1}{1+\gamma}} + c_L^{\frac{1}{1+\gamma}}\right)^{\frac{\gamma-1}{1+\gamma}}}{2^{\frac{1}{1+\gamma}} c_H^{\frac{\gamma}{(1+\gamma)^2}} c_L^{\frac{\gamma}{(1+\gamma)^2}}} = \frac{\left(1 + x^{\frac{1}{1+\gamma}}\right)^{\frac{\gamma-1}{1+\gamma}} x^{\frac{1}{(1+\gamma)^2}}}{2^{\frac{1}{1+\gamma}}} c_H^{-\frac{1}{1+\gamma}},$$

and

$$\mathcal{Y}^N := \mathcal{Y}_1^N + \mathcal{Y}_2^N = \frac{\left(c_H^{\frac{\rho}{1+\gamma-\rho}} + c_L^{\frac{\rho}{1+\gamma-\rho}}\right)^{\frac{1+\gamma-\rho}{(1+\gamma)\rho}}}{2^{\frac{(1-\gamma)\rho+1+\gamma}{(1+\gamma)\rho}} c_H^{\frac{1}{1+\gamma}} c_L^{\frac{1}{1+\gamma}}} = \frac{\left(1 + x^{\frac{\rho}{1+\gamma-\rho}}\right)^{\frac{1+\gamma-\rho}{(1+\gamma)\rho}}}{2^{\frac{(1-\gamma)\rho+1+\gamma}{(1+\gamma)\rho}}} c_H^{-\frac{1}{1+\gamma}}.$$

Positive sorting is optimal if $\mathcal{Y}^P > \mathcal{Y}^N$, while negative sorting prevails otherwise.

Proposition 1 ***(Optimal Sorting Pattern)*** *Fixing $x \equiv c_H/c_L > 1$ and $\gamma > 0$, there exists a threshold $\rho^*(\gamma; x) \in (-\infty, 1)$ for the degree of effort complementarity, such that positive sorting (i.e., $\mathcal{Y}^P > \mathcal{Y}^N$) is optimal if $\rho < \rho^*(\gamma; x)$, and negative sorting (i.e., $\mathcal{Y}^P < \mathcal{Y}^N$) is optimal if $\rho > \rho^*(\gamma; x)$. Moreover, the threshold $\rho^*(\gamma; x)$ is strictly increasing in γ for given x.*

Proposition 1 predicts that positive sorting is more likely to emerge in the optimum when workers' efforts are sufficiently complementary in team production—i.e., with a small ρ. The threshold $\rho^*(\gamma; x)$ depends on both worker heterogeneity x and the convexity of effort cost

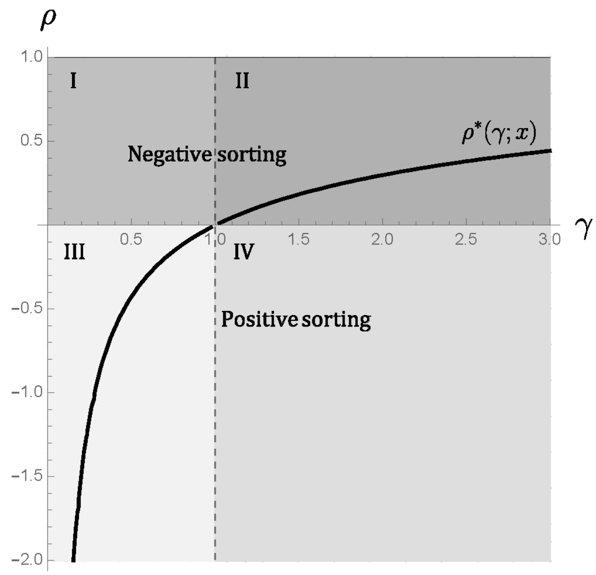


Figure 1: Optimal Sorting Pattern in the (γ, ρ) Space.

function γ. Proposition 1 further claims that the threshold strictly increases in γ; that is, positive sorting is more likely to prevail when effort cost curves become steeper.

It is noteworthy that the comparison between $\mathcal{Y}^P$ and $\mathcal{Y}^N$ is complicated because the parameter ρ factors into the expression of $\mathcal{Y}^N$ nonlinearly. For instance, the term $(1 + x^{\frac{\rho}{1+\gamma-\rho}})^{\frac{1+\gamma-\rho}{\rho}}$ is in the form of a general mean, whose property has not been uncovered and formally established until Nam and Minh (2008).[6] The CES production production is widely adopted in the economics research, and a wealth of literature relies on the interpretation of ρ as a key parameter for economics implications—e.g., in the studies of business-cycle and growth outcomes, income distribution, stabilization policy, and labor market dynamics. Our proof of Proposition 1, especially that of Lemma 8 in Appendix B, can be useful to future research that conducts similar comparative statics in alternative contexts.[7]

We illustrate the result in Figure 1. The vertical axis measures the level of effort complementary—i.e., ρ—while the horizontal axis measures the difficulty of the task γ. The upward-sloping curve traces the threshold $\rho^*(\gamma; x)$ for a given x. Negative sorting arises in the region above the curve, while positive sorting prevails otherwise. The following observation deserves to be highlighted.

Observation 1 $\rho^*(1; x) = 0$ *for all* $x \in (1, \infty)$.

[6]Consider a CES production function $[\delta K^\phi + (1-\delta)L^\phi]^{1/\phi}$, with $\delta \in (0, 1)$, where K denotes capital, L labor, and $1/(1-\phi)$ the constant elasticity of substitution. de La Grandville and Solow (2006) conjecture that holding inputs constant, the positive relationship between output and ϕ has a unique inflection point in ϕ and is convex before and concave thereafter. This conjecture is formally proved by Nam and Minh (2008).

[7]See León-Ledesma, McAdam, and Willman (2010) and Klump, McAdam, and Willman (2012) for detailed discussion of the CES production function.

The value of $\rho^*(\gamma; x)$ boils down to zero when $\gamma = 1$, regardless of the level of worker heterogeneity, x. Hence, the curve of $\rho^* = \rho^*(\gamma; x)$ in Figure 1 always passes through the point $(1, 0)$.

In what follows, we first elaborate on the logic underlying this result. Note that Proposition 1 and its graphic illustration, Figure 1, are obtained holding the degree of worker heterogeneity x fixed. We then discuss how x plays a role in shaping the optimal sorting pattern.

3.2.1 Intuition for Proposition 1

Recall that the manager must reconcile two fundamental concerns when forming teams: (i) to ensure efficient production within team; and (ii) to promote competition between teams. We interpret our results based on the trade-off between these concerns.

First and foremost, production technology $\mathcal{Y}_i(e_{i1}, e_{i2})$ plays an important role in determining the optimal sorting pattern. It is well known that a supermodular production technology favors positive sorting, which leverages effort complementarity. As discussed above, positive sorting effectively incentivizes the two low-cost workers when efforts are sufficiently complementary. Conversely, when efforts are relatively less complementary (or equivalently, more substitutable)—i.e., with a large ρ—negative sorting may help avert efficiency loss: The low-cost worker on each team, under negative sorting, is disciplined from excessive shirking without a competent coworker to freeride on.

Second, positive sorting limits competition, as it leads to a lopsided competition between a weak team and a strong one. The asymmetry discourages the weak team, which in turn entices the stronger to slack off. In contrast, negative sorting levels the playing field in the tournament, thereby fueling competition.

The tension between these forces shapes the optimum: Positive sorting prevails when efforts are sufficiently complementary—i.e., with ρ falling below the threshold $\rho^*(\gamma; x)$—in which case the gain from intra-team production efficiency outweighs the loss from the uneven race.

Proposition 1 shows that the cutoff $\rho^*(\gamma; x)$ increases with γ. That is, the more convex the effort cost function, the more likely positive sorting arises in the optimum. Steeply rising marginal effort cost would discourage effort supply, which diminishes the unevenness between strong and weak workers. In other words, a large γ plays a role in leveling the playing field under positive sorting, which mitigates the anti-competitive effect caused by an imbalanced competition. The above-mentioned trade-off is thus tilted toward positive sorting as γ rises.

For expositional convenience, we call the case of $\rho > 0$ a weakly complementary technology and that of $\rho < 0$ a strongly complementary one. Recall that the convexity of effort cost function is a measure of the difficulty of the task. We analogously label the case of $\gamma < 1$

an easy task and that of $\gamma > 1$ a difficult one. The parametric space in Figure 1 can then be divided into four regions with each to be described by the combination of two attributes: weakly complementary technology, easy task (Region I); weakly complementary technology, difficult task (Region II); strongly complementary technology, easy task (Region III); and strongly complementary technology, difficult task (Region IV).

As we observe in Figure 1, both attributes tend to favor negative sorting in Region I, which leads to a dominant case of negative sorting. In contrast, positive sorting always arises in Region IV. In Regions II and III, the two forces cause tension, and either sorting pattern may prevail, such that the threshold $\rho^*(\gamma; x)$ applies.

3.2.2 Impact of Worker Heterogeneity

The discussion thus far has implicitly assumed a given level of worker heterogeneity, i.e., a fixed $x \equiv c_H/c_L$. Workers' ability heterogeneity could play a subtle role in shaping the optimal sorting pattern. We first present the following comparative statics of the threshold $\rho^*(\gamma; x)$ with respect to x.

Proposition 2 ***(Impact of Heterogeneity on the Optimal Sorting)*** *The threshold for effort complementarity, $\rho^*(\gamma; x)$, varies with $x \equiv c_H/c_L$ as follows:*

$$\frac{\partial \rho^*(\gamma; x)}{\partial x} \begin{cases} > 0 & \text{if } \gamma < 1 \\ = 0 & \text{if } \gamma = 1 \\ < 0 & \text{if } \gamma > 1 \end{cases},$$

and $\rho^*(\gamma; x) \to 0$ for all γ as $x \to +\infty$.

The result and its implications can be intuitively illustrated in Figure 2. Recall by Observation 1 that $\rho^*(1; x) = 0$ and the curve of $\rho^*(\gamma; x)$ always passes through the point $(1, 0)$. As the degree of worker heterogeneity ascends, the curve rotates clockwise around $(1, 0)$. The threshold falls in Region II—i.e., $\rho > 0, \gamma > 1$—which refers to the observation of $\partial \rho^*(\gamma; x)/\partial x < 0$ if $\gamma > 1$, as Proposition 2 states: The trade-off is tilted toward negative sorting. In contrast, the threshold rises in Region III—i.e., $\rho < 0, \gamma < 1$—which is formally stated as $\partial \rho^*(\gamma; x)/\partial x > 0$ if $\gamma < 1$: Positive sorting gains its appeal in this case, as the asymmetry is enlarged. It is important to note that when x approaches infinity, the threshold $\rho^*(\gamma; x)$ converges to zero. That is, the curve tracing $\rho^*(\gamma; x)$ is entirely flattened, and the parametric space is divided into only two regions by the line of $\rho = 0$: In this limiting case, the role played by γ fades away, and the optimal sorting pattern depends only on the degree of effort complementarity of the production technology.

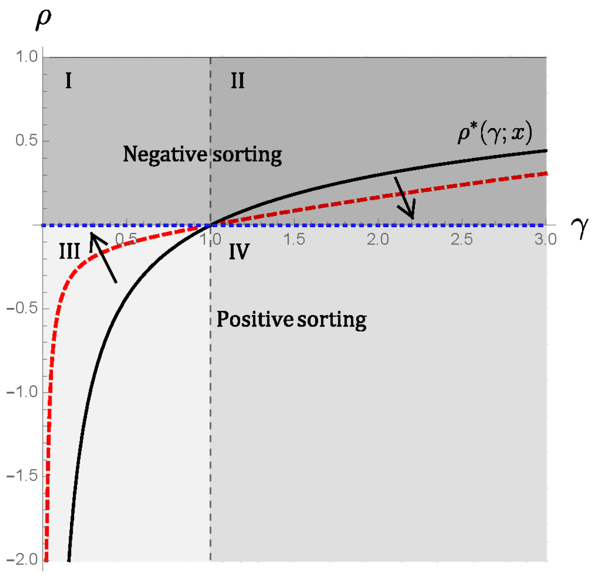


Figure 2: Impact of Worker Heterogeneity on the Optimal Sorting.

An increase in worker heterogeneity triggers competing forces on the optimum and affects both ends of the aforementioned fundamental trade-offs. It generates mixed effects on intra-team production efficiency. First, recall that positive sorting helps leverage effort complementarity with supermodular production functions. Increasing heterogeneity amplifies the efficiency gain of positive sorting vis-à-vis negative sorting. Second, its effect on workers' incentives depends on the nature of production technology. Recall that a low-cost worker under positive sorting can be effectively incentivized by the higher marginal return of his effort—owing to the higher input contributed by his equally competent teammate—when efforts are sufficiently complementary; in contrast, with relatively less complementary effort, a low-cost worker can be prevented from excessively shirking when teamed with a high-cost one (negative sorting) because he is less able to freeride. An increase in heterogeneity strengthens the incentive gain of positive sorting in the former scenario and the opposite in the latter. With strong complementarity, each low-cost worker, under positive sorting, can be motivated further when partnered with a more competent teammate; with weak complementarity, each low-cost worker, under negative sorting, can be disciplined more effectively when he can rely less on his weaker teammate. In summary, an increase in heterogeneity allows positive sorting to gain more in terms of intra-team production efficiency when efforts are strongly complementary, while the advantage may diminish when efforts are less complementary.

The effect of increasing heterogeneity on inter-team competition is unambiguous. It further upsets the competitive balance of the tournament under positive sorting, which clearly favors negative sorting.

As a result, we observe that a larger x affects $\rho^*(\gamma; x)$ in different ways under different parametric regions of Figure 2. Regions I and IV each depict a dominant case of negative or positive sorting, and the optimum does not vary with x. The trade-offs outline above loom large in Regions II and III. Region II presents a case of weakly complementary efforts. A rising x renders negative sorting more appealing. Negative sorting levels the playing field and prevents the cost workers from freeriding: Both gains are amplified by a rising x. As a result, the curve is turned downward, which renders negative sorting more likely. In contrast, Region III depicts a case of strongly complementary efforts. Positive sorting helps leverage effort complementarity and incentivizes low-cost workers under strong complementarity: Both gains are amplified by a larger x. The gain from intra-team production efficiency under positive sorting caused by more significant worker heterogeneity overshadows the loss from imbalanced competition. This causes the curve to be turned upward and renders positive sorting more likely.

4 Extensions

In this section, we examine three extended settings. We first let the manager set prize structure. We then allow the manager to allocate productive resources across teams, which not only boosts recipients' productivity but also manipulates the competitive balance of the playing field. Last, we extend the model to allow for four cost types—in which case workers can be fully heterogeneous—to examine the robustness of our results.

4.1 Optimal Sorting with Endogenous Prize Allocation

The baseline model assumes that the two workers on the winning team equally share the prize. We now consider an alternative setting in which the manager precommits to an identity-dependent prize schedule that specifies the prize share each worker would secure upon winning.

The game proceeds in two stages. In the first stage, the manager forms teams and decides on how to split the prize purse between the workers on each team when it wins. Workers exert efforts and compete in the second stage. We continue to assume a total prize purse of a value two ($V = 2$). The manager specifies and publicly announces a prize schedule $\boldsymbol{v} := (\boldsymbol{v}_1, \boldsymbol{v}_2)$, with $\boldsymbol{v}_i := (v_{i1}, v_{i2})$, $i \in \{1, 2\}$, and $v_{i1} + v_{i2} \leq V = 2$: Namely, a worker ik, $k \in \{1, 2\}$, on a team i receives a prize $v_{ik} \geq 0$ if the team wins. We assume that efforts are not contractible. Thus, the prize schedule $\boldsymbol{v} \equiv (\boldsymbol{v}_1, \boldsymbol{v}_2)$ is independent of individual efforts.

Fix a sorting pattern $\theta \in \{N, P\}$ and prize schedule $\boldsymbol{v} \equiv (\boldsymbol{v}_1, \boldsymbol{v}_2)$. The expected payoff of worker ik is given by

$$\pi_{ik} = \frac{\mathcal{Y}_i}{\mathcal{Y}_1 + \mathcal{Y}_2} v_{ik} - c_{ik} \frac{e_{ik}^{1+\gamma}}{1+\gamma}. \tag{5}$$

In what follows, we first derive the optimal prize schedule under an arbitrary sorting pattern, then the optimal sorting pattern with endogenously allocated prizes.

4.1.1 Optimal Prize Allocation under Positive/Negative Sorting

We first obtain the following result.

Lemma 2 ***(Suboptimality of Money Burning)*** *Fixing* $\theta \in \{N, P\}$, $v_{i1} + v_{i2} = 2$ *for all* $i \in \{1, 2\}$ *in the optimal tournament.*

Lemma 2 claims that "money burning" will not arise in the optimum: The manager must deplete her prize purse to maximally incentivize the workers. We are then ready to solve for the optimal prize schedule and associated equilibrium outcome under each sorting pattern. For the sake of brevity and expositional efficiency, we only present the expression of equilibrium total output, which paves the way to the search for optimal sorting pattern.

We first consider positive sorting. Without loss of generality, the stronger team is labeled team 1, i.e., $c_{11} = c_{12} = c_L$ and $c_{21} = c_{22} = c_H$.

Lemma 3 ***(Optimal Prize Allocation and Equilibrium Total Output under Positive Sorting)*** *Under positive sorting,*

(i) for $\rho \in \left(-\infty, \min\{\frac{1+\gamma}{2}, 1\}\right)$, *the manager evenly splits the prize purse within the winning team, i.e.,* $v_{ik} = 1, \forall i, k \in \{1, 2\}$; *so the equilibrium coincides with that in the baseline model, in which case the tournament generates a total output*

$$\mathcal{Y}_a^P = \frac{\left(c_H^{\frac{1}{1+\gamma}} + c_L^{\frac{1}{1+\gamma}}\right)^{\frac{\gamma-1}{1+\gamma}}}{2^{\frac{1}{1+\gamma}} c_H^{\frac{\gamma}{(1+\gamma)^2}} c_L^{\frac{\gamma}{(1+\gamma)^2}}};$$

(ii) the manager otherwise allocates all the prize to only one worker on the winning team, i.e., $v_{11} = v_{21} = 2$, *without loss of generality given the symmetry within each team; the tournament generates a total output*

$$\mathcal{Y}_a^P = \frac{\left(c_H^{\frac{1}{1+\gamma}} + c_L^{\frac{1}{1+\gamma}}\right)^{\frac{\gamma-1}{1+\gamma}}}{2^{\frac{1+\gamma-\rho}{\rho(1+\gamma)}} c_H^{\frac{\gamma}{(1+\gamma)^2}} c_L^{\frac{\gamma}{(1+\gamma)^2}}}.$$

Lemma 3 is intuitive. When efforts are sufficiently complementary—i.e., $\rho < \min\{\frac{1+\gamma}{2}, 1\}$—efficient production requires that workers on each team join force. Given the symmetry within each team, the manager evenly splits the prize purse to engage both workers. In contrast, when efforts are relatively less complementary—i.e., $\rho > \min\{\frac{1+\gamma}{2}, 1\}$—total output maximization relies less on joint and balanced input. The manager concentrates the entire prize purse on one worker on the winning team to provide the maximal incentive. The tournament boils down to a head-to-head competition between two individual workers—one strong and one weak—from rival teams.

We then consider negative sorting, in which case teams are symmetric and each consists of a low-cost worker and a high-cost one. Without loss of generality, the low-cost worker on each team $i \in \{1, 2\}$ is labeled $i1$, which yields $c_{11} = c_{21} = c_L$ and $c_{12} = c_{22} = c_H$.

Lemma 4 ***(Optimal Prize Allocation and Equilibrium Total Output under Negative Sorting)*** *Under negative sorting,*

(i) For $\rho \in \left(-\infty, \min\{\frac{1+\gamma}{2}, 1\}\right)$, the manager splits the prize purse between workers such that

$$v_{11} = v_{21} = \frac{2c_H^{\frac{\rho}{2\rho-\gamma-1}}}{c_H^{\frac{\rho}{2\rho-\gamma-1}} + c_L^{\frac{\rho}{2\rho-\gamma-1}}}, \text{ and } v_{12} = v_{22} = \frac{2c_L^{\frac{\rho}{2\rho-\gamma-1}}}{c_H^{\frac{\rho}{2\rho-\gamma-1}} + c_L^{\frac{\rho}{2\rho-\gamma-1}}};$$

the tournament generates a total output

$$\mathcal{Y}_a^N = \frac{2^{\frac{\gamma\rho-\gamma-1}{\rho(1+\gamma)}}}{\left(c_H^{\frac{\rho}{2\rho-\gamma-1}} + c_L^{\frac{\rho}{2\rho-\gamma-1}}\right)^{\frac{2\rho-\gamma-1}{\rho(1+\gamma)}}}.$$

(ii) The manager otherwise allocates all the prize to the low-cost worker on the winning team, i.e., $v_{11} = v_{21} = 2$; the tournament generates a total output

$$\mathcal{Y}_a^N = \frac{2^{\frac{\gamma\rho-\gamma-1}{\rho(1+\gamma)}}}{c_L^{\frac{1}{1+\gamma}}}.$$

When efforts are sufficiently complementary—i.e., when ρ falls below the cutoff $\min\{\frac{1+\gamma}{2}, 1\}$—team production remains but prizes are no longer evenly split as in Lemma 3(i). Workers are instead rewarded by their contribution. Analogous to Lemma 3(ii), Lemma 4(ii) requires that the manager reward only one worker when efforts are moderately complementary—i.e., when ρ exceeds the cutoff $\min\{\frac{1+\gamma}{2}, 1\}$. Under negative sorting, the prize should be entirely awarded to the strong worker on the winning team, which maximally incentivizes his contribution. The tournament, again, reduces to a head-to-head competition between two individual workers. In contrast to the case of positive sorting, the competition unfolds between two low-cost workers, and a symmetric tournament arises.

Lemmas 3(ii) and 4(ii) yield useful implications and can be interpreted more broadly. The flexibility of prize allocation allows for an additional degree of freedom: The manager is able to choose the organizational format of her labor force, i.e., dismantling team production when efforts are relatively less complementary and engaging only one worker on each team.

4.1.2 Optimal Sorting with Endogenous Prize Allocation

Lemmas 3 and 4 enable us to identify the optimal sorting pattern with endogenous prize allocation. Positive sorting prevails if $\mathcal{Y}_a^P > \mathcal{Y}_a^N$, and negative sorting stands out otherwise. For notational convenience, let $\overline{\gamma} := \sqrt{5} - 2 < 1$. The following result ensues.

Proposition 3 ***(Optimal Sorting Pattern with Endogenous Prize Allocation)*** *Fix $x \equiv c_H/c_L > 1$ and $\gamma > 0$. There exists a unique threshold $\rho_a^*(\gamma; x)$, such that positive sorting prevails—i.e., $\mathcal{Y}_a^P > \mathcal{Y}_a^N$—for $\rho < \rho_a^*(\gamma; x)$ and negative sorting arises otherwise.*

(i) For $0 < \gamma < \bar{\gamma}$, there exists a unique $\hat{x}(\gamma) > 1$, such that for $1 < x \leq \hat{x}(\gamma)$, $\mathcal{Y}_a^P < \mathcal{Y}_a^N$ holds for all $\rho \in (-\infty, 1]$; and we define the threshold $\rho_a^*(\gamma; x) := -\infty$.

(ii) Otherwise—i.e., $0 < \gamma < \bar{\gamma}$ and $x > \hat{x}(\gamma)$, or $\gamma \geq \bar{\gamma}$—the threshold $\rho_a^*(\gamma; x)$ lies within the interval $(-\infty, 1]$.

The prediction of Proposition 3 qualitatively resembles that of Proposition 1: Positive sorting prevails in the presence of strong complementarity, while negative sorting arises otherwise. As in Proposition 1, Proposition 3(ii) states that there exists a threshold $\rho_a^*(\gamma; x) \in (-\infty, 1]$. In contrast to Proposition 1, Proposition 3(i) shows that the threshold degenerates to $-\infty$, when effort cost curve is sufficiently flat ($0 < \gamma < \bar{\gamma}$) and workers are not excessively heterogeneous ($1 < x \leq \hat{x}(\gamma)$): A dominant case arises, such that negative sorting outperforms positive sorting regardless.

We compare the threshold $\rho_a^*(\gamma; x)$ to that without endogenous prize allocation—i.e., $\rho^*(\gamma; x)$—which allows us to learn how the optimal sorting pattern differs from that in the baseline model.

Proposition 4 ***(Impact of Prize Allocation on Optimal Sorting)*** *Negative sorting is more likely to prevail with endogenous prize allocation. That is, $\rho_a^*(\gamma; x) \leq \rho^*(\gamma; x)$ for all $x > 1$, with the equality holding if, and only if, $\gamma = 1$.*

The prediction of $\rho_a^*(\gamma; x) \leq \rho^*(\gamma; x)$ indicates that endogenous prize allocation favors negative sorting. We illustrate our result in Figure 3. The solid curve depicts the threshold

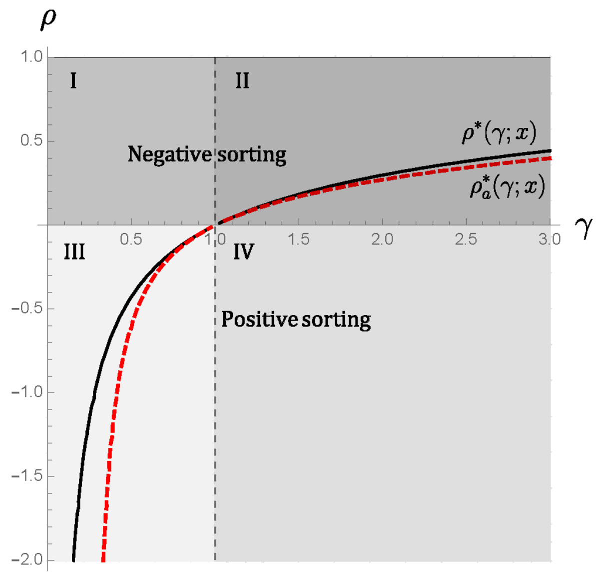


Figure 3: Optimal Sorting Pattern with Endogenous Prize Allocation.

$\rho^*(\gamma; x)$ in the baseline model, and the dashed curve traces the threshold $\rho_a^*(\gamma; x)$ under endogenous prize allocation. The latter curve still passes through the point $(1, 0)$ but remains below that for $\rho^*(\gamma; x)$. Negative sorting is more likely to emerge in the optimum.

We briefly interpret the logic. The freedom to set prize schedule allows the manager to incentivize workers more effectively in team production. By Lemmas 3(ii) and 4(ii), the manager abandons team structure and rewards only one worker when effort complementarity is sufficiently weak, regardless of the prevailing sorting pattern. Negative sorting obviously outperforms the positive one under this circumstance because the former gives rise to a symmetric tournament. By Lemma 3(i), when the efforts are sufficiently complementary, the endogenized prize schedule under positive sorting coincides with that in the baseline model and thus does not improve the performance of the tournament. By Lemma 4(i), in contrast, a prize differential emerges between the strong and weak workers on each team under negative sorting: The endogenously set prize schedule rewards workers by their contribution, which improves production efficiency. The balance is thus tilted toward negative sorting.

4.2 Optimal Sorting with Endogenous Resource Allocation

We now consider an alternative setting in which the manager not only sorts workers into teams, but also allocates productive resources between teams (Fu, Lu, and Lu, 2012; Gao, Fan, Huang, and Chen, 2022; Deng, Fu, and Wu, 2021), subject to a budget constraint. For instance, a pharmaceutical company can provide research funding, laboratory equipment, or computing facility to selected research task forces. Alternatively, a firm may ration administrative support to sales teams. The resources improve recipients' productivity; however, uneven allocation also varies the balance of the playing field.

We assume that the production function takes the form of

$$\delta_i \mathcal{Y}_i(e_{i1}, e_{i2}) \equiv \delta_i \left(\frac{1}{2} e_{i1}^{\rho} + \frac{1}{2} e_{i2}^{\rho} \right)^{1/\rho}, \text{ with } \rho \leq 1,$$

where $\delta_i > 0$ is the amount of resource allocated to a team $i \in \{1, 2\}$. The manager is subject to a budget constraint, and we normalize the amount of resources available to two; so the budget constraint can be written as $\delta_1 + \delta_2 \leq 2$. The baseline model can then be viewed as a special case in which the manager equally splits the resources between teams.

The game proceeds in two stages. In the first stage, the manager chooses and announces team formation $\theta \in \{N, P\}$ and her resource allocation plan, $\boldsymbol{\delta} := (\delta_1, \delta_2)$ to maximize the total output $\sum_{i=1}^{2} \delta_i \mathcal{Y}_i(e_{i1}, e_{i2})$. In the second stage, workers simultaneous choose their effort input. Similar to the baseline model, a team $i \in \{1, 2\}$ wins the tournament with a probability

$$p_i(\boldsymbol{e}_1, \boldsymbol{e}_2) = \begin{cases} \dfrac{\delta_i \mathcal{Y}_i(e_{i1}, e_{i2})}{\delta_1 \mathcal{Y}_1(e_{11}, e_{12}) + \delta_2 \mathcal{Y}_2(e_{21}, e_{22})} & \text{if } \delta_1 \mathcal{Y}_1(e_{11}, e_{12}) + \delta_2 \mathcal{Y}_2(e_{21}, e_{22}) > 0, \\ \dfrac{\delta_i}{\delta_1 + \delta_2} & \text{otherwise.} \end{cases}$$

We first derive the optimal resource plan and the associated equilibrium outcome under each sorting pattern, then the optimal sorting pattern.

4.2.1 Optimal Resource Allocation under Positive/Negative Sorting

First, consider the case of negative sorting, i.e., $c_{11} = c_{21} = c_L$ and $c_{12} = c_{22} = c_H$. The following results ensues.

Lemma 5 ***(Optimal Resource Allocation and Equilibrium Total Output under Negative Sorting)*** *Under negative sorting, the manager evenly splits the resources between the two teams—i.e., $\delta_1^N = \delta_2^N = 1$—and in the equilibrium, the tournament generates a total output*

$$\mathcal{Y}_r^N = \frac{\left(c_H^{\frac{\rho}{1+\gamma-\rho}} + c_L^{\frac{\rho}{1+\gamma-\rho}} \right)^{\frac{1+\gamma-\rho}{\rho(1+\gamma)}}}{2^{\frac{1+\gamma+\rho-\gamma\rho}{\rho(1+\gamma)}} c_H^{\frac{1}{1+\gamma}} c_L^{\frac{1}{1+\gamma}}} = \mathcal{Y}^N. \tag{6}$$

The resultant tournament coincides with that in the baseline model: Teams remain symmetric, and the total output, which we denote by $\mathcal{Y}_r^N$, equals $\mathcal{Y}^N$ in the baseline setting.

Teams are ex ante identical, and the manager evenly splits the resources, which retains an even race without upsetting the competitive balance of the playing field.

The manager, however, faces a trade-off between production efficiency and competitive balance under positive sorting. The teams are ex ante asymmetric, with one strong and one weak—i.e., $c_{11} = c_{12} = c_L$ and $c_{21} = c_{22} = c_H$. On the one hand, allocating more resources to the ex ante stronger team enables more efficient production: They presumably contribute a higher level of effort, so additional amount of resources—i.e., a larger δ—scales up the productivity gain. On the other hand, this further upsets the playing field, thereby discouraging competition and weakening workers' incentives. Our analysis yields the following.

Lemma 6 ***(Optimal Resource Allocation and Equilibrium Total Output under Positive Sorting)*** *Under positive sorting, the optimal resource allocation plan is*

$$\delta_1^P = \frac{\gamma(X-1) - 2 + \sqrt{\gamma^2(X-1)^2 + 4(X-1) + 4}}{(\gamma+1)(X-1)}, \delta_2^P = 2 - \delta_1^P,$$

where $X = \left(c_H/c_L\right)^{\frac{1}{1+\gamma}}$*. Further,* $\delta_1^P > 1$ *if* $\gamma > 1$ *and* $\delta_1^P < 1$ *if* $\gamma < 1$*. The total output in the equilibrium amounts to*

$$\mathcal{Y}_r^P = \frac{\left[\delta_1^P \delta_2^P \left(\delta_1^P c_H^{\frac{1}{1+\gamma}} + \delta_2^P c_L^{\frac{1}{1+\gamma}}\right)^{\gamma-1}\right]^{\frac{1}{1+\gamma}}}{2^{\frac{1}{1+\gamma}} c_H^{\frac{\gamma}{(1+\gamma)^2}} c_L^{\frac{\gamma}{(1+\gamma)^2}}} \geq \mathcal{Y}^P, \tag{7}$$

with the equality holding if, and only if, $\gamma = 1$*.*

By Lemma 6, the ex ante stronger team under positive sorting receives more (less) resources when the task is difficult (easy), which further enlarges (reduces) the asymmetry between the teams. Note that the rationale for leveling the playing field requires that the weak team be effectively motivated to step up its effort, which, in turn, prevents the stronger team from shirking. A steeper cost curve—i.e., a larger γ—implies that it is harder to convert additional input into higher output and thus increases the cost to elicit efforts from the workers. This limits the benefit of leveling the playing field. Efficient resource allocation must prioritize the ex ante stronger team for a larger productivity gain. The opposite takes place when $\gamma < 1$, in which case additional incentives come at a lower cost, thereby generating a larger gain from leveling the playing field.

Obviously, the resultant total output $\mathcal{Y}_r^P$ strictly exceeds that in the baseline model for $\gamma \neq 1$.

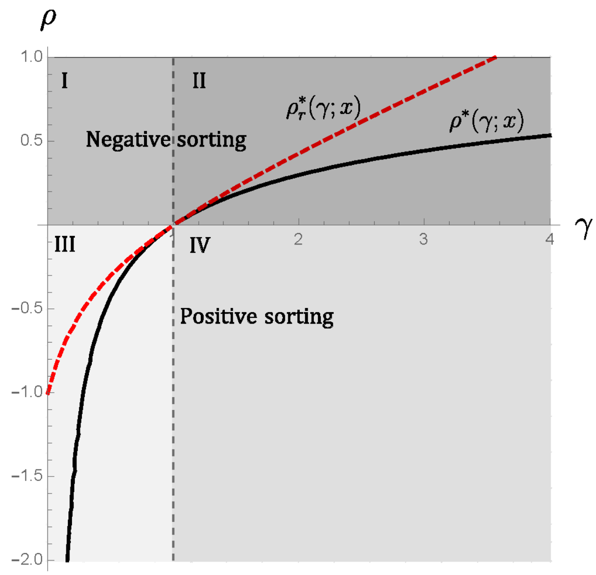


Figure 4: Optimal Sorting Pattern with Endogenous Resource Allocation.

4.2.2 Optimal Sorting with Endogenous Resource Allocation

We can simply compare $\mathcal{Y}_r^P$ with $\mathcal{Y}_r^N$ to obtain the optimal sorting, with positive sorting to prevail if $\mathcal{Y}_r^P > \mathcal{Y}_r^N$ and negative sorting to stand out otherwise.

Proposition 5 ***(Optimal Sorting with Endogenous Resource Allocation)*** *Fix $x = c_H/c_L > 1$ and $\gamma > 0$. There exists a threshold $\rho_r^*(\gamma; x)$ such that positive sorting prevails—i.e., $\mathcal{Y}_r^P > \mathcal{Y}_r^N$—for $\rho < \rho_r^*(\gamma; x)$ and negative sorting arises otherwise.*

The prediction of Proposition 5 is qualitatively similar to that of Proposition 1: Positive sorting arises when efforts are sufficiently complementary, while negative sorting emerges otherwise. We compare the threshold $\rho_r^*(\gamma; x)$ to $\rho^*(\gamma; x)$, the threshold in the baseline model. The following can be obtained.

Proposition 6 ***(Impact of Resource Allocation on Optimal Sorting)*** *With endogenous resource allocation, positive sorting is more likely to emerge in the optimum. More formally, $\rho_r^*(\gamma; x) \geq \rho^*(\gamma; x)$ for all $x > 1$, with the equality holding if, and only if, $\gamma = 1$.*

Proposition 6 shows that endogenous resource allocation favors positive sorting. This result is intuitive. Recall by Lemma 5 that under negative sorting, the tournament remains the same as in the baseline model; the freedom to allocate resources does not affect the performance of the tournament. However, resource allocation allows the manager to exploit the asymmetry between teams under positive sorting; she may prioritize the ex ante stronger team for larger productivity gain at the cost of reduced competition or handicap the stronger team to stimulate competition at the cost of inefficient resource allocation, which improves the total output except for the knife-edge case of $\gamma = 1$. As a result, the trade-off is tilted toward positive sorting, as it can arise in the optimum with a lower degree of effort complementarity.

The result is illustrated in Figure 4. The curve that traces $\rho_r^*(\gamma; x)$ continues to pass through the point $(1, 0)$. However, compared to the curve for $\rho^*(\gamma; x)$, it is stretched upward, which enlarges the parametric space below the curve, i.e., the set of parameterizations for positive sorting to prevail.

4.3 Four Types of Workers

The baseline model assumes two types of workers, with cost parameters c_H and c_L, respectively. We now extend the model to allow for entirely heterogeneous workers, i.e., four cost types. Specifically, let the four workers be ranked in increasing order of cost parameters: 1, 2, 3, 4, with $0 < c_1 \leq c_2 \leq c_3 \leq c_4$. For simplicity, we parametrize the effort cost function and assume $(c_1, c_2, c_3, c_4) = (c - \chi_1 - \chi_2, c - \chi_1, c + \chi_1, c + \chi_1 + \chi_2)$, with $c > 0$, $\chi_1 \geq 0$, $\chi_2 \geq 0$, and $\chi_1 + \chi_2 \leq c$. The extended model degenerates to the baseline model if $\chi_2 = 0$. Further, an increase in χ_1 or χ_2 would lead to a more dispersed type distribution, and we say that workers are more heterogeneous in this case.

Let $\langle(1, 2), (3, 4)\rangle$ indicate a sorting profile under which players 1 and 2 are assigned into one team and 3 and 4 into the other. Other sorting profiles can similarly be defined. Now the manager is given a larger choice set, as three sorting arrangements are possible: $\langle(1, 2), (3, 4)\rangle$, $\langle(1, 4), (2, 3)\rangle$, and $\langle(1, 3), (2, 4)\rangle$. The first arrangement—pools the strongest (weakest) into one team—exemplifies positive sorting, while the second leads to negative sorting. The third, $\langle(1, 3), (2, 4)\rangle$, lies in the middle.

This nuance largely complicates the analysis, and we resort to numerical exercises. The observations demonstrate that the third sorting arrangement is always suboptimal. As a result, the manager chooses either positive sorting or negative sorting, as in the baseline model. We construct the following numerical example to illustrate our observations. Set $(c, \chi_1, \chi_2) = (2.5, 0.5, 1)$. So the profile of workers' cost parameters is $(c_1, c_2, c_3, c_4) = (1, 2, 3, 4)$. Fixing $\gamma = 0.5$, we plot the total output associated with each of the three sorting arrangements for different values of ρ. As Figure 5 shows, $\langle(1, 3), (2, 4)\rangle$ is strictly dominated by either of the other two arrangements.

The manager's decision is thus reduced to the binary choice between positive and negative sortings, which reinstates the relevance of our analysis in this extended setting. Our results remain qualitatively robust by the numerical exercises. First, a cutoff of $\rho^*(\gamma; c, \chi_1, \chi_2)$ exists in the four-type model: Negative sorting arises for $\rho > \rho^*(\gamma; c, \chi_1, \chi_2)$, while positive sorting prevails otherwise. Figures 6(a) and 6(b) illustrate this claim. Second, the variation in

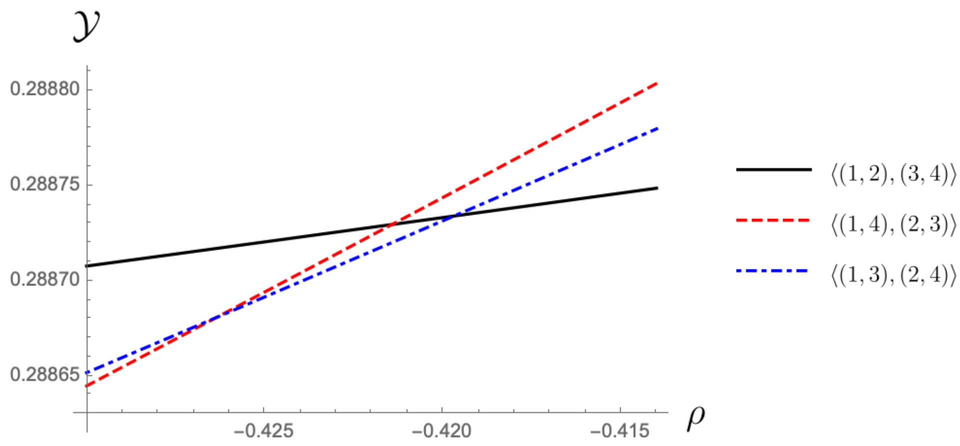


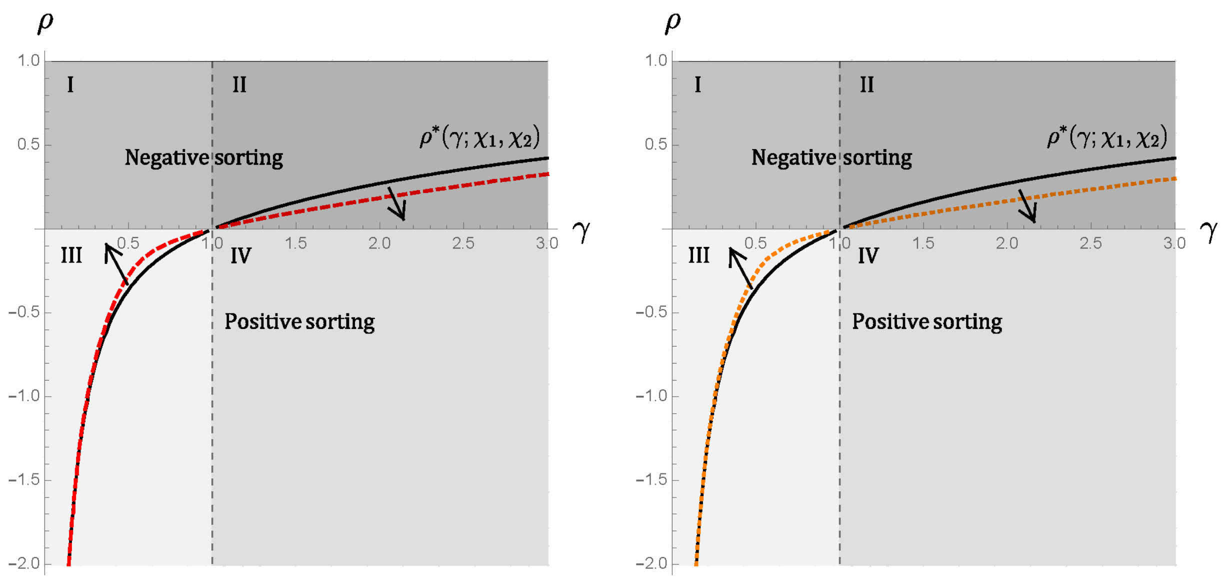
Figure 5: Equilibrium Total Outputs under Different Sorting Patterns.

heterogeneity (expressed as an increase in χ_1 or χ_2 while holding c constant) generates an effect similar to that in the baseline model. As χ_1 or χ_2 increases, the curve that traces the threshold $\rho^*(\gamma; c, \chi_1, \chi_2)$ rotates around the point of $(1, 0)$ (see Figures 6(a) and 6(b)), which resembles the observation in the baseline model when $x = c_H/c_L$ increases.

5 Practical Implications

Our results not only generate theoretical implications, but also offer an intuitive playbook for managerial practice. First, Figure 1 provides a lucid account of how heterogeneous players should be sorted into teams given the nature of team production process (weakly versus strongly complementary efforts) and the difficulty of the task, which is depicted by the convexity of the effort cost functions. Consider, for instance, a complex and challenging research project that requires fundamental breakthrough on multiple components and calls for cross-functional collaboration. This scenario can be mapped into Region IV in the figure, which suggests that the organization pools its finest personnel into a single team (positive sorting). In contrast, consider an established firm that provides team incentives to regional sales teams; sales persons work in parallel within each region. This scenario is analogous to those collected in Region I, which suggests that the firm matches veterans with novices in each team.

Second, our results provide full comparative statics with respect to the degree of worker heterogeneity. The impact of increasing heterogeneity depends subtly on the nature of the team production technology, which is illuminated in Figure 2. It shows that increasing heterogeneity favors positive sorting under strongly complementary efforts, while encouraging negative sorting when effort complementarity is weak. Imagine that a firm has experienced a wave of turnover, with a substantial portion of mid-career employees to have been re-



(a) The impact of χ_1 on optimal sorting. (b) The impact of χ_2 on optimal sorting.

Figure 6: Impact of Worker Heterogeneity on the Optimal Sorting.

placed by recruits. The observations of Figure 2 provide straightforward implications for the reorganization of the more heterogeneous workforce.

Third, our analysis shows that when the manager is allowed to allocate prizes (e.g., bonus packages) between the workers on the winning team, negative sorting is more likely to prevail in the optimum (see Figure 3). This suggests that a manager should create more diverse teams, while tolerating more dispersed compensation structures inside teams. This incentivizes workers more effectively and creates more competition between teams.

Fourth, we demonstrate that positive sorting is more likely when the manager distributes productive resources between teams (see Figure 4). Further, our analysis demonstrates that a manager, with this flexibility, is encouraged to (i) create favorite team and champion it through prioritized resource allocations when facing a difficult and challenging project (e.g., research for fundamental breakthrough), although this results in a more uneven race and loss of competition; and (ii) handicap the strong team through resource allocation to level the playing filed and intensify competition when pursuing with a routine task (e.g., process optimization for cost reduction in manufacturing).

6 Concluding Remarks

In this paper, we analyze the optimal sorting of heterogeneous workers in a team tournament. We identify a fundamental trade-off faced by an output-maximizing manager between intra-team production efficiency and inter-team competition. The optimal sorting depends on the complex combination of the nature of production technology, the convexity of the effort cost functions, as well as the degree of worker heterogeneity. We fully characterize the optimal sorting pattern and analyze how it varies with the environmental factors. Further, we consider extended settings in which the manager can set prize schedule or allocate productive resources across teams.

Large room for future research remains. In this paper, we assume that the manager employs relative performance evaluation (RPE)—i.e., tournaments—to incentivize workers. Relatedly, Franco, Mitchell, and Vereshchagina (2011) consider a standard model of moral hazard with team production, in which a profit-maximizing principal decides on how to sort agents into teams, as well as a wage scheme that exhibits independent performance evaluation (IPE). Their model focuses on the within-team moral hazard problems and thus abstracts away the use of inter-team competition to further save agency costs. It would be interesting to reexamine optimal sorting of workers, while endogenizing the choice of RPE vis-à-vis IPE.

References

Baik, K. H. (2008): "Contests with group-specific public-good prizes," *Social Choice and Welfare*, 30, 103–117.

Barbieri, S., D. A. Malueg, and I. Topolyan (2014): "The best-shot all-pay (group) auction with complete information," *Economic Theory*, 57, 603–640.

Baye, M. R. and H. C. Hoppe (2003): "The strategic equivalence of rent-seeking, innovation, and patent-race games," *Games and Economic Behavior*, 44, 217–226.

Bergeron, A., P. Bessone, J. K. Kabeya, G. Z. Tourek, and J. L. Weigel (2022): "Optimal assignment of bureaucrats: Evidence from randomly assigned tax collectors in the DRC," *NBER Working Paper No. 30413*.

Birkinshaw, J. (2001): "Strategies for managing internal competition," *California Management Review*, 44, 21–38.

Brookins, P., J. P. Lightle, and D. Ryvkin (2015): "Optimal sorting in group contests with complementarities," *Journal of Economic Behavior & Organization*, 112, 311–323.

Chade, H. and J. Eeckhout (2018): "Matching information," *Theoretical Economics*, 13, 377–414.

Che, Y.-K. and S.-W. Yoo (2001): "Optimal incentives for teams," *American Economic Review*, 91, 525–541.

Choi, J. P., S. M. Chowdhury, and J. Kim (2016): "Group contests with internal conflict and power asymmetry," *Scandinavian Journal of Economics*, 118, 816–840.

Chowdhury, S. M., D. Lee, and R. M. Sheremeta (2013): "Top guns may not fire: Best-shot group contests with group-specific public good prizes," *Journal of Economic Behavior & Organization*, 92, 94–103.

Chowdhury, S. M., D. Lee, and I. Topolyan (2016): "The max-min group contest: Weakest-link (group) all-pay auction," *Southern Economic Journal*, 83, 105–125.

Chowdhury, S. M. and I. Topolyan (2016a): "Best-shot versus weakest-link in political lobbying: an application of group all-pay auction," *Social Choice and Welfare*, 47, 959–971.

——— (2016b): "The attack-and-defense group contests: Best shot versus weakest link," *Economic Inquiry*, 54, 548–557.

Clark, D. J. and C. Riis (1996): "On the win probability in rent-seeking games," *Mimeo*.

Dasgupta, P. and J. Stiglitz (1980): "Industrial structure and the nature of innovative activity," *Economic Journal*, 90, 266–293.

de La Grandville, O. and R. M. Solow (2006): "A conjecture on general means," *Journal of Inequalities in Pure and Applied Mathematics*, 7, 3.

Deng, S., Q. Fu, and Z. Wu (2021): "Optimally biased Tullock contests," *Journal of Mathematical Economics*, 92, 10–21.

Eliaz, K. and Q. Wu (2018): "A simple model of competition between teams," *Journal of Economic Theory*, 176, 372–392.

Esteban, J. and D. Ray (1999): "Conflict and distribution," *Journal of Economic Theory*, 87, 379–415.

——— (2001): "Collective action and the group size paradox," *American Political Science Review*, 95, 663–672.

——— (2008): "On the salience of ethnic conflict," *American Economic Review*, 98, 2185–2202.

Franco, A. M., M. Mitchell, and G. Vereshchagina (2011): "Incentives and the structure of teams," *Journal of Economic Theory*, 146, 2307–2332.

Fu, Q., J. Lu, and Y. Lu (2012): "Incentivizing R&D: Prize or subsidies?" *International Journal of Industrial Organization*, 30, 67–79.

Fullerton, R. L. and R. P. McAfee (1999): "Auctioning entry into tournaments," *Journal of Political Economy*, 107, 573–605.

Gao, P., X. Fan, Y. Huang, and Y.-J. Chen (2022): "Resource allocation among competing innovators," *Management Science*, 68, 6059–6074.

Glover, J. and E. Kim (2021): "Optimal team composition: Diversity to foster implicit team incentives," *Management Science*, 67, 5800–5820.

Imberman, S. A. and M. F. Lovenheim (2015): "Incentive strength and teacher productivity: Evidence from a group-based teacher incentive pay system," *Review of Economics and Statistics*, 97, 364–386.

Imhof, L. A. and M. Kräkel (2022): "Team diversity and incentives," *Management Science*, forthcoming.

Kambhampati, A. and C. Segura-Rodriguez (2022): "The optimal assortativity of teams inside the firm," *RAND Journal of Economics*, 53, 484–515.

Kaya, A. and G. Vereshchagina (2014): "Partnerships versus corporations: Moral hazard, sorting, and ownership structure," *American Economic Review*, 104, 291–307.

——— (2015): "Moral hazard and sorting in a market for partnerships," *Economic Theory*, 60, 73–121.

——— (2022): "Sorting expertise," *Journal of Economic Theory*, 204, 105497.

Klump, R., P. McAdam, and A. Willman (2012): "The normalized CES production function: Theory and empirics," *Journal of Economic Surveys*, 26, 769–799.

Kolmar, M. and H. Rommeswinkel (2013): "Contests with group-specific public goods and complementarities in efforts," *Journal of Economic Behavior & Organization*, 89, 9–22.

Lamberson, P. and S. E. Page (2012): "Optimal forecasting groups," *Management Science*, 58, 805–810.

León-Ledesma, M. A., P. McAdam, and A. Willman (2010): "Identifying the elasticity of substitution with biased technical change," *American Economic Review*, 100, 1330–57.

Li, J., N. Lim, and H. Chen (2020): "Examining salesperson effort allocation in teams: A randomized field experiment," *Marketing Science*, 39, 1122–1141.

Loury, G. C. (1979): "Market structure and innovation," *Quarterly Journal of Economics*, 93, 395–410.

McChrystal, G. S., T. Collins, D. Silverman, and C. Fussell (2015): *Team of Teams: New Rules of Engagement for a Complex World*, Penguin.

McFadden, D. (1973): "Conditional logit analysis of qualitative choice behavior," in *Frontier in Econometrics*, ed. by Z. P.

——— (1974): "The measurement of urban travel demand," *Journal of Public Economics*, 3, 303–328.

Nam, P. T. and M. N. Minh (2008): "Proof for a conjecture on general means," *Journal of Inequalities in Pure and Applied Mathematics*, 9, 86.

Nitzan, S. (1991a): "Collective rent dissipation," *Economic Journal*, 101, 1522–1534.

——— (1991b): "Rent-seeking with non-identical sharing rules," *Public Choice*, 71, 43–50.

Ryvkin, D. (2011): "The optimal sorting of players in contests between groups," *Games and Economic Behavior*, 73, 564–572.

Appendix A: Proofs

Proof of Lemma 1

Proof. See the main text. ■

Proof of Proposition 1

Proof. For $\rho \neq 0$, let us define

$$
\begin{aligned}
\mathcal{W}(x,\rho,\gamma) :=& \log\left(\mathcal{Y}^P/\mathcal{Y}^N\right) \\
=& \frac{1}{(1+\gamma)^2}\log(x) + \frac{\gamma-1}{1+\gamma}\log\left(1+x^{\frac{1}{1+\gamma}}\right) \\
&- \frac{\gamma\rho-\gamma-1}{(1+\gamma)\rho}\log 2 - \frac{1+\gamma-\rho}{(1+\gamma)\rho}\log\left(1+x^{\frac{\rho}{1+\gamma-\rho}}\right),
\end{aligned} \tag{8}
$$

and let $\mathcal{W}(x,0,\gamma) := \lim_{\rho\to 0}\log\left(\mathcal{Y}^P/\mathcal{Y}^N\right)$. It can be verified that $\mathcal{W}(x,\rho,\gamma)$ is continuous and differentiable with respect to $\rho \leq 1$. Further, simple algebra would verify that $\mathcal{Y}^P > \mathcal{Y}^N$ is equivalent to $\mathcal{W}(x,\rho,\gamma) > 0$. It is useful to prove two intermediate results.

Lemma 7 *$\mathcal{W}(x,\rho,\gamma)$ is strictly decreasing in ρ. Moreover, fixing $x > 1$ and $\gamma > 0$, there exists a unique solution to $\mathcal{W}(x,\rho^*,\gamma) = 0$, which we denote by $\rho^*(\gamma;x)$.*

Proof. The proof consists of two steps.

Step I We show that $\frac{\partial \mathcal{W}}{\partial \rho}(x,\rho,\gamma) < 0$ for $\rho \in (-\infty, 1]$. Note that $\lim_{\rho\to 0}\partial\mathcal{W}/\partial\rho = -\frac{\log^2(x)}{8(\gamma+1)^2} < 0$, and it suffices to show that $\partial\mathcal{W}/\partial\rho < 0$ for $\rho \neq 0$. Taking the derivative of $\mathcal{W}(x,\rho,\gamma)$ with respect to ρ, we can obtain that

$$
\frac{\partial \mathcal{W}(x,\rho,\gamma)}{\partial \rho} = \frac{1}{\rho^2}\left[\log\left(\frac{1}{2}\left(x^{\frac{\rho}{\gamma-\rho+1}}+1\right)\right) - \frac{\rho\log(x)x^{\frac{\rho}{\gamma-\rho+1}}}{(\gamma-\rho+1)\left(x^{\frac{\rho}{\gamma-\rho+1}}+1\right)}\right].
$$

Let $y := x^{\frac{\rho}{\gamma-\rho+1}} > 0$. It follows immediately from $\rho \neq 0$, $x > 1$, and $\gamma > 0$ that $y \neq 1$. Now we have that

$$
\left.\frac{\partial \mathcal{W}(x,\rho,\gamma)}{\partial \rho}\right|_{x=y^{\frac{\gamma-\rho+1}{\rho}}} = \frac{1}{\rho^2}\left[\log\left(\frac{y+1}{2}\right) - \frac{y\log(y)}{y+1}\right].
$$

It suffices to show that $\log\left(\frac{y+1}{2}\right) - \frac{y\log(y)}{y+1} < 0, \forall\, y > 0, y \neq 1$. Note that

$$
\frac{d}{dy}\left[\log\left(\frac{y+1}{2}\right) - \frac{y\log(y)}{y+1}\right] = -\frac{\log(y)}{(y+1)^2},
$$

and the right-hand side of the above equation is strictly positive if and only if $y < 1$. Therefore, the term $\log\left(\frac{y+1}{2}\right) - \frac{y\log(y)}{y+1}$ achieves its maximum at $y = 1$, which is equal to zero and in turn implies that $\partial\mathcal{W}/\partial\rho < 0$.

Step II We show that $\mathcal{W}(x, -\infty, \gamma) > 0$ and $\mathcal{W}(x, 1, \gamma) < 0$.

(i) We first show that $\mathcal{W}(x, -\infty, \gamma) > 0$. Note that

$$\mathcal{W}(x, -\infty, \gamma) = -(\gamma+1)\left[\gamma\log 2 - (\gamma-1)\log\left(x^{\frac{1}{\gamma+1}}+1\right)\right] - \gamma\log(x) + (\gamma+1)\log(x+1).$$

Carrying out the algebra, we can obtain that

$$w(x,\gamma) := \underbrace{x(x+1)\left(x^{-\frac{1}{\gamma+1}}+1\right)}_{>0} \times \frac{\partial\mathcal{W}(x,-\infty,\gamma)}{\partial x} = -\gamma x^{-\frac{1}{\gamma+1}} + x^{\frac{\gamma}{\gamma+1}} + \gamma x - 1,$$

and

$$\partial w/\partial x = \frac{\gamma x^{-\frac{\gamma+2}{\gamma+1}}\left[(\gamma+1)x^{\frac{1}{\gamma+1}+1} + x + 1\right]}{\gamma+1} > 0, \text{ for } \gamma > 0.$$

From $w(1,\gamma) = 0$, we can conclude that $w(x,\gamma) > 0$, which in turn implies that $\mathcal{W}(x, -\infty, \gamma)$ is strictly increasing in x. Therefore, $\mathcal{W}(x, -\infty, \gamma) > \mathcal{W}(1, -\infty, \gamma) = 0$ for all $x > 1$ and $\gamma > 0$.

(ii) Next, we show that $\mathcal{W}(x, 1, \gamma) < 0$. It is straightforward to verify that $\mathcal{W}(1, 1, \gamma) < 0$. Further, we have that

$$\frac{\partial}{\partial x}\mathcal{W}(x,1,\gamma) = \frac{\left(1 - x^{\frac{1}{\gamma+1}+\frac{1}{\gamma}}\right) + \gamma\left(x^{\frac{1}{\gamma+1}} - x^{\frac{1}{\gamma}}\right)}{x\left(x^{\frac{1}{\gamma+1}}+1\right)\left(x^{1/\gamma}+1\right)} < 0,$$

where the inequality follows from the facts that $1 - x^{\frac{1}{\gamma+1}+\frac{1}{\gamma}} < 0$ and $x^{\frac{1}{\gamma+1}} - x^{\frac{1}{\gamma}} < 0$ for all $x > 1$ and $\gamma > 0$. Therefore, $\mathcal{W}(x, 1, \gamma) < 0$ for all $x > 1$ and $\gamma > 0$.

In summary, $\partial\mathcal{W}/\partial\rho < 0$, $\mathcal{W}(x, -\infty, \gamma) > 0$, and $\mathcal{W}(x, 1, \gamma) < 0$. Therefore, fixing $x > 1$ and $\gamma > 0$, there exists a unique solution to $\mathcal{W}(x, \rho^*, \gamma) = 0$. This concludes the proof. ■

Lemma 8 *Fix $x > 1$ and $\rho \leq 1$. There exists a unique solution to $\mathcal{W}(x, \rho, \gamma) = 0$, which we denote by $\gamma^*(\rho; x) \in (0, \infty)$. Moreover, $\mathcal{W}(x, \rho, \gamma) < 0$ for $\gamma \in \big(0, \gamma^*(\rho; x)\big)$ and $\mathcal{W}(x, \rho, \gamma) > 0$ for $\gamma \in \big(\gamma^*(\rho; x), +\infty\big)$.*

Proof. See Appendix B. ■

Now we are ready to prove the proposition. It remains to show that $\rho^*(\gamma;x)$ is strictly increasing in γ. By the implicit function theorem, we can obtain that

$$\frac{d\rho^*(\gamma;x)}{d\gamma} = -\frac{\partial\mathcal{W}/\partial\gamma}{\partial\mathcal{W}/\partial\rho}\bigg|_{\rho=\rho^*(\gamma;x)}.$$

By Lemma 7, we have that $\partial\mathcal{W}/\partial\rho < 0$. Further, note that $\gamma^*(\rho^*(\gamma;x);x) = \gamma$ by definition, which in turn implies that

$$\frac{\partial\mathcal{W}}{\partial\gamma}\bigg|_{\rho=\rho^*} = \frac{\partial\mathcal{W}}{\partial\gamma}\bigg|_{\gamma=\gamma^*} > 0,$$

where the inequality follows from Lemma 8. Therefore, $d\rho^*(\gamma;x)/d\gamma > 0$. This concludes the proof. ■

Proof of Proposition 2

Proof. The proof consists of three steps.

Step I By Lemma 7, $\partial\mathcal{W}/\partial\rho < 0$ for any $x > 1$ and $\gamma > 0$. Holding fixed γ and applying the implicit function theorem, we can obtain that

$$\frac{d\rho^*}{dx} = -\frac{\partial\mathcal{W}/\partial x}{\partial\mathcal{W}/\partial\rho}\bigg|_{\rho=\rho^*}.$$

Recall $\partial\mathcal{W}/\partial\rho < 0$ from Lemma 7. Therefore, $d\rho^*/dx$ and $\partial\mathcal{W}/\partial x$ have the same sign. Define

$$\begin{aligned} m(\rho) :=& \underbrace{(\gamma+1)^2 x \left(x^{\frac{1}{\gamma+1}}+1\right)\left(x^{\frac{\rho}{\gamma-\rho+1}}+1\right)}_{>0} \times \frac{\partial\mathcal{W}}{\partial x} \\ =& -\left(\gamma + x^{\frac{1}{\gamma+1}}\right) x^{\frac{\rho}{\gamma-\rho+1}} + \left(\gamma x^{\frac{1}{\gamma+1}}+1\right). \end{aligned}$$

It is evident that $m(\rho^*)$ and $d\rho^*/dx$ have the same sign.

It can be verified that $m(\rho)$ is strictly decreasing in ρ. Further, there exists a unique solution to $m(\rho) = 0$, which is given by

$$\hat{\rho} = \frac{(\gamma+1)\left[\log\left(\gamma x^{\frac{1}{\gamma+1}}+1\right) - \log\left(\gamma + x^{\frac{1}{\gamma+1}}\right)\right]}{\log\left(\gamma x^{\frac{1}{\gamma+1}}+1\right) - \log\left(\gamma + x^{\frac{1}{\gamma+1}}\right) + \log(x)}. \tag{9}$$

The above analysis altogether indicates that if $\mathcal{W}(x, \hat{\rho}, \gamma) \gtreqless 0$—which implies that $\hat{\rho} \lesseqgtr \rho^*$—then $m(\rho^*) \lesseqgtr 0$. Therefore, to prove the proposition, it suffices to show that

$$\mathcal{W}(x, \hat{\rho}, \gamma) \begin{cases} < 0 & \text{if } \gamma < 1, \\ = 0 & \text{if } \gamma = 1, \\ > 0 & \text{if } \gamma > 1. \end{cases} \tag{10}$$

Step II Plugging (9) into (8), we can obtain that

$$\begin{aligned}\mathcal{W}(x, \hat{\rho}, \gamma) = \frac{1}{(\gamma+1)^2} \Bigg\{ & (\gamma^2 - 1) \left[\log\left(x^{\frac{1}{\gamma+1}} + 1 \right) - \log 2 \right] \\ & - \frac{\log(x)}{\log\left(\gamma + x^{\frac{1}{\gamma+1}}\right) - \log\left(\gamma x^{\frac{1}{\gamma+1}} + 1\right)} \Bigg[\gamma \log 2 - (\gamma+1)\log(\gamma+1) \\ & - (\gamma+1)\log\left(x^{\frac{1}{\gamma+1}} + 1\right) + \gamma \log\left(\gamma + x^{\frac{1}{\gamma+1}}\right) + \log\left(2\gamma x^{\frac{1}{\gamma+1}} + 2\right) \Bigg] \Bigg\}.\end{aligned}$$

It is evident that $\mathcal{W}(x, \hat{\rho}, 1) = 0$. For $\gamma \neq 1$, define $h := x^{\frac{1}{\gamma+1}} > 1$ and

$$T(h, \gamma) := (\gamma+1) \frac{\log(\gamma+h) - \log(\gamma h + 1)}{\log(h)} \mathcal{W}(x, \hat{\rho}, \gamma) \Big|_{x = h^{1+\gamma}}.$$

It can be verified that $\log(\gamma + h) - \log(\gamma h + 1) \gtreqless 0$ is equivalent to $\gamma \lesseqgtr 1$. Further, simple algebra would verify that $T(1, \gamma) = 0$. Therefore, to prove (10), it suffices to show that $T(h, \gamma) < 0 = T(1, \gamma)$, for $h > 1$, $\gamma > 0$, and $\gamma \neq 1$.

Step III Carrying out the algebra, we have that

$$\begin{aligned}T(h, \gamma) = & (\gamma - 1) \frac{\left[\log(h+1) - \log 2\right] \left[\log(\gamma + h) - \log(\gamma h + 1)\right]}{\log(h)} - (\gamma+1)\log 2 \\ & + (\gamma+1)\log(\gamma+1) + (\gamma+1)\log(h+1) - \gamma\log(\gamma+h) - \log(\gamma h + 1).\end{aligned}$$

Further, we can obtain that

$$\begin{aligned}\tau(h, \gamma) := & \underbrace{h(h+1)(\gamma+h)(\gamma h+1)\log^2(h)}_{>0} \times \frac{\partial T(h, \gamma)}{\partial h} \\ = & (\gamma - 1)\Big[-(h+1)\log(h+1) + h\log(h) + (h+1)\log 2 \Big] \\ & \times \Big\{ \left(\gamma + \gamma h^2 + \gamma^2 h + h\right) \left[\log(\gamma+h) - \log(\gamma h+1)\right] + \left(\gamma^2 - 1\right) h \log(h) \Big\}\end{aligned}$$

$$=(\gamma-1)S_1(h)S_2(h,\gamma),$$

where $S_1(h)$ and $S_2(h,\gamma)$ are defined as

$$S_1(h) := -(h+1)\log(h+1) + h\log(h) + (h+1)\log 2,$$

and

$$S_2(h,\gamma) := \left(\gamma+\gamma h^2+\gamma^2 h+h\right)\left[\log(\gamma+h)-\log(\gamma h+1)\right] + \left(\gamma^2-1\right)h\log(h).$$

First, note that $S_1'(h) = \log(h) - \log(h+1) + \log 2 > 0$, which implies that $S_1(h) > S_1(1) = 0$. Next, consider $S_2(h,\gamma)$. Simple algebra would verify that

$$\frac{\partial}{\partial h}S_2(h,\gamma) = \left(\gamma^2-1\right)\log(h) + \left(\gamma^2+2\gamma h+1\right)\left[\log(\gamma+h)-\log(\gamma h+1)\right],$$

and

$$\frac{\partial^2}{\partial h^2}S_2(h,\gamma) = -\frac{\gamma\left(\gamma^2-1\right)\left(h^2-1\right)}{h(\gamma+h)(\gamma h+1)} + 2\gamma\left[\log(\gamma+h)-\log(\gamma h+1)\right]\begin{cases} >0 & \text{if } \gamma<1, \\ <0 & \text{if } \gamma>1. \end{cases}$$

Further, $S_2(1,\gamma)=0$ and $\frac{\partial}{\partial h}S_2(1,\gamma)=0$. These altogether enable us to conclude that

$$S_2(h,\gamma)\begin{cases} >0 & \text{if } \gamma<1, \\ <0 & \text{if } \gamma>1. \end{cases}$$

Therefore, $\tau(h,\gamma) \equiv (\gamma-1)S_1(h)S_2(h,\gamma) < 0$ for $\gamma \neq 1$, which implies that $\partial T/\partial h < 0$ for all $h > 1$ and thus $T(h,\gamma) < T(1,\gamma) = 0$.

To conclude the proof, note that fixing $\gamma > 0$, $\mathcal{W}(x,\rho,\gamma) \to +\infty$ as $x \to +\infty$ for $\rho < 0$ and $\mathcal{W}(x,\rho,\gamma) \to -\infty$ as $x \to +\infty$ for $\rho > 0$. This suggests that $\rho^* \to 0$ as $x \to +\infty$. ■

Proof of Lemmas 2 to 4

Proof. By standard technique, the equilibrium aggregate output—which we denote by $\mathcal{Y}$ with slight abuse of notation—for an arbitrary cost profile is $\boldsymbol{c} := \langle(c_{11},c_{12}),(c_{21},c_{22})\rangle$ and prize schedule $\boldsymbol{v} := \langle(v_{11},v_{12}),(v_{21},v_{22})\rangle$ can be derived as

$$\mathcal{Y} = \left(\frac{1}{2}\right)^{\frac{1}{\rho}}\mathcal{K}_1^{\frac{1}{\gamma+1}}\mathcal{K}_2^{\frac{1}{\gamma+1}}\left(\mathcal{K}_1+\mathcal{K}_2\right)^{\frac{\gamma-1}{\gamma+1}},$$

where

$$\mathcal{K}_i = \left(\frac{1}{c_{i1}c_{i2}}\right)^{\frac{1}{\gamma+1}}\left[(v_{i1}c_{i2})^{\frac{\rho}{\gamma-\rho+1}} + (v_{i2}c_{i1})^{\frac{\rho}{\gamma-\rho+1}}\right]^{\frac{\gamma-\rho+1}{\rho(\gamma+1)}}, i \in \{1,2\}.$$

Define $v_i := v_{i1} + v_{i2} \leq 2, i = 1, 2$. Therefore, $v_{i2} = v_i - v_{i1}$. In the subsequent analysis, we treat (v_i, v_{i1}) instead of (v_{i1}, v_{i2}) as the design variables.

The aggregate output under positive sorting—which we denote by $\mathcal{Y}_a^P(v_1, v_2, v_{11}, v_{21})$—and that under negative sorting—which we denote by $\mathcal{Y}_a^N(v_1, v_2, v_{11}, v_{21})$—can be derived as follows:

$$\mathcal{Y}_a^P(v_1, v_2, v_{11}, v_{21}) = \frac{\Lambda_1^{\frac{1}{1+\gamma}} \Lambda_2^{\frac{1}{1+\gamma}} \left(c_H^{\frac{1}{1+\gamma}} \Lambda_1 + c_L^{\frac{1}{1+\gamma}} \Lambda_2 \right)^{\frac{\gamma-1}{\gamma+1}}}{2^{\frac{1}{\rho}} c_H^{\frac{\gamma}{(1+\gamma)^2}} c_L^{\frac{\gamma}{(1+\gamma)^2}}},$$

and

$$\mathcal{Y}_a^N(v_1, v_2, v_{11}, v_{21}) = \frac{\Omega_1^{\frac{1}{1+\gamma}} \Omega_2^{\frac{1}{1+\gamma}} (\Omega_1 + \Omega_2)^{\frac{\gamma-1}{\gamma+1}}}{2^{\frac{1}{\rho}} c_H^{\frac{1}{1+\gamma}} c_L^{\frac{1}{1+\gamma}}},$$

where $\Lambda_i := \left[v_{i1}^{\frac{\rho}{1+\gamma-\rho}} + (v_i - v_{i1})^{\frac{\rho}{1+\gamma-\rho}} \right]^{\frac{1+\gamma-\rho}{\rho(1+\gamma)}}$ and $\Omega_i := \left[(c_H v_{i1})^{\frac{\rho}{1+\gamma-\rho}} + \left[c_L (v_i - v_{i1}) \right]^{\frac{\rho}{1+\gamma-\rho}} \right]^{\frac{1+\gamma-\rho}{\rho(1+\gamma)}}$.

It is evident that $\mathcal{Y}_a^P(v_1, v_2, v_{11}, v_{21})$ is increasing in Λ_i and Λ_i is increasing in v_i. Therefore, $v_i = 2$ for $i \in \{1, 2\}$ in the optimum under positive sorting. Similarly, we can show that $v_i = 2$ for $i \in \{1, 2\}$ in the optimum under negative sorting. That is, "money burning" will not arise in the optimum.

Positive Sorting In what follows, we write Λ_i as a function of v_{i1}. It can be verified that Λ_i is maximized at $v_{i1} \in \{0, 1, 2\}$, depending on the value of ρ. For $\rho > 0$, we have

$$\Lambda_i(1) = 2^{\frac{1+\gamma-\rho}{\rho(1+\gamma)}}, \ \Lambda_i(0) = \Lambda_i(2) = 2^{\frac{1}{1+\gamma}}.$$

Simple algebra would verify that $\Lambda_i(1) > \Lambda_i(0) = \Lambda_i(2)$ if and only if $0 < \rho < \frac{1+\gamma}{2}$. For $\rho \leq 0$, we have $\Lambda_i(0) = \Lambda_i(2) = 0 < \Lambda_i(1)$.

In summary, for $\rho \in \left(-\infty, \min\{\frac{1+\gamma}{2}, 1\} \right)$, it is optimal for the manager to set $v_{11} = v_{21} = 1$; otherwise, the manager should set $v_{11} = v_{21} \in \{0, 2\}$. The expression of total output under positive sorting can be derived accordingly, as shown in Lemma 3.

Negative Sorting We write Ω_i as a function of v_{i1}. Simple algebra would verify that Ω_i is maximized at $v_{i1} = \frac{2c_H^{\frac{\rho}{2\rho-\gamma-1}}}{c_H^{\frac{\rho}{2\rho-\gamma-1}} + c_L^{\frac{\rho}{2\rho-\gamma-1}}}$ or at $v_{i1} = 2$, depending on the value of ρ.

For $\rho > 0$, we have that

$$\Omega_i\left(\frac{2c_H^{\frac{\rho}{2\rho-\gamma-1}}}{c_H^{\frac{\rho}{2\rho-\gamma-1}}+c_L^{\frac{\rho}{2\rho-\gamma-1}}}\right)=\frac{2^{\frac{1}{1+\gamma}}c_H^{\frac{1}{1+\gamma}}c_L^{\frac{1}{1+\gamma}}}{\left(c_H^{\frac{\rho}{2\rho-\gamma-1}}+c_L^{\frac{\rho}{2\rho-\gamma-1}}\right)^{\frac{2\rho-\gamma-1}{\rho(1+\gamma)}}},$$

and

$$\Omega_i(2)=2^{\frac{1}{1+\gamma}}c_H^{\frac{1}{1+\gamma}}.$$

It can be verified that $\Omega_i\left(\frac{2c_H^{\frac{\rho}{2\rho-\gamma-1}}}{c_H^{\frac{\rho}{2\rho-\gamma-1}}+c_L^{\frac{\rho}{2\rho-\gamma-1}}}\right) > \Omega_i(2)$ and thus the manager should optimally set $v_{11} = v_{21} = \frac{2c_H^{\frac{\rho}{2\rho-\gamma-1}}}{c_H^{\frac{\rho}{2\rho-\gamma-1}}+c_L^{\frac{\rho}{2\rho-\gamma-1}}}$ if $0 < \rho < \frac{1+\gamma}{2}$. Otherwise, setting $v_{11} = v_{21} = 2$ maximizes the aggregate output.

For $\rho \leq 0$, it can be verified that $\Omega_i\left(\frac{2c_H^{\frac{\rho}{2\rho-\gamma-1}}}{c_H^{\frac{\rho}{2\rho-\gamma-1}}+c_L^{\frac{\rho}{2\rho-\gamma-1}}}\right) > 0 = \Omega_i(2)$ and thus the manager should optimally set $v_{11} = v_{21} = \frac{2c_H^{\frac{\rho}{2\rho-\gamma-1}}}{c_H^{\frac{\rho}{2\rho-\gamma-1}}+c_L^{\frac{\rho}{2\rho-\gamma-1}}}$.

In summary, for $\rho \in \left(-\infty, \min\{\frac{1+\gamma}{2}, 1\}\right)$, it is optimal for the manager to set $v_{11} = v_{21} = \frac{2c_H^{\frac{\rho}{2\rho-\gamma-1}}}{c_H^{\frac{\rho}{2\rho-\gamma-1}}+c_L^{\frac{\rho}{2\rho-\gamma-1}}}$; otherwise, the manager should set $v_{11} = v_{21} = 2$. The expression of total output under negative sorting can be derived accordingly, as shown in Lemma 4. ■

Proof of Proposition 3

Proof. By Lemmas 3 and 4, for $0 < \gamma < 1$ and $\frac{1+\gamma}{2} < \rho \leq 1$, we have that

$$\mathcal{Y}_a^P=\frac{\left(c_H^{\frac{1}{1+\gamma}}+c_L^{\frac{1}{1+\gamma}}\right)^{\frac{\gamma-1}{1+\gamma}}}{2^{\frac{1}{\rho}}c_H^{\frac{\gamma}{(1+\gamma)^2}}c_L^{\frac{\gamma}{(1+\gamma)^2}}},\text{ and }\mathcal{Y}_a^N=\frac{2^{\frac{\gamma\rho-\rho-\gamma-1}{\rho(1+\gamma)}}}{c_L^{\frac{1}{1+\gamma}}}.$$

It can be verified that for $x > 1$ and $0 < \gamma < 1$, we have $\mathcal{Y}_a^P < \mathcal{Y}_a^N$—i.e., negative sorting outperforms positive sorting. For $\rho \in \left(-\infty, \min\{\frac{1+\gamma}{2}, 1\}\right)$, we have that

$$\mathcal{Y}_a^P=\frac{\left(c_H^{\frac{1}{1+\gamma}}+c_L^{\frac{1}{1+\gamma}}\right)^{\frac{\gamma-1}{1+\gamma}}}{2^{\frac{2}{1+\gamma}}c_H^{\frac{\gamma}{(1+\gamma)^2}}c_L^{\frac{\gamma}{(1+\gamma)^2}}},\text{ and }\mathcal{Y}_a^N=\frac{2^{\frac{\gamma\rho-\rho-\gamma-1}{\rho(1+\gamma)}}}{\left(c_H^{\frac{\rho}{2\rho-\gamma-1}}+c_L^{\frac{\rho}{2\rho-\gamma-1}}\right)^{\frac{2\rho-\gamma-1}{\rho(1+\gamma)}}}.$$

Similar to the proof of Proposition 1, for $\rho \neq 0$, define

$$\begin{aligned}\mathcal{V}(x,\rho,\gamma) :=& \log\left(\mathcal{Y}_a^P/\mathcal{Y}_a^N\right)\\ =& \frac{\gamma-1}{1+\gamma}\log\left(1+x^{\frac{1}{1+\gamma}}\right)+\frac{2\rho-\gamma-1}{\rho(1+\gamma)}\log\left(1+x^{\frac{\rho}{2\rho-\gamma-1}}\right)\\ &-\frac{\rho-1}{\rho}\log 2-\frac{\gamma}{(1+\gamma)^2}\log(x),\end{aligned} \tag{11}$$

and let $\mathcal{V}(x,0,\gamma) := \lim_{\rho\to 0}\log\left(\mathcal{Y}_a^P/\mathcal{Y}_a^N\right)$. Evidently, $\mathcal{Y}_a^P > \mathcal{Y}_a^N$ is equivalent to $\mathcal{V}(x,\rho,\gamma) > 0$. Further, we have that

$$\begin{aligned}\mathcal{V}(x,-\infty,\gamma) =& -\frac{1}{(\gamma+1)^2}\Big[(\gamma+1)^2\log 2-(\gamma^2-1)\log\left(x^{\frac{1}{\gamma+1}}+1\right)\\ &+\gamma\log(x)-2(\gamma+1)\log\left(\sqrt{x}+1\right)\Big].\end{aligned}$$

The following lemma can be established.

Lemma 9 *Let $\bar{\gamma} = \sqrt{5}-2$. The following statements hold:*

(i) For $\gamma \geq \bar{\gamma}$, $\mathcal{V}(x,-\infty,\gamma) > 0$ for all $x > 1$.

(ii) For $0 < \gamma < \bar{\gamma}$, there exists a unique threshold $\hat{x}(\gamma) > 1$ such that $\mathcal{V}(x,-\infty,\gamma) < 0$ for $x \in \left(1,\hat{x}(\gamma)\right)$; $\mathcal{V}(\hat{x}(\gamma),-\infty,\gamma) = 0$ for $x = \hat{x}(\gamma)$; and $\mathcal{V}(x,-\infty,\gamma) > 0$ for $x > \hat{x}(\gamma)$.

Proof. First, it is straightforward to verify that $\mathcal{V}(1,-\infty,\gamma) = 0$ and $\mathcal{V}(+\infty,-\infty,\gamma) = +\infty$. Define $V(x,\gamma) = \mathcal{V}(x,-\infty,\gamma)$ with slight abuse of notation. It follows that

$$\frac{\partial V(x,\gamma)}{\partial x} = \frac{-\gamma+\gamma x^{\frac{1}{\gamma+1}+\frac{1}{2}}-x^{\frac{1}{\gamma+1}}+\sqrt{x}}{(\gamma+1)^2\left(\sqrt{x}+1\right)x\left(x^{\frac{1}{\gamma+1}}+1\right)}.$$

Let $V_1(x,\gamma) := -\gamma+\gamma x^{\frac{1}{\gamma+1}+\frac{1}{2}}-x^{\frac{1}{\gamma+1}}+\sqrt{x}$. Then $V_1(1,\gamma) = 0$ and

$$\frac{\partial V_1(x,\gamma)}{\partial x} = \frac{\gamma(\gamma+3)x^{\frac{1}{\gamma+1}}-2x^{\frac{1}{\gamma+1}-\frac{1}{2}}+(\gamma+1)}{2(\gamma+1)x^{1/2}}.$$

Let $V_2(x,\gamma) := \gamma(\gamma+3)x^{\frac{1}{\gamma+1}}-2x^{\frac{1}{\gamma+1}-\frac{1}{2}}+(\gamma+1)$. It follows that $V_2(1,\gamma) = \gamma^2+4\gamma-1$, which is greater than 0 if and only if $\gamma > \sqrt{5}-2$. Further, we have that

$$\frac{\partial V_2(x,\gamma)}{\partial x} = \frac{x^{\frac{1}{\gamma+1}-\frac{3}{2}}\left(\gamma^2x^{1/2}+3\gamma x^{1/2}+\gamma-1\right)}{\gamma+1}.$$

Let $V_3(x,\gamma) := \gamma^2 x^{1/2} + 3\gamma x^{1/2} + \gamma - 1$. It is evident that $V_3(x,\gamma)$ is increasing in x. Further, $V_3(1,\gamma) = \gamma^2 + 4\gamma - 1$, which is greater than 0 if and only if $\gamma > \sqrt{5} - 2$.

For $\gamma \geq \sqrt{5} - 2$, we have that $V_3(1,\gamma) \geq 0$ and thus $V_3(x,\gamma) > 0$ for all $x > 1$, which implies that $\frac{\partial V_2(x,\gamma)}{\partial x} > 0$. Therefore, $V_2(x,\gamma)$ is increasing in x; together with the fact that $V_2(1,\gamma) \geq 0$ for $\gamma > \sqrt{5}-2$, we can conclude that $V_2(x,\gamma) > 0$ for all $x > 1$ and thus $V_1(x,\gamma)$ is increasing in x. Recall that $V_1(1,\gamma) = 0$. Therefore, $V_1(x,\gamma) > 0$ and thus $V(x,\gamma) > 0$ for all $x > 1$. This concludes the proof of Lemma 9(i).

For $\gamma < \sqrt{5} - 2$, we have that $V_3(1,\gamma) < 0$. Recall that $V_3(x,\gamma)$ increases with x and turns positive as x becomes sufficiently large. This indicates that $V_2(x,\gamma)$ is U-shaped in x.

Next, note that $V_2(1,\gamma) < 0$ for $\gamma < \sqrt{5} - 2$. Further, $V_2(+\infty,\gamma) = +\infty$. Therefore, $V_2(x,\gamma)$ is U-shaped in x, first being negative and turning positive as x ascends. This in turn implies that $V_1(x,\gamma)$ is U-shaped in x. Note that $V_1(1,\gamma) = 0$ and $V_1(+\infty,\gamma) = +\infty$. By similar argument, we can show that $V(x,\gamma)$ is also U-shaped in x. Note that $V(1,\gamma) = 0$. Therefore, there exists another solution to $V(x,\gamma) = 0$—which we denote by $\hat{x}(\gamma)$—such that $V(x,\gamma) < 0$ for $x \in (1,\hat{x}(\gamma))$ and $V(x,\gamma) \geq 0$ otherwise. This concludes the proof of Lemma 9(ii). ■

Lemma 10 *$\mathcal{V}(x,\rho,\gamma)$ strictly decreases with ρ. If $0 < \gamma < \bar{\gamma}$ and $1 < x \leq \hat{x}(\gamma)$, $\mathcal{V}(x,\rho,\gamma) < 0$. Otherwise, there exists a unique solution to $\mathcal{V}(x,\rho,\gamma) = 0$, which we denote by $\rho_a^*(\gamma;x)$; moreover, $\rho_a^*(\gamma;x) \in \left(-\infty, \min\{\frac{1+\gamma}{2}, 1\}\right)$.*

Proof. For $\rho \neq 0$, we have that

$$\begin{aligned}\frac{\partial \mathcal{V}(x,\rho,\gamma)}{\partial \rho} =& \frac{x^{\frac{\rho}{2\rho-\gamma-1}}}{\rho^2(1+\gamma-2\rho)\left(1+x^{\frac{\rho}{2\rho-\gamma-1}}\right)} \\ &\times \left\{\rho\log(x) + (1+\gamma-2\rho)\left(1+x^{\frac{\rho}{\gamma-2\rho+1}}\right)\left[\log\left(1+x^{\frac{\rho}{2\rho-\gamma-1}}\right) - \log 2\right]\right\}.\end{aligned}$$

Let $z := x^{\frac{\rho}{2\rho-\gamma-1}}$. It can be verified that $z > 1$ for $\rho < 0$ and $0 < z < 1$ for $0 < \rho < \min\{\frac{1+\gamma}{2}, 1\}$. Simple algebra would verify that

$$\left.\frac{\partial \mathcal{V}(x,\rho,\gamma)}{\partial \rho}\right|_{x=z^{\frac{2\rho-\gamma-1}{\rho}}} = \frac{1}{\rho^2}\left[\log\left(\frac{z+1}{2}\right) - \frac{z\log(z)}{z+1}\right] < 0, \text{ for all } z > 0, z \neq 1,$$

where the last inequality follows from the proof of Lemma 7. Further, $\lim_{\rho\to 0}\frac{\partial \mathcal{V}(x,\rho,\gamma)}{\partial \rho} = -\frac{\log^2(x)}{8(\gamma+1)^2} < 0$. Therefore, $\mathcal{V}(x,\rho,\gamma)$ is strictly decreasing in ρ.

Suppose that $\mathcal{V}(x,-\infty,\gamma) \leq 0$. Then $\mathcal{V}(x,\rho,\gamma) < 0$ for all $\rho \leq 1$. Suppose otherwise that $\mathcal{V}(x,-\infty,\gamma)>0$; together with the fact that

$$\mathcal{V}\left(x,\frac{1+\gamma}{2},\gamma\right)=-\frac{\gamma}{(1+\gamma)^2}\log(x)-\frac{1-\gamma}{1+\gamma}\left[\log\left(x^{\frac{1}{\gamma+1}}+1\right)-\log 2\right]<0,\text{ for }0<\gamma<1,$$

and

$$\mathcal{V}(x,1,\gamma)=-\frac{1}{(1+\gamma)^2}\left\{\left(\gamma^2-1\right)\left[\log\left(x^{\frac{1}{1-\gamma}}+1\right)-\log\left(x^{\frac{1}{\gamma+1}}+1\right)\right]+\gamma\log(x)\right\}<0,\text{ for }\gamma>1,$$

we can conclude that there exists a unique solution $\rho_a^*(\gamma;x)$ to $\mathcal{V}(x,\rho,\gamma)=0$, and $\rho_a^*(\gamma;x)\in\left(-\infty,\min\{\frac{1+\gamma}{2},1\}\right)$. ■

Proposition 3 follows immediately from Lemmas 9 and 10. This concludes the proof. ■

Proof of Proposition 4

Proof. The proof follows from the facts that (i) $\mathcal{Y}_a^P=\mathcal{Y}^P$; (ii) $\mathcal{Y}_a^N\geq\mathcal{Y}^N$; and (iii) when the condition stated in Proposition 3(ii) is satisfied, solving for $\mathcal{Y}_a^P=\mathcal{Y}^P=\mathcal{Y}_a^P=\mathcal{Y}^N$ gives $(\rho,\gamma)=(0,1)$. ■

Proof of Lemmas 5 and 6

Proof. Fixing an arbitrary cost profile $\boldsymbol{c}:=\langle(c_{11},c_{12}),(c_{21},c_{22})\rangle$ and a resource allocation plan $\boldsymbol{\delta}:=(\delta_1,\delta_2)$, the first-order conditions that govern workers' equilibrium effort are

$$\frac{\delta_1\delta_2}{2}e_{ik}^{\rho-1}\left(\frac{1}{2}e_{i1}^{\rho}+\frac{1}{2}e_{i2}^{\rho}\right)^{\frac{1}{\rho}-1}\frac{\mathcal{Y}_{-i}}{(\delta_1\mathcal{Y}_1+\delta_2\mathcal{Y}_2)^2}=c_{ik}e_{ik}^{\gamma},i\in\{1,2\},k\in\{1,2\},\tag{12}$$

from which we can derive equilibrium individual efforts under positive sorting ($c_{11}=c_{12}=c_L$ and $c_{21}=c_{22}=c_H$) and those under negative sorting ($c_{11}=c_{21}=c_L$ and $c_{12}=c_{22}=c_H$), respectively. Note that δ_1 refers to the amount of resource allocated to the strong team under positive sorting. The associated equilibrium aggregate output can then be derived as

$$\mathcal{Y}_r^P(\delta_1,\delta_2)=\frac{\left[\delta_1\delta_2\left(\delta_1c_H^{\frac{1}{1+\gamma}}+\delta_2c_L^{\frac{1}{1+\gamma}}\right)^{\gamma-1}\right]^{\frac{1}{1+\gamma}}}{2^{\frac{1}{1+\gamma}}c_H^{\frac{\gamma}{(1+\gamma)^2}}c_L^{\frac{\gamma}{(1+\gamma)^2}}},\tag{13}$$

and

$$\mathcal{Y}_r^N(\delta_1,\delta_2)=(\delta_1\delta_2)^{\frac{1}{1+\gamma}}\frac{\left(c_H^{\frac{\rho}{1+\gamma-\rho}}+c_L^{\frac{\rho}{1+\gamma-\rho}}\right)^{\frac{1+\gamma-\rho}{\rho(1+\gamma)}}}{2^{\frac{1+\gamma+\rho-\gamma\rho}{\rho(1+\gamma)}}c_H^{\frac{1}{1+\gamma}}c_L^{\frac{1}{1+\gamma}}}.\tag{14}$$

It is straightforward to show that $\delta_1 + \delta_2 = 2$ in the optimum. Further, the optimal amount of resource allocated to the first team is $\delta_1^P = \frac{\gamma(X-1)-2+\sqrt{\gamma^2(X-1)^2+4(X-1)+4}}{(\gamma+1)(X-1)}$ under positive sorting—where $X = \left(c_H/c_L\right)^{\frac{1}{1+\gamma}}$—and is $\delta_1^N = 1$ under negative sorting. Plugging δ_i^P and δ_i^N into (13) and (14) respectively, we can derive the equilibrium outputs as shown in Lemma 5 and Lemma 6. This concludes the proof. ■

Proof of Proposition 5

Proof. The proof is similar to that of Proposition 1, and is omitted for brevity. ■

Proof of Proposition 6

Proof. The proof is similar to that of Proposition 4, and is omitted for brevity. ■

Appendix B: Additional Proofs

Proof of Lemma 8

Proof. Define

$$\mathfrak{A}(x,\rho,\gamma) := \frac{\partial}{\partial\gamma}\left[(1+\gamma)^2\mathcal{W}(x,\rho,\gamma)\right]$$
$$= -\frac{2\left\{\left[\gamma(\rho-1)-1\right]\log 2+\rho(1-\gamma)\log\left(x^{\frac{1}{\gamma+1}}+1\right)+(\gamma-\rho+1)\log\left(x^{\frac{\rho}{\gamma-\rho+1}}+1\right)\right\}}{\rho}$$
$$+\frac{\log(x)\left[-2(\gamma-\rho+1)+(\gamma+1)\rho x^{\frac{\rho}{\gamma-\rho+1}+\frac{1}{\gamma+1}}+\left(\gamma^2+2\rho-1\right)x^{\frac{\rho}{\gamma-\rho+1}}-(\gamma+1)(\gamma-\rho+1)x^{\frac{1}{\gamma+1}}\right]}{(\gamma+1)(\gamma-\rho+1)\left(x^{\frac{1}{\gamma+1}}+1\right)\left(x^{\frac{\rho}{\gamma-\rho+1}}+1\right)}$$
$$+\frac{2\log(x)}{\gamma+1}-\log 2-\log\left(x^{\frac{\rho}{\gamma-\rho+1}}+1\right)+2\log\left(x^{\frac{1}{\gamma+1}}+1\right).$$

The key to the proof of the lemma is to show that $\mathfrak{A}(x,\rho,\gamma)>0$.

To proceed, define $k := x^{\frac{1}{\gamma+1}}$ and $\ell := x^{\frac{\rho}{\gamma-\rho+1}}$. It follows immediately that $\rho = 1\Big/\left(\frac{\log(k)}{\log(\ell)}+\frac{1}{1+\gamma}\right)$. Replacing x and ρ with k and ℓ respectively, we can obtain that

$$\mathfrak{B}(k,\ell,\gamma) := (k+1)(\ell+1)\log(\ell)\times\mathfrak{A}(x,\rho,\gamma)\Big|_{x=k^{1+\gamma},\,\rho=1\big/\left(\frac{\log(k)}{\log(\ell)}+\frac{1}{1+\gamma}\right)}$$
$$=\log(k)\left\{\log(\ell)\left(-\gamma k+2k\ell+k+\gamma\ell+\ell\right)+(\gamma+1)(k+1)(\ell+1)\left[\log 4-2\log(\ell+1)\right]\right\}$$
$$-(k+1)\log(\ell)\left\{-2\gamma(\ell+1)\log(k+1)+(\ell+1)\left[(2\gamma-1)\log 2+\log(\ell+1)\right]-\ell\log(\ell)\right\}.$$

We consider the three following cases, depending on ρ relative to 0.

Case I: $0<\rho\leq 1$. Then we have $\ell\in(1,+\infty)$ and $k\in\left[\ell^{\frac{\gamma}{1+\gamma}},+\infty\right)$. Let $h := \ell^{\frac{\gamma}{\gamma+1}}$. It follows immediately that $\ell>h>1$. Define

$$B(h,\ell) := \underbrace{\frac{\log(\ell)-\log(h)}{\log(\ell)}}_{>0}\times\mathfrak{B}\left(\ell^{\frac{\gamma}{1+\gamma}},\ell,\gamma\right)\Big|_{\ell^{\frac{\gamma}{\gamma+1}}=h}$$
$$=-2h(\ell+1)\log^2(h)+(h+1)\log(\ell)\left\{\ell\log(\ell)+(\ell+1)\left[\log 2-\log(\ell+1)\right]\right\}$$
$$-(\ell+1)\log(h)\left[-h\log(\ell)+h\log(\ell+1)+h\log 2-2(h+1)\log(h+1)+\log(2\ell+2)\right].$$

Carrying out the algebra, we have that

$$B(h,h)=0,$$

$$\frac{\partial B}{\partial \ell}(h,h) = \frac{(h+1)^2}{h}\left[\log 2 - \log(h+1)\right] + (h+1)\log(h) + \log^2(h) > 0,$$
$$\frac{\partial^2 B}{\partial \ell^2}(h,h) = \frac{(h+1)\left\{h\log(h) + (h-1)\left[\log 2 - \log(h+1)\right]\right\}}{h^2} > 0,$$
$$\frac{\partial}{\partial \ell}\left(\ell^2(\ell+1)\frac{\partial^2 B}{\partial \ell^2}\right)(h,h) = 2\left(h^2+h+1\right)\log(h) + 2h(h+1)\left[\log 2 - \log(h+1)\right] + 3(h+1) > 0,$$
$$\frac{\partial^2}{\partial \ell^2}\left(\ell^2(\ell+1)\frac{\partial^2 B}{\partial \ell^2}\right)(h,+\infty) = 2(h+1)\log 2 - 2\log(h) > 0,$$
$$\frac{\partial^3}{\partial \ell^3}\left(\ell^2(\ell+1)\frac{\partial^2 B}{\partial \ell^2}\right)(h,\ell) = -\frac{2(h+1)\left(\ell^2+\ell+1\right)}{\ell^2(\ell+1)^2} < 0.$$

The above derivatives and conditions, together with the fact that $\ell > h > 1$, imply that $B(h,\ell) > 0$, which in turn implies that

$$\mathfrak{B}\left(\ell^{\frac{\gamma}{1+\gamma}}, \ell, \gamma\right) > 0. \tag{15}$$

Next, consider $\partial(k\mathfrak{B})/\partial k$. Define

$$\begin{aligned} B_1(h,\ell) :=& \underbrace{\frac{\log(\ell)-\log(h)}{\log(\ell)}}_{>0} \times \frac{\partial(k\mathfrak{B})}{\partial k}\left(\ell^{\frac{\gamma}{1+\gamma}}, \ell, \gamma\right)\Big|_{\ell^{\frac{\gamma}{\gamma+1}}=h} \\ =& -4h(\ell+1)\log^2(h) + (2h\ell+\ell)\log^2(\ell) - (\ell+1)\log(h)\Big[-2h\log(\ell) + 2h\log(\ell+1) \\ &+ h\log 4 - 2(2h+1)\log(h+1) + \log(2\ell+2)\Big] + (h+1)(\ell+1)\left[\log 4 - 2\log(\ell+1)\right] \\ &+ \log(\ell)\left\{h\left[\ell\left(2+\log 4\right) + 1 + \log 4\right] - (2h+1)(\ell+1)\log(\ell+1) + \ell + (\ell+1)\log 2\right\}. \end{aligned}$$

Carrying out the algebra, we can obtain that

$$B_1(h,h) = 2(h+1)^2\left[\log 2 - \log(h+1)\right] - h\log^2(h) + 2(h+1)h\log(h) > 0,$$
$$\frac{\partial B_1}{\partial \ell}(h,h) = \log^2(h) + (4h+3)\log(h) + \frac{(h+1)(4h+1)}{h}\left[\log 2 - \log(h+1)\right] > 0,$$
$$\frac{\partial^2 B_1}{\partial \ell^2}(h,h) = \frac{2(h+2)h^2\log(h) + \left(h^2-1\right)(2h+1)\left[\log 2 - \log(h+1)\right]}{h^2(h+1)} > 0,$$
$$\frac{\partial}{\partial \ell}\left(\ell^2(\ell+1)\frac{\partial^2 B_1}{\partial \ell^2}\right)(h,h) = \left(4h^2+4h+2\right)\log(h) + 2(2h+1)h\left[\log 2 - \log(h+1)\right] + 5h + 4 > 0,$$
$$\frac{\partial^2}{\partial \ell^2}\left(\ell^2(\ell+1)\frac{\partial^2 B_1}{\partial \ell^2}\right)(h,+\infty) = 4h\log 2 - 2\log(h) - 2 + 2\log 2 > 0,$$

$$\frac{\partial^3}{\partial \ell^3}\left(\ell^2(\ell+1)\frac{\partial^2 B_1}{\partial \ell^2}\right)(h,\ell) = -\frac{2(2h+1)\left(\ell^2+\ell+1\right)}{\ell^2(\ell+1)^2} < 0.$$

From the above derivatives and conditions, we can conclude that $B_1(h,\ell) > 0$, which in turn implies that

$$\frac{\partial (k\mathfrak{B})}{\partial k}\left(\ell^{\frac{\gamma}{1+\gamma}}, \ell, \gamma\right) > 0. \tag{16}$$

Next, let us define

$$\begin{aligned}
B_2(h,\ell) &:= \underbrace{\frac{h(h+1)\left[\log(\ell) - \log(h)\right]}{\log(\ell)}}_{>0} \times \frac{\partial^2 (k\mathfrak{B})}{\partial k^2}\left(\ell^{\frac{\gamma}{1+\gamma}}, \ell, \gamma\right)\Bigg|_{\ell^{\frac{\gamma}{\gamma+1}}=h} \\
&= -h(\ell+1)\log(h)\Big[-2(h+1)\log(\ell) + 2h\log(\ell+1) + h\log 4 - 4(h+1)\log(h+1) \\
&\quad + 2\log(\ell+1) + 2 + \log 4\Big] - 4h(h+1)(\ell+1)\log^2(h) + 2h(h+1)\ell\log^2(\ell) \\
&\quad + (h+1)\log(\ell)\Big\{h\left[\ell\left(6+\log 4\right) + 3 + \log 4\right] - 2h(\ell+1)\log(\ell+1) + \ell\Big\} \\
&\quad + (\ell+1)\left(3h^2+4h+1\right)\left[-2\log(\ell+1) + \log 4\right].
\end{aligned}$$

Carrying out the algebra, we can obtain that

$$\begin{aligned}
&B_2(h,h) = 2(h+1)\Big\{\left(3h^2+4h+1\right)\left[\log 2 - \log(h+1)\right] - h\log^2(h) + (3h+1)h\log(h)\Big\} > 0, \\
&\frac{\partial B_2}{\partial \ell}(h,h) = \left(8h^2+9h+3\right)\log(h) + 4\left(2h^2+3h+1\right)\left[\log 2 - \log(h+1)\right] + 2h + 2 > 0, \\
&\frac{\partial^2 B_2}{\partial \ell^2}(h,h) = \frac{2\left(h^2+2h-1\right)\log(h) + 2(h-1)(h+1)\left[\log 2 - \log(h+1)\right] + 2(h-1)}{h} > 0, \\
&\frac{\partial}{\partial \ell}\left(\frac{\ell^2(\ell+1)}{h+1}\frac{\partial^2 B_2}{\partial \ell^2}\right)(h,h) = 4h^2\left[\log 2 - \log(h+1)\right] + 4(h+1)h\log(h) + 7h + 1 > 0, \\
&\frac{\partial^2}{\partial \ell^2}\left(\frac{\ell^2(\ell+1)}{h+1}\frac{\partial^2 B_2}{\partial \ell^2}\right)(h,+\infty) = 4h\log 2 - 2 > 0, \\
&\frac{\partial^3}{\partial \ell^3}\left(\frac{\ell^2(\ell+1)}{h+1}\frac{\partial^2 B_2}{\partial \ell^2}\right)(h,\ell) = -\frac{4h\left(\ell^2+\ell+1\right)}{\ell^2(\ell+1)^2} < 0.
\end{aligned}$$

From the above derivatives and conditions, we can conclude that $B_2(h,\ell) > 0$ and thus

$$\frac{\partial^2 (k\mathfrak{B})}{\partial k^2}\left(\ell^{\frac{\gamma}{1+\gamma}}, \ell, \gamma\right) > 0. \tag{17}$$

Next, consider the third derivative of $k\mathfrak{B}$ with respect to k and define

$$\begin{aligned}
B_3(h,\ell) :=& \underbrace{\frac{h^2(h+1)^2\left[\log(\ell)-\log(h)\right]}{\log(\ell)}}_{>0} \times \frac{\partial^3(k\mathfrak{B})}{\partial k^3}\left(\ell^{\frac{\gamma}{1+\gamma}},\ell,\gamma\right)\Big|_{\ell^{\frac{\gamma}{\gamma+1}}=h} \\
=& -(\ell+1)\left[-2h^3\log 4 - 3h^2\log 4 + 2(h+1)^2(2h-1)\log(\ell+1)+\log 4\right] \\
&+(h+1)^2\left[h(4\ell+2)-\ell\right]\log(\ell) - 2h(h+2)(\ell+1)\log(h).
\end{aligned}$$

Notice that

$$\begin{aligned}
B_3(h,h) &= (h+1)\left\{\left(4h^2+3h-3\right)h\log(h) + 2(h+1)^2(2h-1)\left[\log 2 - \log(h+1)\right]\right\} > 0, \\
\frac{\partial B_3}{\partial \ell}(h,h) &= 2(2h^3+3h^2-1)\left[\log 2 - \log(h+1)\right] \\
&\quad + \left(4h^3+5h^2-2h-1\right)\log(h) + 3h^2+6h+3 > 0, \\
\frac{\partial^2 B_3}{\partial \ell^2}(h,\ell) &= \frac{(h+1)^2(\ell-1)(2h+\ell)}{\ell^2(\ell+1)} > 0.
\end{aligned}$$

The above derivatives and conditions enable us to conclude that $B_3(h,\ell)>0$ and thus

$$\frac{\partial^3(k\mathfrak{B})}{\partial k^3}\left(\ell^{\frac{\gamma}{1+\gamma}},\ell,\gamma\right) > 0. \tag{18}$$

Next, define

$$\begin{aligned}
B_4(h,\ell) :=& \frac{\log(\ell)-\log(h)}{2(h+1)\log(\ell)} \times \frac{\partial}{\partial k}\left(k^2(k+1)^2\frac{\partial^3(k\mathfrak{B})}{\partial k^3}\right)\left(\ell^{\frac{\gamma}{1+\gamma}},\ell,\gamma\right)\Big|_{\ell^{\frac{\gamma}{\gamma+1}}=h} \\
=& -2(\ell+1)\log(h) + (6h\ell+3h+\ell+1)\log(\ell) + 6h(\ell+1)\left[\log 2 - \log(\ell+1)\right].
\end{aligned}$$

Note that

$$\begin{aligned}
B_4(h,h) &= \left(6h^2+2h-1\right)\log(h) + 6h(h+1)\left[\log 2-\log(h+1)\right] > 0, \\
\frac{\partial B_4}{\partial \ell}(h,h) &= \frac{1}{h} + 4 + (6h-1)\log(h) + 6h\left[\log 2 - \log(h+1)\right] > 0, \\
\frac{\partial^2 B_4}{\partial \ell^2}(h,\ell) &= \frac{(\ell-1)(3h+\ell+1)}{\ell^2(\ell+1)} > 0.
\end{aligned}$$

Hence, $B_4(h,\ell)>0$ and thus

$$\frac{\partial}{\partial k}\left(k^2(k+1)^2\frac{\partial^3(k\mathfrak{B})}{\partial k^3}\right)\left(\ell^{\frac{\gamma}{1+\gamma}},\ell,\gamma\right) > 0. \tag{19}$$

Next, define

$$B_5(h,\ell) := \frac{\log(\ell)-\log(h)}{2\log(\ell)} \times \frac{\partial^2}{\partial k^2}\left(k^2(k+1)^2\frac{\partial^3(k\mathfrak{B})}{\partial k^3}\right)\left(\ell^{\frac{\gamma}{1+\gamma}},\ell,\gamma\right)\Bigg|_{\ell^{\frac{\gamma}{\gamma+1}}=h}$$
$$=6(\ell+1)(2h+1)\left[\log 2-\log(\ell+1)\right]-2(\ell+1)\log(h)+\left[6h(2\ell+1)+7\ell+4\right]\log(\ell).$$

Note that

$$B_5(h,h)=\left(12h^2+11h+2\right)\log(h)+6(h+1)(2h+1)\left[\log 2-\log(h+1)\right]>0,$$
$$\frac{\partial B_5}{\partial \ell}(h,h)=\frac{4}{h}+7+(12h+5)\log(h)+6(2h+1)\left[\log 2-\log(h+1)\right]>0,$$
$$\frac{\partial^2 B_5}{\partial \ell^2}(h,\ell)=\frac{(\ell-1)(6h+\ell+4)}{\ell^2(\ell+1)}>0.$$

The above derivatives and conditions enable us to conclude that $B_5(h,\ell)>0$ and thus

$$\frac{\partial^2}{\partial k^2}\left(k^2(k+1)^2\frac{\partial^3(k\mathfrak{B})}{\partial k^3}\right)\left(\ell^{\frac{\gamma}{1+\gamma}},\ell,\gamma\right)>0. \tag{20}$$

Finally, we can show that

$$\frac{\partial^3}{\partial k^3}\left(k^2(k+1)^2\frac{\partial^3(k\mathfrak{B})}{\partial k^3}\right)(k,\ell,\gamma)$$
$$=12(\gamma+1)\left\{(2\ell+1)\log(\ell)+(\ell+1)\left[\log 4-2\log(\ell+1)\right]\right\}>0. \tag{21}$$

Combining (15)-(21), we have that $\mathfrak{B}(k,\ell,\gamma)>0$; together with the fact that $\log(\ell)>0$, we can conclude that $\mathfrak{A}(x,\rho,\gamma)>0$ for the case of $0<\rho\leq 1$.

Case II: $\boldsymbol{\rho<0}$. Then we have $\ell\in(0,1)$ and $k\in(1,\infty)$. Note that $\mathfrak{B}(1,\ell,\gamma)=0$. Further, we have that

$$\frac{\partial\mathfrak{B}}{\partial k}(1,\ell,\gamma)=\log(\ell)\left\{\gamma+3\gamma\ell+3\ell+1+(\ell+1)\left[\log 2-\log(\ell+1)\right]\right\}$$
$$+4(\gamma+1)(\ell+1)\left[\log 2-\log(\ell+1)\right]+\ell\log^2(\ell).$$

Denote the right-hand side of the above equation by $\widehat{B}_1(\ell,\gamma)$. Simple algebra would verify that

$$\widehat{B}_1(\ell,0)=\ell\log^2(\ell)+\left\{3\ell+1+(\ell+1)\left[\log 2-\log(\ell+1)\right]\right\}\log(\ell)$$
$$+4(\ell+1)\left[\log 2-\log(\ell+1)\right]<0,$$

and

$$\frac{\partial \widehat{B}_1}{\partial \gamma}(\ell,\gamma) = (3\ell+1)\log(\ell) + 4(\ell+1)\left[\log 2 - \log(\ell+1)\right] < 0,$$

from which we can conclude that

$$\frac{\partial \mathfrak{B}}{\partial k}(1,\ell,\gamma) \equiv \widehat{B}_1(\ell,\gamma) < 0. \tag{22}$$

Similarly, we have that

$$\frac{\partial^2 \mathfrak{B}}{\partial k^2}(1,\ell,\gamma) = (\ell+1)\log(\ell) < 0. \tag{23}$$

Next, note that

$$\frac{\partial}{\partial k}\left(k^2(k+1)\frac{\partial^2 \mathfrak{B}}{\partial k^2}\right)(1,\ell,\gamma) = \underbrace{\left[\gamma + (3\gamma+5)\ell + 3\right]\log(\ell) + (\gamma+1)(\ell+1)\left[2\log 4 - 4\log(\ell+1)\right]}_{\widehat{B}_3(\ell,\gamma)}.$$

Denote the right-hand side of the above equation by $\widehat{B}_3(\ell,\gamma)$. Simple algebra would verify that

$$\widehat{B}_3(\ell,0) = (5\ell+3)\log(\ell) + 4(\ell+1)\left[\log 2 - \log(\ell+1)\right] < 0,$$

and

$$\frac{\partial \widehat{B}_3}{\partial \gamma}(\ell,\gamma) = (3\ell+1)\log(\ell) + 4(\ell+1)\left[\log 2 - \log(\ell+1)\right] < 0,$$

from which we can obtain that

$$\frac{\partial}{\partial k}\left(k^2(k+1)\frac{\partial^2 \mathfrak{B}}{\partial k^2}\right)(1,\ell,\gamma) \equiv \widehat{B}_3(\ell,\gamma) < 0. \tag{24}$$

Finally, we have that

$$\begin{aligned} &\frac{\partial^2}{\partial k^2}\left(k^2(k+1)\frac{\partial^2 \mathfrak{B}}{\partial k^2}\right)(k,\ell,\gamma) \\ =&2(\gamma+1)\left\{(2\ell+1)\log(\ell) + 2(\ell+1)\left[\log 2 - \log(\ell+1)\right]\right\} < 0. \end{aligned} \tag{25}$$

Note that $k > 1$; together with (22)-(25), we can conclude that $\mathfrak{B}(k,\ell,\gamma) < \mathfrak{B}(1,\ell,\gamma) = 0$ for the case of $\rho < 0$. Note that $\log(\ell) < 0$. Therefore, we have that $\mathfrak{A}(x,\rho,\gamma) > 0$.

Case III: $\boldsymbol{\rho = 0}$. Then we have $\ell = 1$ ($\ell \to 1$) and $k \in (1, +\infty)$. Note that

$$\begin{aligned}
\mathfrak{C}(k,\gamma) :=& \lim_{\ell \to 1} \left[\mathfrak{A}(x,\rho,\gamma) \Big|_{x=k^{1+\gamma},\, \rho=1 / \left(\frac{\log(k)}{\log(\ell)} + \frac{1}{1+\gamma} \right)} \right] \\
=& - \frac{\left[\gamma + (3\gamma - 1)k + 1 \right] \log(k)}{2(k+1)} - 2\gamma \left[\log 2 - \log(k+1) \right], \\
\mathfrak{C}(k,0) =& \frac{(k-1)\log(k)}{2(k+1)} > 0, \\
\frac{\partial \mathfrak{C}}{\partial \gamma}(k,\gamma) =& 2 \left[\log(k+1) - \log 2 \right] - \frac{(3k+1)\log(k)}{2(k+1)} > 0.
\end{aligned}$$

The above derivatives and conditions enable us to conclude that $\mathfrak{C}(k,\gamma) > 0$ for $k > 1, \gamma > 0$ and thus $\mathfrak{A}(x,\rho,\gamma) > 0$.

In summary, $\mathfrak{A}(x,\rho,\gamma) > 0$ for all $\rho \leq 1$, $\gamma > 0$, and $x > 1$. Further, it can be verified that $\mathcal{W}(x,\rho,0) < 0$ and $\lim_{\gamma \to +\infty} (1+\gamma)^2 \mathcal{W}(x,\rho,\gamma) > 0$. Therefore, there exists a unique solution to $\mathcal{W}(x,\rho,\gamma) = 0$, which we denote by $\gamma^*(\rho; x) \in (0,\infty)$. Moreover, $\mathcal{W}(x,\rho,\gamma) < 0$ for $\gamma \in \big(0, \gamma^*(\rho;x)\big)$ and $\mathcal{W}(x,\rho,\gamma) > 0$ for $\gamma \in \big(\gamma^*(\rho;x), +\infty\big)$. This concludes the proof. ■