On Rent Dissipation in Dynamic Multi-battle Contests*

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Abstract

We study dynamic multi-battle contests and examine how the contest structure shapes dynamic incentives and determines the extent of rent dissipation. A discouragement effect often arises—such as in tug-of-war and best-of-K contests—preventing full rent dissipation even when the series can extend infinitely. We identify a structural property, exchangeability, that contributes to the effect. Leveraging this insight, we establish a necessary and sufficient condition for almost-full rent dissipation. As an application, we introduce the iterated incumbency contest, which illustrates how volatility in the surrounding environment sustains dynamic incentives and generates almost-full rent dissipation, and thus offers insights into various competitive phenomena.

Keywords: dynamic competition, contests, rent dissipation

JEL Classification Codes: D72, C73, D86.

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1 Introduction

Competitions often unfold over multiple phases rather than being decided by a single, decisive action. Contenders confront each other repeatedly in a sequence of battles, and final victory requires accumulating sufficiently many intermediate successes, rather than a single stroke of effort. Such dynamics are pervasive in the socioeconomic landscape. Military warfare provides a close analogy: Outcomes on individual battlefields may shift momentum or local control, yet rarely determine ultimate victory on their own. U.S. presidential primaries offer another intuitive example. Campaigns are conducted sequentially across states, and party nomination is conferred only after a candidate has accumulated enough delegate victories over time. High-profile patent litigations exhibit similar multi-battle dynamics. Final outcomes emerge only after numerous rulings across jurisdictions and potentially repeated appeals within individual lawsuits, as exemplified by the canonical litigation between Apple and Samsung from 2011 to 2018, which spanned multiple countries and legal venues.

A substantial body of research studies the strategic foundations of dynamic multi-battle contests (see, e.g., Harris and Vickers, 1987; Klumpp and Polborn, 2006; Konrad and Kovenock, 2009; Fu, Lu, and Pan, 2015b), with a sustained emphasis on contenders' dynamic incentives and on how the structure of the contest shapes effort provision. This literature has long highlighted the discouragement effect. Consider a dynamic contest between two ex ante symmetric contestants. An accidental early win by one contestant grants the lucky frontrunner an advantage and could turn an initially even race into a lopsided competition. This ex post asymmetry discourages effort provision by both contestants: The laggard is deterred by bleak winning prospects, while the reduced resistance affords the frontrunner an easy win (see, e.g., Klumpp and Polborn, 2006). This limits rent dissipation—i.e., total effort relative to the prize awarded to the winner—and generates inefficiency in terms of effort supply.

In this paper, we examine rent dissipation in a general dynamic contest. More specifically, we investigate whether and under what conditions the contest would or would not fully dissipate its rent, particularly when the contest may involve a long series of encounters. We identify a structural feature common in many dynamic contests—*exchangeability*—that plays a critical role in generating the discouragement effect and limiting rent dissipation. Further, we establish that contests with a *transient dominance property* in their equilibrium can achieve almost-full rent dissipation in the limit.

Consider, for instance, a tug-of-war contest, which is often used to model protracted litigation and competition for political control. Final victory requires one contestant to establish a sufficiently large lead. Yet any lead can be offset by an opponent's win, so the contest can, in principle, be prolonged indefinitely. Two opposing forces are at play in determining its equilibrium rent dissipation. On the one hand, a potentially infinite horizon attenuates incentives to exert effort for immediate advancement; on the other hand, this allows effort to accumulate. The answer is not obvious a priori. Further, both forces are strengthened when the margin required for final victory under the contest rule increases. Suppose that the required margin grows very large. Will the contest fully dissipate its rent in equilibrium?

We examine a general dynamic contest that accommodates a broad array of contest architectures. Two players compete head-to-head in a sequence of component battles that take place successively. In each battle, the players invest effort. A contest rule specifies the overall winner based on the history of component battles, and a prize is awarded accordingly. We impose no restriction on the specific form of the contest rules. Popular examples include the best-of-$(2K + 1)$ contest, with $K \in \mathbb{N}_{++}$, in which a player secures overall victory by winning at least $K+1$ battles. The best-of-$(2K+1)$ format can also be viewed as a first-past-the-post (FPTP) race, which awards the prize to the contestant who reaches a predetermined target number of battle wins before the opponent does. Another example is a tug-of-war contest with margin K, in which a player prevails if and only if they win $K \in \mathbb{N}_{++}$ *more* battles than his rival.

We first examine a benchmark case of *finite* contests. A contest is *finite* if it has a finite *minimum length*, that is, the minimum number of battles required for the contest to decide its winner is finite. For instance, a tug-of-war contest with margin K has a minimum length of exactly K, although the contest may last infinitely long. We establish that rent can never fully dissipate in any finite contest (Theorem 1), such as a tug-of-war contest. Its expected equilibrium total effort is bounded from above by an upper limit strictly below the prize value.

We then proceed to explore the limiting properties of the contest. That is, consider a series of finite contests, and let the minimum lengths of these contests approach infinity. We demonstrate in the case of tug-of-war contests that full rent dissipation does not arise even in the limit, i.e., when the required margin K grows to infinity. As the lead possessed by the frontrunner continues to grow, a discouragement effect comes into play: Even if a large number of additional battles remain from the finish line, the laggard becomes fully discouraged, and the probability of the frontrunner's winning future battles converges to one (Proposition 2). Owing to this discouragement effect, the equilibrium rent dissipation ratio—defined by the ratio of equilibrium total effort to prize value—remains bounded by a constant that is strictly less than one and uniform to all $K \in \mathbb{N}_{++}$ (Theorem 2).

This striking observation compels us to delve deeper into the nature of contestants' dynamic incentives. The aforementioned *exchangeability* underpins the discouragement effect.

An exchangeable contest does not distinguish between two histories of battle outcomes if they only differ in the order of past wins and losses, such that the state of the dynamics can be represented simply by the win counts. For example, in a best-of-three contest (such as a tennis match), the same deciding set arises regardless of whether a player wins the first game and loses the second or vice versa. Similarly, in a tug-of-war contest, only the net number of battles won by each player matters, while the order of each contestant's wins and losses does not matter. This feature induces a form of "long memory" in a dynamic contest, whereby early battle outcomes exert a persistent influence on future play. The frontrunner can leverage his early successes and accumulate an advantage without bound, which cannot easily be overturned even if the laggard wins some later battles. This leads to excessive discouragement over time and limits effort provision. We establish that an exchangeable contest can never fully dissipate its rent regardless of its minimum length (Theorem 3). For instance, in a best-of-$(2K + 1)$ contests, total equilibrium effort is bounded away from its prize value even as K approaches infinity.

These findings inspire us to explore under what conditions rent can be fully dissipated asymptotically. We propose a *transient dominance property* (Definition 4) and show that it is sufficient (and also necessary) for almost-full rent dissipation (Theorem 5). The property imposes two requirements on the equilibrium of the contest. First, the frontrunner has a significant advantage against the laggard. Second, the competition remains persistently "fluid," in the sense that one's lead can always be reversed with sufficiently high likelihood, so that no player's advantage can accumulate without a bound. Together, these conditions defy the long memory enabled by exchangeability and prevent persistent impact of early outcomes. This keeps the contest competitive as long as final victory is not yet formally awarded.

This transient dominance property can be readily illustrated by a tug-of-war contest with random resets, in which the contest may revert to its initial state with a nonzero probability after each battle. More importantly, we propose an *iterated incumbency contest* that reveals the fundamentals of the transient dominance property. This contest provides an intuitive framework and novel insights into a wide range of dynamic competitions in uncertain environments, such as competitions over emerging technologies and evolutionary competition between species. Our analysis verifies that such contests satisfy the transient dominance property and lead to almost-full rent dissipation in the limit (Theorem 7). The result provides a natural rationale for the pervasively observed Red Queen effect in technological or biological evolutionary processes. More details are provided in Section 4.

Related literature Harris and Vickers (1987) lay the groundwork for research on dynamic contests with successive, disjoint component battles. They propose two modeling approaches that differ in their prevailing winning rules. The first is a race model, in which a player prevails if he is the first to win a prespecified number of battles. The best-of-$(2K+1)$ contest is its most intuitive variant, whereby a player must win a majority of component battles for the final prize. Klumpp and Polborn (2006), Konrad and Kovenock (2009, 2010), Gelder (2014), and Klumpp, Konrad, and Solomon (2019) extend this research stream. The second is a tug-of-war model, in which the ultimate winner must accumulate a sufficiently large lead over his opponent along the path. McAfee (2000), Konrad and Kovenock (2005), and Agastya and McAfee (2006) further analyze contests with this feature.

This literature has long recognized the discouragement effect that arises in dynamic multi-battle contests, whereby early outcomes stifle future competition, disincentivize effort, and limit rent dissipation. Klumpp and Polborn (2006) saliently interpret the New Hampshire effect in U.S. presidential primaries as a manifestation of this discouragement effect. A number of studies propose environmental or structural features that may mitigate the discouragement effect and sustain effort supply over time, such as intermediate prizes in Konrad and Kovenock (2009)[1] and uncertainty in effort cost functions in Konrad and Kovenock (2010). Gelder (2014) shows that a player may drastically outperform the frontrunner when falling behind if the loser faces a severe penalty.

However, these studies are all situated within race models with a finite number of component battles. In contrast, our analysis accommodates a general setup without imposing specific winning rules and sheds light on the asymptotic properties of equilibrium rent dissipation when the length of a contest approaches infinity. In this general setting, our analysis reveals the fundamental role of exchangeability and the long-memory it enables in driving the discouragement effect and underdissipation of rents; this, in turn, motivates conditions of transient dominance that yield almost-full rent dissipation.

The notion of exchangeability was first proposed by Ewerhart and Teichgräber (2019). Allowing for infinite horizon and without imposing specific winning rules, Ewerhart and Teichgräber (2019) study dynamic multi-battle contests that satisfy certain properties, i.e., exchangeability, monotonicity, and centeredness. They show that these contests eventually enter a tie-breaking phase isomorphic to a tug-of-war game. Our paper differs in that we focus on dynamic rent dissipation, whereas their main contribution lies in establishing the connection between contests with infinite horizons and tug-of-war games.[2]

All of these studies examine contentions between individual players. In contrast, Fu, Lu,

[1] Fu, Ke, and Tan (2015a) allow a player to derive a utility from winning each component battles. The "utility of winning" plays a similar role to the intermediate prizes in Konrad and Kovenock (2009).

[2] They also provide a comprehensive analysis of the tug-of-war game, extending the Tullock success function case considered in Karagözoğlu, Sağlam, and Turan (2021) to more general success functions.

and Pan (2015b) study multi-battle races between teams, with members from rival groups matched in head-to-head competitions. Häfner and Konrad (2016) and Häfner (2017) study tug-of-war contests between teams with a structure similar to that in Fu et al. (2015b).

The economics literature has considered a wide array of dynamic contests in alternative forms. All aforementioned studies assume that players participate in disjoint battles, each of which yields a winner. In contrast, another stream of the literature allows each player's effort to accumulate over time, with cumulative output determining the final winner at the end of the contest. Notable examples include Meyer (1992); Yildirim (2005); Gershkov and Perry (2009); Aoyagi (2010); Ederer (2010); Gürtler and Harbring (2010); Goltsman and Mukherjee (2011).

2 Model and Preliminaries

Two risk-neutral players A and B compete head-to-head for a prize of common value $v > 0$.[3] The contest consists of a sequence of successive *component battles*, and the final winner is determined by the history of battle outcomes.

2.1 Component Battles

In each component battle, player $\ell \in \{A, B\}$ simultaneously exerts an effort $x_\ell \in \mathbb{R}_+ := [0, +\infty)$. Effort is measured directly in units of disutility, so exerting effort x_ℓ entails a cost of x_ℓ. For a given effort profile (x_A, x_B), player A wins the current battle with a probability $p(x_A, x_B) : \mathbb{R}_+ \times \mathbb{R}_+ \to [0, 1]$, and player B wins with complementary probability $p(x_B, x_A) = 1 - p(x_A, x_B)$. We call $p(x, x')$ the *success function* (SF) for component battles.

Our baseline model focuses on a homogeneous (of degree zero) success function. However, the analysis applies in a broad context and the results remain largely intact when allowing for alternative contest technologies, which we will elaborate on in Section 5. More specifically, we assume the following.

Assumption 1 (Homogeneous Success Function)**.** *For a given effort profile* $(x, x') \in \mathbb{R}_+^2 \setminus \{0, 0\}$*, a player, by exerting an effort* x*, wins the battle with a probability*

$$p(x, x') = \gamma\left(\frac{x}{x'}\right),$$

[3] We do not normalize the value to 1 because our setting does not require the winning probability of each battle (battle success function) to be homogeneous. Prize value does affect equilibrium outcomes without homogeneity.

where $\gamma : [0, +\infty] \to [0, 1]$ *is a continuous and twice-differentiable function with* $\gamma(0) = 0$*,* $\gamma(+\infty) = 1$*,* $\gamma' > 0$*,* $\gamma'' \leq 0$*, and* $\gamma(x) + \gamma(1/x) = 1$*.*

The popularly adopted Tullock success function provides a classic example within this family of models, whereby

$$p(x, x') = \begin{cases} \frac{x^r}{x^r + (x')^r}, & \text{if } (x, x') \neq (0, 0), \\ \frac{1}{2}, & \text{if } (x, x') = (0, 0), \end{cases}$$

with $r \in (0, 1]$. A serial SF (Alcalde and Dahm, 2007) also satisfies the homogeneous-of-degree-zero requirement: For $\alpha \in (0, 1)$, $p(x, x') = \frac{1}{2}(x/x')^\alpha$ if $0 \leq x \leq x'$ and $p(x, x') = 1 - \frac{1}{2}(x'/x)^\alpha$ if $x > x'$, with $p(0, 0) = 1/2$.

2.2 Contest Architecture

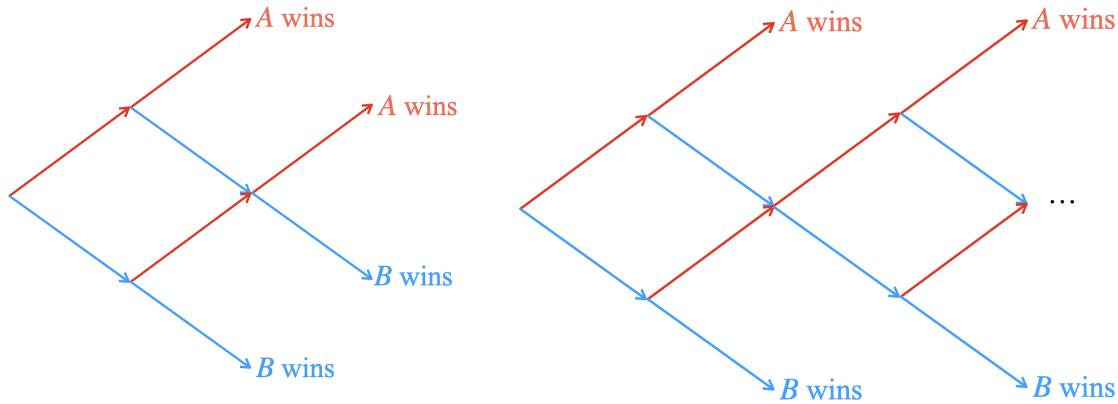
At the beginning of each battle, the outcomes of all previous battles are commonly known. An *outcome path* is denoted by $\ell^t := (\ell_s)_{s=1}^t$ for $t > 1$, where each element $\ell_s \in \{A, B\}$ indicates the winner of battle s. Let $\widetilde{H} := \{\ell^t : t \in \mathbb{N}_{++} \bigcup \{+\infty\}\}$ be the set of all possible outcome paths. Further, let $H^\dagger \in \widetilde{H}$ denote the set of *terminal histories*: Each terminal history is an outcome path ending with the battle that determines the ultimate winner of the contest according to the contest rule.

Given a path $\ell^t = (\ell_1, \ldots, \ell_t)$, let $\ell^{t'|t} := (\ell_1, \ldots, \ell_{t'})$ denote its prefix of length $t' \leq t$. We can define the *history set* of the contest as $H := \{\ell^{t'|t} : \ell^t \in H^\dagger\}$. Each element of H is called a *history* of the contest: That is, a history is an outcome path that does not extend beyond any terminal history.

We construct Figure 1 to illustrate the terminologies: Figure 1(a) depicts a best-of-three contest, while Figure 1(b) presents a tug-of-war with margin two, i.e., a tug-of-war in which a player must lead by two wins to secure final victory. Each branch represents a possible battle outcome, and an upward (resp., downward) branch corresponds to a battle won by player A (resp., player B). The set of terminal histories for the best-of-three contest is $\{AA, BAA, ABB, BB\}$, and terminal histories of the tug-of-war with margin 2 include AA, BB, $ABBB$, $BAAA$, and so on.

2.3 Payoffs and Continuation Values

For a finite terminal history of length t, the winner's net payoff is $v - \sum_{s=1}^t x_{\text{winner},s}$, and the loser's payoff is $-\sum_{s=1}^t x_{\text{loser},s}$. For an infinite terminal history, we assign a payoff



(a) Best-of-3 Contest. (b) Tug-of-war with Margin 2.

Figure 1: Example Contest Architectures.

$\frac{v}{2} - \sum_{s=1}^{+\infty} x_{\ell,s}$ to each player $\ell \in \{A, B\}$.

Denote by $V_\ell(h)$ a player ℓ's continuation value in the contest given a history $h \in H$, i.e., the equilibrium payoff the player expects from the subsequent competition. The history ends up as (h, ℓ) if a player ℓ wins the current battle and would otherwise evolve into (h, ℓ'). Hence, the value generated by winning the battle is given by

$$\Delta_\ell(h) = V_\ell(h, \ell) - V_\ell(h, \ell').$$

Given an effort profile (x_A, x_B), a player $\ell \in \{A, B\}$ receives an expected payoff $p(x_\ell, x_{-\ell})\Delta_\ell - x_\ell$ from the battle. By standard results in the contest literature (e.g., Malueg and Yates, 2005), a unique pure-strategy Nash equilibrium exists in each component battle.

Further, let V_ℓ^0 be a player ℓ's continuation value at the beginning of the contest, which is equal to his equilibrium expected payoff in the contest. The expected total effort the contest can elicit in equilibrium is thus given by $v - \sum_{\ell \in \{A,B\}} V_\ell^0$. The contest fully dissipates its rent if $\sum_{\ell \in \{A,B\}} V_\ell^0$ reduces to zero, i.e., the equilibrium total effort approaching the prize value v.

2.4 Benchmark: Bounded Rent Dissipation in Finite Contests

Let $L(\mathcal{M}) := \min\{t : \ell^t \in H^\dagger\}$ denote the minimum length of the contest $\mathcal{M}$, i.e., the length of the shortest terminal history. We formally define the notion of *finite contests* as follows.

Definition 1. A contest is *finite* if $L(\mathcal{M}) := \min\{t : \ell^t \in H^\dagger\} < +\infty$.

We view the case of finite contests as a benchmark and establish that, in any such contest, rent dissipation is capped by an upper bound, which is stated below. Recall that for a given effort profile (x, x'), a player, by exerting an effort x, wins a battle with a probability $p(x, x') = \gamma(x/x')$. We define $\phi(\theta) := \gamma(\theta) - \theta\gamma'(\theta)$ for $\theta \geq 0$.[4] The following ensues.

Theorem 1. *Under Assumption 1, the expected total effort in any equilibrium (if one exists) of a finite contest $\mathcal{M}$ is less than $\left\{1 - [\phi(1)]^{L(\mathcal{M})}\right\} v$. Therefore, rent does not fully dissipate in finite contests.*

By Theorem 1, rent is never fully dissipated in a finite contest. The equilibrium rent dissipation rate, $\left\{1 - [\phi(1)]^{L(\mathcal{M})}\right\}$, is determined by the shortest terminal history. For instance, a standard best-of-$(2K+1)$ contest can end after $K+1$ battles, so the upper bound is $\left\{1 - [\phi(1)]^{K+1}\right\}$. Consider a tug-of-war contest with a margin $N < +\infty$, in which a player must win $N \geq 1$ *more* battles than his opponent to secure final victory. The contest could last infinitely long if neither party manages to establish a sufficient margin. However, the length of its shortest terminal history is exactly N, so its expected total effort cannot exceed $\left\{1 - [\phi(1)]^{N}\right\} v$ by Theorem 1.

3 Asymptotically Bounded Rent Dissipation and Exchangeability

Theorem 1 inspires a natural question. Consider a sequence of contests $\{\mathcal{M}_k\}_{k \in \mathbb{N}_{++}}$ where each $\mathcal{M}_k$ is a finite contest. Let $L(\mathcal{M}_k) \to +\infty$ as $k \to +\infty$, and we examine the limiting property of this sequence. Does the sequence of expected equilibrium total efforts converge to v?

The upper bound $1 - [\phi(1)]^{L(\mathcal{M}_k)}$ It approaches 1 as $L(\mathcal{M}_k) \to +\infty$. However, this does not necessarily mean that rent full rent dissipation is approached in the limit. Our analysis first demonstrates in a tug-of-war contest model that rent may remain underdissipated even asymptotically. We then identify a structural property of contest rules that limits rent dissipation in dynamic contests.

3.1 Case of Tug-of-War Contests

Consider a tug-of-war contests with margin $N \geq 1$, whereby a player must win N more battles than his opponent to secure the final victory. We adopt symmetric Markov Perfect

[4] This is the single-battle equilibrium gain ratio: For a given battle, when a player's value of winning is θ times his opponent's, $\phi(\theta)$ is his equilibrium payoff as a share of his value of winning.

Equilibrium (MPE) as the solution concept of the game and first establish its existence and uniqueness.

An MPE is also a subgame-perfect equilibrium, but requires that players' strategies depend on history only through the most recent state. The state of a tug-of-war contest can be summarized by $i \in \mathbb{N}$ with $-N \leq i \leq N$, where i stands for a player's lead over this opponent, i.e., the number of extra wins he currently possesses. A symmetric MPE can then be described by the value function $V(i; N, v)$ for i over $\{i \in \mathbb{N} | -N \leq i \leq N\}$, which represents the expected value of a player who leads by i battles. For brevity, we omit N, v in the expressions when there is no confusion. The following ensues.

Proposition 1 (Existence of Symmetric MPE in Tug-of-War)**.** *Under Assumption 1, a unique symmetric MPE exists in a tug-of-war with margin $N \geq 1$. In the equilibrium, $0 = V(-N) < V(-N+1) < \ldots < V(N-1) < V(N) = v$.*[5]

The equilibrium result paves the way for analysis of equilibrium rent dissipation in the game. Let $Q(i; N, v)$ denote the probability that a player who leads by i battles wins the entire contest. We have the following.

Proposition 2 (Discouragement Effect)**.** *Impose Assumption 1. In the symmetric MPE of the tug-of-war with margin N, it holds that $\lim_{i \to +\infty} \inf_{N > i} Q(i; N, v) = 1$; that is, the frontrunner secures an almost-sure win once its lead is sufficiently large.*

By Proposition 2, as the lead i is sufficiently large, the final outcome of the contest boils down to certainty. This implies a discouragement effect that often arises in dynamic contest. The laggard will be completely discouraged when the gap is sufficiently large; he would give up the competition even if the contest has yet to end, i.e., $N > i$. This leads to $\lim_{i \to +\infty} \inf_{N > i} Q(i; N, v) = 1$. This discouragement effect disincentivizes players: The laggard concedes, which further affords the frontrunner easy wins. A bounded rent dissipation result follows.

Theorem 2 (Bounded Rent Dissipation in Tug-of-War)**.** *Under Assumption 1, there exists a constant $\alpha > 0$ such that $V(0; N, v) \geq \alpha v$ for all $N \geq 1$ and $v > 0$. Consequently, the expected total effort in any symmetric MPE of the tug-of-war with any margin $N \geq 1$ is bounded from above by $(1 - 2\alpha)v$.*

[5] Ewerhart and Teichgräber (2019) establish existence and uniqueness of a symmetric MPE for tug-of-war contests under the additional assumption that $\frac{\phi(1/\theta)}{\theta[1-\phi(\theta)]}$ is strictly declining in $\theta \in (0, 1)$. We relax this assumption by establishing the required monotonicity directly in our proof.

Theorem 2 establishes a uniform upper bound on rent dissipation in tug-of-war contests that applies for all $N \geq 1$. This formally verifies that rent cannot fully dissipate in a tug-of-war contest, even when the margin requirement for victory becomes arbitrarily large and the lengths of its terminal histories grow without bound.

3.2 Exchangeable Contests

We now develop a general sufficient condition—exchangeability—for rent underdissipation that drives discouragement effect in dynamic contests. We first provide a formal definition of this notion and then elaborate on its role in shaping players' dynamic effort incentives and limiting effort provision. We then provide a counterexample showing that, absent exchangeability, rent can fully dissipate asymptotically.

3.2.1 Exchangeability and Discouragement Effect

An exchangeable contest is defined as follows.

Definition 2. A contest is *exchangeable* if any two histories that differ only in the order of the battle outcomes lead to the same subgame or final outcome.

Exchangeability is satisfied by many dynamic contests, such as tug-of-war contests and best-of-$(2K+1)$ with $K \in \mathbb{N}_{++}$. Consider, for instance, a best-of-$(2K+1)$ contest with $K > 1$ or a tug-of-war with $N > 2$. Two histories, (A, B, A) and (A, A, B), lead to the same state of the contest.

Exchangeability allows a frontrunner to accumulate advantage, thereby catalyzing the discouragement effect. For player A, given past outcomes $h = (\ell_1, \ldots, \ell_t)$, his value of winning the next battle is given by $\Delta_A(\ell_1, \ldots, \ell_t) = V(\ell_1, \ldots, \ell_t, A) - V(\ell_1, \ldots, \ell_t, B)$, which motivates the effort he would invest in winning the impending battle. With exchangeability, the history $(\ell_1, \ldots, \ell_t, A)$ leads to the same subgame as the alternative history $(A, \ell_1, \ldots, \ell_t)$; similarly, $(\ell_1, \ldots, \ell_t, B)$ leads to the same subgame as $(B, \ell_1, \ldots, \ell_t)$. This equivalence implies that thanks to exchangeability, the impact of an early outcome does not vanish as the contest unfolds, even after a large number of subsequent battles. Exchangeability results in long memory along the dynamics: One's early success can carry forward into a persistent advantage, and an early disadvantage, possibly arising from an accidental loss, can be perpetuated indefinitely. This renders a laggard's catching up more difficult, which discourages him, disincentivizes both players, and leads to rent underdissipation.

This intuition can be developed into a formal analysis of rent dissipation in the limit. For this purpose, we introduce the notion of *(exchangeability-preserving) extension.* For expositional ease, we focus on contests that are symmetric, meaning the contest rules are independent of the player's identity.

Definition 3. For a given symmetric exchangeable contest $\mathcal{M}$, a symmetric contest $\mathcal{M}'$ is said to be the N-*extension* of $\mathcal{M}$, with $N \geq 2$, if both the following conditions are met:

(i) $\mathcal{M}'$ is exchangeable, and its subgame following histories (A, B) or (B, A) is $\mathcal{M}$;

(ii) from the start of $\mathcal{M}'$, a player wins the contest after winning exactly N battles in a row.

For instance, the best-of-$(2K+1)$ contest is the $(K+1)$-extension of the best-of-$(2K-1)$ contest. As another example, a tug-of-war with margin-N is the N-extension of itself. With this definition, we explore a sequence of exchangeable contests $\{\mathcal{M}_k\}_{k=1}^{K}$ such that, for each k, the contest $\mathcal{M}_{k+1}$ is an N_{k+1}-extension of $\mathcal{M}_k$. By construction, $L(\mathcal{M}_k) \leq N_k$ for every k, and Theorem 1 therefore implies that rent cannot fully dissipate in any $\mathcal{M}_k$. Letting $K \to +\infty$, we ask whether the expected equilibrium total effort of the contest can converge to v along an infinite sequence. Further, since we focus on the limit property of the game, it is without loss to assume that the initial contest $\mathcal{M}_1$ is a single-battle contest. Let $V_{0;k}$ denote the expected equilibrium payoff of a player in $\mathcal{M}_k$. The following ensues.

Theorem 3 (Bounded Rent Dissipation in Exchangeable Contests)**.** *Impose Assumption 1. For any sequence of exchangeable extensions* $\{\mathcal{M}_k\}_{k=1}^{+\infty}$ *such that* $\lim_{k\to+\infty} V_{0;k}$ *exists, it holds that* $\lim_{k\to+\infty} V_{0;k} \geq \widetilde{\alpha}v$*, where* $\widetilde{\alpha} > 0$ *is constant that depends only on the success function. That is, rent does not fully dissipate in the limit of the sequence of exchangeable contests, and the expected total effort is less than* $(1-2\widetilde{\alpha})v$*.*

Theorem 3 formalizes the role of exchangeability in limiting rent dissipation. Exchangeability, with its long-memory property, allows early success to translate into a lasting advantage. This persistence discourages the laggard and attenuates competition. The well-known discouragement effect in the dynamic contest literature thus arises.

To further illustrate the role of exchangeability, we now examine a simple dynamic contest with a contest rule that violates exchangeability. We show that, in this setting, full rent dissipation can be approached in the limit.

3.3 Consecutive-win Contests: When Exchangeability is Absent

Consider a K-consecutive-win contest, with $K \in \mathbb{N}_{++}$: That is, the player who first wins K battles in a row is awarded the final prize. Notably, this contest is not exchangeable.

Assume $K = 3$; (A, A, B) and (A, B, A) do not lead to the same subsequent subgame: Player B holds an advantage in the former case, whereas player A is leading in the latter. More importantly, memory is short in this contest: The impact of one's wins is entirely wiped out once he loses a battle before achieving K consecutive wins.

As before, we focus on symmetric MPE with the state space $\{i \in \mathbb{N} : -K \leq i \leq K\}$: if $i > 0$, the player currently holds a winning streak of length i; if $i < 0$, he currently holds a losing streak of length $-i$; and $i = 0$ represents the start of the game. Obviously, if one player is in state i, his opponent must be in state $-i$. The symmetric MPE is characterized by a value function $\widehat{V}(i; K, v)$ that represents a player's continuation value in state i of the K-consecutive-win contest with a prize v. For convenience, when no confusion arises, we suppress the arguments K and v for $\widehat{V}(\cdot)$ and other equilibrium objects defined below.

In Appendix B.7, we establish that the contest yields a unique symmetric MPE. Let $\widehat{Q}(i; K, v)$ denote the probability that a player in state i eventually wins the contest. We further have the following result.

Proposition 3 (Bounded Accumulated Advantage in Consecutive-win Contests)**.** *Impose Assumption 1. In the symmetric MPE of the consecutive-win contest, it holds that*

$$\lim_{K>|i|, K\to+\infty} \widehat{Q}(i; K, v) = \frac{1}{2}$$

for all $i \in \mathbb{N}$.

Proposition 3 stands in sharp contrast to Proposition 2. With a large gap, players' winning odds remain even. As a result, the laggard would not be strongly discouraged. It is worth noting that a lead in the consecutive-win contest is *insecure*: When the laggard wins a battle in a tug-of-war contest, the frontrunner's lead is merely reduced, whereas in a K-consecutive-win contest, a single loss entirely erases the lead and completely turns the table. This feature largely mutes the discouragement effect and leads to the following.

Theorem 4 (Full Rent Dissipation in the Limit of Consecutive-win Contests)**.** *Under Assumption 1, for all $\epsilon > 0$, there exists $K^\dagger$ such that $\widehat{V}(0; K, v) < \epsilon v$ in the equilibrium of the K-consecutive-win contest for $K > K^\dagger$, which implies that the expected total effort is greater than $(1 - 2\epsilon)v$.*

The theorem formally establishes that the K-consecutive-win contest, which violates exchangeability, fully dissipates the rent in the limit. This case further illustrates the nature of exchangeability. The contest rules out exchangeability and renders the impacts of winning a battle vis-à-vis losing one asymmetric: Advantage must accumulate gradually, but leadership can be lost abruptly. This feature keeps the laggard hopeful and incentivized, thereby nullifying the discouragement effect.

This observation further compels us to explore general conditions under which a contest can fully dissipate its rent. In particular, we formalize the idea that dominance is transient in consecutive-win contests as follows. This formulation will lead to a full characterization of almost full rent dissipation in the next section.

Proposition 4 (Dominance Is Transient in Consecutive-win Contests)**.** *Fix any small $\epsilon > 0$. Then there exist integers $K^\dagger > 0$ and $K' > K^\dagger$ such that, for all K-consecutive-win contests with $K > K'$, the following holds. Let H_ℓ^-, for each player $\ell \in \{A, B\}$, denote the set of histories in which ℓ currently has a losing streak of length $K^\dagger$. Then:*

(i) For every $h \in H_\ell^-$, player ℓ's continuation value is small: $V_\ell^-(h) < \epsilon v$.

(ii) Along the equilibrium path, the probability that the realized history passes through both H_A^- and H_B^- exceeds $1 - \epsilon$.

4 Almost-Full Rent Dissipation

In this section, we first provide a condition for almost-full rent dissipation in dynamic multi-battle contests. We then introduce an *iterated incumbency contest*, which satisfies this condition and fully dissipates its rent in the limit. This contest provides an intuitive framework for studying a wide range of real-world competitive phenomena, spanning technological competition and biological evolutionary processes.

4.1 Transient Dominance Property and Almost-Full Rent Dissipation

We now propose the following transient dominance property.

Definition 4. Consider a contest $\mathcal{M}$ and its equilibrium. Fix any small $\epsilon > 0$. We say that the equilibrium has the *transient dominance property* if the following conditions are satisfied:

(i) There exist two subsets of the nonterminal histories, H_A^- and H_B^-, such that at any history $h \in H_\ell^-$, the continuation value $V_\ell(h) \leq \epsilon v$.

(ii) In equilibrium, the probability that the realized outcome history reaches both H_A^- and H_B^- is at least $1 - \epsilon$.

Condition (i) identifies, for each player ℓ, a set H_ℓ^- of nonterminal histories at which the player ℓ is in a "weak" position with a small ϵv. This typically implies that his opponent is in a dominant position in the contest and expects a larger payoff.[6] Condition (ii) nevertheless requires that the realized history of the contest reaches *both* H_A^- and H_B^- with sufficiently high probabilities. It implies that, conditional on the history arriving at one of H_A^- or H_B^- for the first time, a shift in dominance is highly likely down the road, i.e., a transition from H_ℓ^- to $H_{-\ell}^-$ along the subsequent history. Taken together, the definition implies that a player's lead is temporary and will almost certainly be reversed. The laggard is encouraged to strive to leapfrog, while frontrunner strives to protect his dominance. Our analysis yields the following.

Theorem 5 (Transient Dominance Property as a Sufficient Condition for Almost-Full Rent Dissipation)**.** *Fix any $\epsilon > 0$, the expected equilibrium total effort of a contest $\mathcal{M}$ with prize v exceeds $(1-4\epsilon)v$ if its equilibrium exhibits the transient dominance property.*

Theorem 5 shows that the transient dominance property leads to almost-full rent dissipation in the contest. The expected equilibrium total effort approaches v if the equilibrium satisfies this requirement. Three remarks are in order. First, it can easily be verified that the transient dominance property is not only sufficient but also necessary for almost full rent dissipation. In fact, if the expected total effort in an equilibrium of contest $\mathcal{M}$ with prize v exceeds $(1-\epsilon)v$, then the equilibrium has the transient dominance property.[7]

Second, this result is intuitive in light of our discussion in Section 3. Transient dominance clearly defies the long-memory property enabled by exchangeability. One's lead is insecure, since it can easily be reversed. Consequently, dominance cannot be perpetuated, which sustains the contestability of the competition along the dynamics. The laggard is thus encouraged to strive to escape his weak position (low continuation value), which motivates the leader to protect his dominance.

Third, the transient dominance property is not an obscure technical condition and can arise intuitively from a simple modification of standard tug-of-war contests. Suppose that, after each battle, with probability $p \in (0, 1)$, the contest resets to its initial state, provided that no final winner has yet been determined. That is, following a reset, each player must again achieve a lead of N wins over the opponent to secure final victory before another reset occurs. In this environment, exchangeability fails, and any lead is inherently fragile. We

[6]The opponent's continuation value may also be low. However, combined with condition (ii), that case immediately implies almost-full rent dissipation. The economically interesting case is therefore one in which the low continuation value reflects ex post asymmetry generated by a temporary performance gap, and we focus our attention on this case.

[7]This can be seen easily by setting both H_A^- and H_B^- to be the initial (empty) history.

have the following result.

Theorem 6 (Tug-of-war with Random Resets)**.** *There exists a unique symmetric MPE in a tug-of-war contest with margin $N \geq 2$ and a probability of reset $p \in [0,1)$. For any small $\epsilon > 0$, there exists $N^\dagger$ such that the tug-of-war with margin $N \geq N^\dagger$ and reset probability $p \in (0,1)$ satisfies the transient dominance property.*

This tug-of-war with random resets exhibits the transient dominance property when the required margin N is sufficiently large. As predicted by Theorem 5, rent is then almost fully dissipated in equilibrium. In what follows, we present an alternative contest game that also exhibits this property and discuss its implications.

4.2 Application: Iterated Incumbency Contests

We now examine an intuitive dynamic contest that illustrates how Definition 4 applies and leads to full rent dissipation. Economic agents often engage in protracted competition in uncertain environments. Consider, for instance, two firms striving for dominance in an evolving market driven by an emerging technology; technological progress and shifts in consumer tastes continuously reshape the competitive landscape. As a result, a firm's temporary market dominance may not suffice to eliminate its opponent or to establish permanent market leadership. When the underlying fundamentals change, the laggard may regain the opportunity to overtake the frontrunner. We incorporate these features and construct a model of dynamic competition with uncertainty, which we call an *iterated incumbency contest* and describe in detail below.

The dynamic competition unfolds over multiple *rounds*, indexed by $n = 0, 1, \ldots, N$. In round 0, a fair coin toss determines a temporary leader (i.e., the *incumbent*) and a laggard (i.e., the challenger). In each round $n \geq 1$, an exogenous shock determines whether the fundamentals of the contest have shifted and renewed competition emerges: With probability $q \in (0, 1]$, the fundamentals change, and the firms must compete for the incumbency; with the complementary probability $1 - q$, the status quo is preserved, and the incumbent retains his status without a battle. Each round can be viewed as an interim regime in which the incumbent holds power or status within an ongoing evolution process. The firm which exits round N as the incumbent secures the final victory and receives the prize v.

When the status quo remains, we can interpret the situation as an extremely large advantage held by the incumbent, such that no effort can meaningfully weaken his dominance. The competition for incumbency, once triggered, proceeds as a subcontest with multiple successive battles, which we denote by $\mathcal{M}^{\text{sub}}$. The incumbent still holds an advantaged position in competition, but the advantage is not insurmountable. To reflect this, the subcontest is assumed to be biased in favor of the incumbent. This bias can take various forms. One example is that the incumbent can win the subcontest by winning a single battle; while the challenger must win $K > 1$ battles before the incumbent wins one. We refer to such a subcontest as a $\mathcal{M}(K, 1)$ contest. Alternatively, $\mathcal{M}^{\text{sub}}$ can take the form of a tug-of-war contest with margin $K + 1$ and initial state $K \geq 1$, whereby the incumbent enjoys a head start of K.

Subcontests are assumed to be *sufficiently biased* in favor of the incumbent. Consider a subcontest $\mathcal{M}^{\text{sub}}$, and imagine an auxiliary *standalone* contest with the same structure and winning rule as $\mathcal{M}^{\text{sub}}$, but with a unit prize. Let V_+^{sub} and V_-^{sub} denote the respective equilibrium payoffs of the incumbent and the challenger in the auxiliary contest. For any fixed small $\epsilon > 0$, a (sub)contest $\mathcal{M}$ is said to be *sufficiently biased* if

$$\frac{1 - V_+^{\text{sub}}}{V_-^{\text{sub}}} > \frac{1}{\epsilon}.$$

In particular, this impies that $V_-^{\text{sub}} < \epsilon$, which attests to the challenger's unfavorable position in the competition.[8] The two intuitive contest formats mentioned above, $\mathcal{M}(K, 1)$ contest and tug-of-war contest with margin $K + 1$ and initial state K, both satisfy the requirement in the limit, which is formally verified below.

Lemma 1. *Let the subcontest take the form of either an $\mathcal{M}(K, 1)$ contest or a tug-of-war contest with margin $K + 1$ and initial state K. Then $\frac{1-V_+^{sub}}{V_-^{sub}} \to +\infty$ as $K \to +\infty$.*

The structure of an iterated incumbency contest is illustrated by Figure 2 below. If no competition occurs in a given round, the incumbent is deemed to win that round automatically.

4.2.1 Rent Dissipation in Iterated Incumbency Contests

We are now ready to apply Theorem 5 to show that the iterated incumbency contest can approximate full rent dissipation. The following result is obtained.

Theorem 7 (Full Dissipation in Iterated Incumbency Contests)**.** *Consider an N-round iterated incumbency contest with subcontest $\mathcal{M}^{sub}$ and impose Assumption 1. If the subcontest is sufficiently biased, the iterated incumbency contest satisfies the transient dominance property as $N \to +\infty$, which leads to almost full rent dissipation.*

[8]This condition can also be satisfied if the subcontest or the auxiliary contest is unbiased, but both V_+^{sub} and V_-^{sub} are infinitely small. In this case, the analysis is trivial because this auxiliary contest itself would fully dissipate rent in equilibrium.

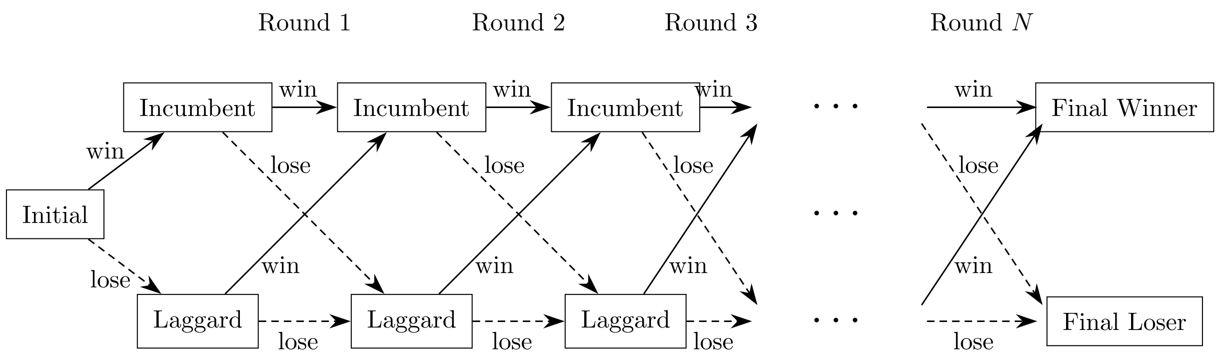


Figure 2: N-round Iterated Incumbency Contest.

The intuition is as follows. First, given homogeneity of each battle's success function, the probability that the incumbent wins a subcontest is independent of the total number of rounds N. As N becomes large, incumbency can almost certainly be overturned in some round. Consequently, the challenger remains hopeful of overtaking the incumbent before the contest ends, rendering leadership fragile and preserving contestability throughout the game. Second, a very small V_-^{sub} encourages the hopeful challenger to improve his status, which, in turn, compels the incumbent to fight to avoid relegation. This sustains competition. Taken together, these features generate transient dominance, which ensures full rent dissipation in the limit.

However, it is worth noting that full rent dissipation may not require very large N in the iterated incumbency contest. We now provide a graphical illustration to clarify the logic. In Figure 3, the blue (diamond) dot near the upper left corner represents their continuation values at the beginning of round N when player B is the incumbent. Since this is the final round and its outcome decides the overall winner of the contest, the location of this point is $(qV_-^{\text{sub}}v, (1-q+qV_+^{\text{sub}})v)$, as implied by the rule of the subcontest $\mathcal{M}^{\text{sub}}$ and the terminal payoff profiles $(0, v)$ or $(v, 0)$. The red (solid) arrow and the black (dotted) arrow, respectively, point to the terminal payoffs when the incumbent wins and when the challenger wins.

Notably, the absolute value of the slope of the red arrow is exactly $\left(1 - V_+^{\text{sub}}\right)/V_-^{\text{sub}}$. Symmetrically, there is a blue dot near the lower right corner corresponding to the case in which A is the incumbent at the beginning of round N.

Then, moving one round backward, the black (round) dot near the upper left corner represents the players' continuation values at the beginning of round $N-1$ when B is the incumbent. Again, the red and black arrows illustrate how the continuation values evolve once that round concludes. As we move further backward, the continuation values of the incumbent and the challenger become progressively closer. Eventually, as shown in Figure 3, they converge to the same point, represented by the circle in the figure.

Theorem 7 implies that when the subcontest is sufficiently biased, this limiting point (i.e., the circle) can be made arbitrarily close to the origin. In other words, their ex ante equilibrium expected payoff in the contest, V_A^0 and V_B^0, converge to zero, implying full rent dissipation.

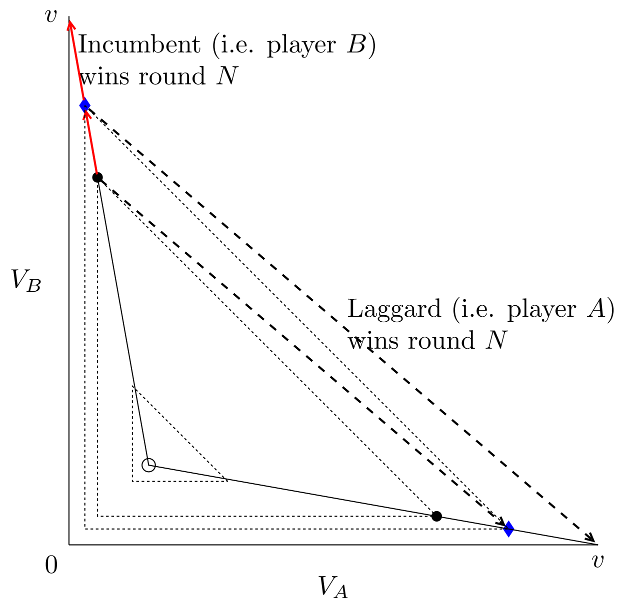


Figure 3: Transition of Continuation Values in an Iterated Incumbency Contest.

Next, we discuss how the model and analysis shed light on a range of real-world competitive phenomena.

4.2.2 Iterated Incumbency Contests: Applications and Relevance

A diverse array of socioeconomic and biological phenomena exhibit a structural pattern resembling an iterated incumbency contest. The model captures protracted rivalries in which exogenous shocks periodically relevel and rejuvenate competition, preventing temporary advantages from compounding into permanent supremacy. In such environments, leadership is inherently contingent: While incumbency confers a short-term dominance, the advantage operates does not accumulate indefinitely across regimes. Dominance is therefore transient because the surrounding landscape shifts and repeatedly resets the contest.

Schumpeterian competition provides a natural economic illustration of this mechanism. Radical innovations and technological paradigm shifts often function as de facto reset mechanisms, narrowing the effective gap between established firms and emerging challengers and renew competitions. Nokia's dominance in feature phones, Kodak's leadership in film photography, and Blockbuster's command of physical video rental all rested on capabilities finely tuned to a particular technological environment. When the smartphone era, digital imaging, and high-speed internet transformed those environments, competition was redefined along new dimensions. Although incumbents retained residual strengths, e.g., brand recognition, distribution networks, organizational capital, the rules of performance were sufficiently altered such that challengers could regain their opportunity to overtake the incumbent. In this sense, incumbency provided a bounded advantage within each regime, but regime shifts prevented that advantage from accumulating without limit and sustained the competition. The insecurity of leadership maintains incentives both for challengers to leapfrog and for incumbents to reinvest in its status.

A similar logic arises in biological evolution. In host-parasite arms races, an immune system may successfully neutralize a pathogen strain, thereby establishing a temporary lead. Yet the pathogen's rapid mutation generates new variants, effectively resetting the contest and restoring competitive pressure. Previous immunological victories do not accumulate indefinitely; their effectiveness is contingent on the prevailing biological environment. Predator-prey dynamics display the same structure: a predator may evolve to gain an edge, but selective pressure on prey narrows that advantage over time.

The finches of Daphne Major provide a particularly vivid example (Grant and Grant, 1980). During prolonged periods of soft-seed abundance, small-beaked finches prospered. A severe drought altered seed availability, weakening the advantage of small-beaked finches and enabling large-beaked finches to dominate. Environmental volatility thus redefines the basis of success, ensuring that advantages remain conditional rather than permanently compounding.

More broadly, this structure aligns with the Red Queen hypothesis (Van Valen, 1973): Species must continuously adapt merely to preserve relative position in an ever-changing ecosystem. Persistent environmental fluctuation sustains contestability by preventing any participant from converting temporary superiority into irreversible dominance. In such systems, incentives can be sustained precisely because the lead is ultimately ephemeral. The iterated incumbency framework provides a parsimonious way to formalize this logic: Incumbency confers a bounded advantage within each phase of competition, but exogenous shifts repeatedly reopen the contest, maintaining long-run rivalry and sustained investment. The volatility in the coevolutionary process leads to sustained effort, forcing every species to "run on a treadmill" of adaptation simply to avoid extinction. A natural incentive structure endogenously emerges without exogenous design.

5 Extension: General Success Functions

Our baseline analysis assumes a homogeneous success function (Assumption 1) for expositional efficiency. We now demonstrate that our results hold for more general success functions that admit common properties for component battles. Our discussion proceeds in three steps. We first extract the key properties for component battles that underpin our main results. Second, we demonstrate that these properties also emerge for other success functions. Third, we explain how our results extend to the broader setting with more general success functions.

5.1 Key Properties for Component Battles

Consider a component battle in the contest with a winning value $\Delta_\ell > 0$ for player $\ell \in \{A, B\}$. Let $x_\ell^* = x^*(\Delta_\ell, \Delta_{-\ell})$ denote the player's equilibrium effort. We have the following.

Lemma 2 (Key Properties for Component Battles)**.** *Consider a single battle between two players with respective winning values $\Delta, \Delta' > 0$. With homogeneous success functions as described in Assumption 1, the following hold.*

(i) The battle yields a unique pure-strategy Nash equilibrium; the equilibrium strategy $x^*(\Delta, \Delta')$ is continuous in the winning values, and $x^*(\Delta, \Delta') > 0$.

(ii) There exists a constant $C > 0$ such that $\frac{x^*(\Delta', \Delta)}{x^*(\Delta, \Delta')} < C\frac{\Delta'}{\Delta}$ for all $\Delta' \geq \Delta$.

The property laid out in Lemma 2(ii) requires that the ratio of players' equilibrium efforts be bounded by a constant multiplied by the ratio of winning values Δ_ℓ. Further, define

$$\pi_\ell^* = \pi^*(\Delta_\ell, \Delta_{-\ell}) := \frac{p(x_\ell^*, x_{-\ell}^*)\Delta_\ell - x_\ell^*}{\Delta_\ell} \equiv p(x_\ell^*, x_{-\ell}^*) - \frac{x_\ell^*}{\Delta_\ell}, \tag{1}$$

which is the ratio of player ℓ's equilibrium payoff in the battle to his value of winning, and we call it the battle's *gain ratio function.* With homogeneous success function, $\pi_\ell^* = \phi(\Delta_\ell/\Delta_{-\ell})$.

Lemma 3. *Consider a single battle between two players with respective winning values $\Delta, \Delta' > 0$. With homogeneous success functions as described in Assumption 1, the gain ratio function of the battle has the following properties.*

(i) For all $\epsilon > 0$, $\pi^*(\Delta', \Delta)$ uniformly converges to a continuous function $\pi_0(\Delta') > 0$ as $\Delta \to 0^+$ across all $\Delta' \in [\epsilon, +\infty)$.

(ii) For all $R > 0$, there exists $\pi^*_R > 0$ such that if $\Delta' \geq R\Delta$, then $\pi^*(\Delta', \Delta) \geq \pi^*_R$. Moreover, there exists $R' > 0$ and $\widetilde{\pi} > \frac{1}{2}$ such that if $\Delta' \geq R'\Delta$, then $\pi^*(\Delta', \Delta) \geq \widetilde{\pi}$.

(iii) For all $R'' > 1$, there exists $d_{R''} > 0$ such that if $1 \leq \frac{\Delta'}{\Delta} \leq R''$, $\pi^*(\Delta', \Delta) + \pi^*(\Delta, \Delta') \leq 1 - d_{R''}$.

Lemma 3 concerns the fraction of winning value that a player can extract as his equilibrium payoff in the battle. Lemma 3(i) characterizes a standard continuity property. Lemma 3(ii), in words, states that for any $R > 0$, if a player's winning value Δ' does not fall below a fraction R of his opponent's winning value Δ, then his equilibrium payoff must be at least a fraction π^*_R of Δ'. Further, it states that a player's equilibrium payoff is more than half of his winning value Δ', as long as he is sufficiently strong compared to his opponent, i.e., $\Delta' \geq R'\Delta$ for a constant R', which can be chosen to be arbitrarily large. It is noteworthy that $\pi^*(\Delta', \Delta) + \pi^*(\Delta, \Delta')$ in Lemma 3(iii), as the sum of the two players' gain ratios, is an intuitive indicator of rent dissipation in the battle. The property says that when the battle is not too unbalanced, i.e., Δ'/Δ falls within $[1, R'']$, the fractions of winning value the players can extract must also be bounded, i.e., $\pi^*(\Delta', \Delta) + \pi^*(\Delta, \Delta') \leq 1 - d_{R''}$.

The equilibrium properties laid out in Lemma 2 and Lemma 3 play critical roles in shaping our main results.

5.2 Alternative Success Functions

We derive the properties in Lemma 2 and Lemma 3 from our baseline setting of homogeneous success function. However, they can also arise in alternative contexts. For example, consider the popularly adopted ratio-form success functions, which are specified as follows. Given an effort profile (x, x'), a player exerting an effort x wins the battle with a probability

$$p(x, x') = \begin{cases} \frac{f(x)}{f(x)+f(x')}, & \text{if } (x, x') \neq (0, 0), \\ \frac{1}{2}, & \text{if } (x, x') = (0, 0), \end{cases} \tag{2}$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and twice differentiable with $f(0) = 0$, $f' > 0$, $f'' < 0$, and $\inf_{x>0} \frac{xf'(x)}{f(x)} > 0$. We obtain the following.

Proposition 5. *Ratio-form success functions have the properties in Lemma 2 and Lemma 3.*

5.3 How the results extend

Our results largely hold as long as the success functions for component battles admit the properties laid out in Lemma 2 and Lemma 3. For example, Theorem 1 remains intact with the upper bound $\left[1-(\pi_1^*)^{L(\mathcal{M})}\right]v$ for rent dissipation in finite contests. Similarly, Proposition 2, Theorem 2, Proposition 3, Theorem 4, Theorem 5, Lemma 1, and Theorem 7 are also unaffected.

Theorem 3 requires an additional property: There exist $\widehat{\theta}>1$ and $M>0$ such that, for any single battle in which the winning values satisfy $\frac{\Delta_\ell}{\Delta_{-\ell}}>\widehat{\theta}$, it holds that $\frac{p(x^*_{-\ell},x^*_\ell)\Delta_\ell}{p(x^*_\ell,x^*_{-\ell})\Delta_{-\ell}}>M$. This additional property, together with Lemma 2 and Lemma 3, reinstates Theorem 3. It can be verified that this property (and thus Theorem 3) holds for both the homogeneous and ratio-form success functions. The details are provided in Appendix A.4.

A caveat is worth noting for Theorem 6. To extend Theorem 6 beyond homogeneous success functions, we need to verify the existence of an MPE. This is technically challenging: We conjecture that this equilibrium exists but cannot prove it in general. However, assuming equilibrium existence, the transient dominance property can still be established for tug-of-war with resets.

Finally, we provide the following remark to close this section.

Remark 1. *Suppose that, for an effort profile (x',x), a success function $p(x',x)$ admits the properties in Lemma 2 and Lemma 3. Then the success function $\tilde{p}(x',x)=q\times p(x',x)+(1-q)\times\frac{1}{2}$ for any $q\in(0,1]$ also has those properties.*

Remark 1 states that the properties in Lemma 2 and Lemma 3 are preserved when the success function is mixed with an arbitrarily large fair-coin component. Consequently, our main results continue to hold with additional noise in the single-battle technology.

6 Conclusion

This paper develops a general analysis of dynamic multi-battle contests to explore how contest architecture governs rent dissipation. We first show that a contest cannot fully dissipate its rent as long as its minimum length remains finite. Further, we establish in the case of tug-of-war contests that full rent dissipation may not arise asymptotically even if the minimum length of the contest approaches infinity.

We identify a structural property common in many dynamic contests, exchangeability, which plays a critical role in limiting rent dissipation in equilibrium. Exchangeability enables each battle outcome to exert a long-lasting impact along the dynamics. The long memory of each battle cements one's lucky early wins into persistent advantage, which, in turn, discourages the opponent and disincentivizes both players.

Further, we propose a transient dominance property that defies the long memory caused by exchangeability and establish that dynamic contests exhibiting such a property approach full rent dissipation in the limit. An iterated incumbency contest model is then proposed to demonstrate the principle of transient dominance, which provides a natural and intuitive account of dynamic competitions with random exogenous shocks, such as Schumpeterian competition with technological disruptions and coevolutionary processes in nature.

Our main analysis is set up in two-player settings. We conjecture that the qualitative insights—e.g., the roles played by exchangeability and transient dominance—may extend to multiplayer contests, but a thorough investigation is left for future research.

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Appendix A Single-battle Properties

In this appendix, we establish single-battle properties (i.e., Lemma 2, Lemma 3, and Lemma A2) under the homogeneous (and also the ratio-form) success function. Proofs of Proposition 5 and Remark 1 are also included here. Then in Appendix B, we prove the results for a general success function based on these single-battle properties wherever possible.

A.1 Single-battle Equilibrium under Assumption 1

We first present the equilibrium solution for a single battle under the assumption of homogeneous success function.

Lemma A1 (Malueg and Yates, 2005)**.** *Under Assumption 1, in a single battle with winning values $\Delta_A, \Delta_B > 0$, there exists a unique pure-strategy Nash equilibrium. Equilibrium effort levels are*

$$x_A^* = \Delta_B \gamma'\left(\frac{\Delta_B}{\Delta_A}\right), \qquad x_B^* = \Delta_A \gamma'\left(\frac{\Delta_A}{\Delta_B}\right),$$

and players' expected equilibrium utilities are

$$\Pi_A^* = \Delta_A \phi\left(\frac{\Delta_A}{\Delta_B}\right), \qquad \Pi_B^* = \Delta_B \phi\left(\frac{\Delta_B}{\Delta_A}\right).$$

A.2 Proof of Lemma 2 and Lemma 3

Proof. Suppose that the battle success function is homogeneous as given by Assumption 1. By Lemma A1, we have $x_A = \Delta_B \gamma'(\frac{\Delta_B}{\Delta_A})$ and $x_B = \Delta_A \gamma'(\frac{\Delta_A}{\Delta_B})$ in equilibrium, so Lemma 2(i) is satisfied. Taking derivative of $\gamma(x) + \gamma(1/x) = 1$ yields that $\gamma'(x)x^2 = \gamma'(1/x)$. As a result,

$$\frac{x_A}{x_B} = \frac{\Delta_A}{\Delta_B} \frac{\frac{\Delta_B}{\Delta_A}\gamma'\left(\frac{\Delta_B}{\Delta_A}\right)}{\frac{\Delta_A}{\Delta_B}\gamma'\left(\frac{\Delta_A}{\Delta_B}\right)} = \frac{\Delta_A}{\Delta_B},$$

and Lemma 2(ii) is naturally satisfied.

For Lemma 3, we have

$$\pi^*(\Delta', \Delta) = \phi\left(\frac{\Delta'}{\Delta}\right) := \gamma\left(\frac{\Delta'}{\Delta}\right) - \frac{\Delta'}{\Delta}\gamma'\left(\frac{\Delta'}{\Delta}\right),$$

where ϕ is continuous and strictly increasing, with $\phi(0) = 0$, $\phi(1) \in (0, \frac{1}{2})$, and $\lim_{x \to +\infty} \phi(x) = 1$. Parts (i) and (ii) follow immediately. For part (iii), since $1 - \phi(x) - \phi(1/x)$ is a continuous and positive function on the closed interval $[1, R'']$, letting $d_{R''} := \min_{x\in[1,R'']}\{1 - \phi(x) - \phi(1/x)\}$ finishes the proof. $\square$

A.3 Proof of Proposition 5

Proof. Suppose that the battle success function is given by (2). Lemma 2(i) is satisfied because the equilibrium is determined by the FOC:

$$\frac{\gamma'(x_A)\gamma(x_B)}{(\gamma(x_A)+\gamma(x_B))^2}\Delta_A = \frac{\gamma'(x_B)\gamma(x_A)}{(\gamma(x_A)+\gamma(x_B))^2}\Delta_B = 1. \tag{A1}$$

For Lemma 2(ii), since $\gamma(\cdot)$ is concave and $\inf_{x>0}\frac{x\gamma'(x)}{\gamma(x)} > 0$, there exists constant $C > 1$ such that $\frac{x\gamma'(x)}{\gamma(x)} \in [1/C, 1]$. Therefore,

$$\frac{x_A}{x_B} = \underbrace{\frac{x_A\gamma'(x_A)}{\gamma(x_A)}}_{\text{bounded in }[1/C,1]} \underbrace{\frac{\gamma(x_A)\gamma'(x_B)}{\gamma(x_B)\gamma'(x_A)}}_{=\frac{\Delta_A}{\Delta_B}\text{ by FOC}} \underbrace{\frac{\gamma(x_B)}{x_B\gamma'(x_B)}}_{\text{bounded in }[1,C]} \in \left[\frac{1}{C}\frac{\Delta_A}{\Delta_B}, C\frac{\Delta_A}{\Delta_B}\right]. \tag{A2}$$

For Lemma 3, we first show that $\pi^*(\Delta',\Delta) \in \left[p(x',x)^2, p(x',x)\right]$. Let $x' = x^*(\Delta',\Delta), x = x^*(\Delta,\Delta')$, we have that

$$\begin{aligned}
\pi^*(\Delta',\Delta) \quad &= p(x',x) - \frac{x'}{\Delta'} = \frac{\gamma(x')}{\gamma(x')+\gamma(x)} - \frac{x'}{\Delta'} \\
&\underbrace{=}_{\text{FOC}} \frac{\gamma(x')}{\gamma(x')+\gamma(x)} - \frac{x'\gamma'(x')\gamma(x)}{(\gamma(x)+\gamma(x'))^2} \\
&= \frac{\gamma(x')^2 + \gamma(x)\gamma(x') - x'\gamma'(x')\gamma(x)}{(\gamma(x)+\gamma(x'))^2} \underbrace{\geq}_{\gamma(x')\geq x'\gamma'(x')} p(x',x)^2.
\end{aligned}$$

When $\Delta' \geq \Delta$, by (A1), $x' \geq x$, then it follows that

$$\log\frac{\gamma(x')}{\gamma(x)} = \int_x^{x'}\frac{\gamma'(t)}{\gamma(t)}dt \in \left[\int_x^{x'}\frac{1}{Ct}dt, \int_x^{x'}\frac{1}{t}dt\right] \underbrace{\subseteq}_{\text{by (A2)}} \left[\frac{1}{C}\log\left(\frac{1}{C}\frac{\Delta'}{\Delta}\right), \log\left(C\frac{\Delta'}{\Delta}\right)\right]. \tag{A3}$$

Let $\pi_R^* := \left[G\left(\frac{1}{C}\log\left(\frac{1}{C}R\right)\right)\right]^2$, where $G(z) = \frac{e^z}{1+e^z}$. Then we have $\pi^*(\Delta',\Delta) \geq [p(x',x)]^2 \geq \pi_R^*$. Moreover, since $\pi_R^* \to 1$ as $R \to +\infty$, parts (i) and (ii) are proved. For part (iii), notice that

$$1-\pi^*(\Delta',\Delta)-\pi^*(\Delta,\Delta')=\frac{x'\gamma'(x')\gamma(x)+x\gamma'(x)\gamma(x')}{(\gamma(x)+\gamma(x'))^2}\geq\frac{1}{C}p(x,x')p(x',x).$$

Since $p(x,x')p(x',x)\geq(1-G(\log(CR)))G(\log(CR))$ by (A3), the proof is completed. $\square$

A.4 Lemma A2 and Proof

Lemma A2. *Both the homogeneous and ratio-form success functions admit the following single-battle property: There exist* $\widehat{\theta}>1$ *and* $M>0$ *such that* $\frac{p(x^*_{-\ell},x^*_\ell)\Delta_\ell}{p(x^*_\ell,x^*_{-\ell})\Delta_{-\ell}}>M$ *for all* $\frac{\Delta_\ell}{\Delta_{-\ell}}>\widehat{\theta}$.

Proof. **Ratio-form SF.** For any $M\in(0,1)$, suppose that $\Delta_\ell>\Delta_{-\ell}$. If $x^*_\ell>x^*_{-\ell}$, we have that $\gamma'(x^*_\ell)<\gamma'(x^*_{-\ell})$ and

$$\begin{aligned}\frac{p(x^*_{-\ell},x^*_\ell)\Delta_\ell}{p(x^*_\ell,x^*_{-\ell})\Delta_{-\ell}}&=\frac{\gamma(x^*_{-\ell})\Delta_\ell}{\gamma(x^*_\ell)\Delta_{-\ell}}\underbrace{=}_{\text{FOC}}\frac{\gamma(x^*_{-\ell})\gamma(x^*_\ell)\gamma'(x^*_{-\ell})}{\gamma(x^*_\ell)\gamma'(x^*_\ell)\gamma(x^*_{-\ell})}\\&=\frac{\gamma'(x^*_{-\ell})}{\gamma'(x^*_\ell)}>1>M.\end{aligned}$$

Otherwise if $x^*_\ell\leq x^*_{-\ell}$, it is clear that

$$\frac{p(x^*_{-\ell},x^*_\ell)\Delta_\ell}{p(x^*_\ell,x^*_{-\ell})\Delta_{-\ell}}\geq\frac{\Delta_\ell}{\Delta_{-\ell}}\geq 1>M.$$

Homogeneous SF. Let $\theta=\frac{\Delta_\ell}{\Delta_{-\ell}}$, then in this case $\frac{p(x^*_{-\ell},x^*_\ell)\Delta_\ell}{p(x^*_\ell,x^*_{-\ell})\Delta_{-\ell}}$ becomes $\frac{\theta(1-\phi(\theta))}{1-\phi(1/\theta)}$. We have that

$$\frac{\theta(1-\phi(\theta))}{1-\phi(1/\theta)}=\frac{\theta\big(1-\gamma(\theta)+\theta\gamma'(\theta)\big)}{1-\gamma\big(\frac{1}{\theta}\big)+\frac{1}{\theta}\gamma'\big(\frac{1}{\theta}\big)}\underbrace{=}_{w=1/\theta}\frac{\gamma(w)+w\gamma'(w)}{w\big(1-\gamma(w)+w\gamma'(w)\big)}.$$

Carrying out algebra, $\frac{\theta(1-\phi(\theta))}{1-\phi(1/\theta)}>M$ is equivalent to

$$\frac{\gamma(w)}{w}(1+Mw)+\gamma'(w)(1-Mw)>M. \tag{A4}$$

Let $M=\gamma'(1)$. For any $w\in\big(0,\min\{1,\frac{1}{2M}\}\big)$, concavity implies $\frac{\gamma(w)}{w}\geq\gamma'(w)$ and monotonicity of γ' implies $\gamma'(w)>\gamma'(1)=M$. Hence

$$\frac{\gamma(w)}{w}(1+Mw)+\gamma'(w)(1-Mw)\ \geq\ \frac{\gamma(w)}{w}+\frac{1}{2}\gamma'(w)\ \geq\ \frac{3}{2}\gamma'(w)\ >\ \frac{3}{2}M,$$

so (A4) holds for all sufficiently small w, completing the proof. $\square$

A.5 Proof of Remark 1

Proof. Let $p'(x', x) = q \times p(x', x) + (1 - q) \times \frac{1}{2}$ for $q \in (0, 1]$. Suppose p satisfies Lemma 2 and Lemma 3. We proceed to prove that p' also satisfies Lemma 2 and Lemma 3. For the winning value profile (Δ_A, Δ_B), notice that

$$\arg\max_{x_\ell}\{p'(x_\ell, x_{-\ell})\Delta_\ell - x_\ell\} \;\Leftrightarrow\; \arg\max_{x_\ell}\{p(x_\ell, x_{-\ell})q\Delta_\ell - x_\ell\} \tag{A5}$$

Therefore, the equilibrium strategy under p' is $x^*_{p'}(\Delta', \Delta) = x^*_p(q\Delta', q\Delta)$, and Lemma 2(i) naturally holds. Moreover, we have

$$\frac{x^*_{p'}(\Delta', \Delta)}{x^*_{p'}(\Delta, \Delta')} = \frac{x^*_p(q\Delta', q\Delta)}{x^*_p(q\Delta, q\Delta')} < C\frac{q\Delta'}{q\Delta} = C\frac{\Delta'}{\Delta},$$

which proves Lemma 2(ii). Lemma 3 follows immediately from the fact that

$$\pi^*_{p'}(\Delta', \Delta) = \frac{1-q}{2} + q\pi^*_p(\Delta', \Delta).$$

$\square$

Appendix B Proofs for Main Results

This appendix build upon the results in Appendix A. In this appendix, we prove the results for a general success function that admits single-battle properties given in Lemma 2 and Lemma 3, except for the uniqueness part of Proposition 1, Theorem 3, and Theorem 6: We prove Theorem 3 with Lemma A2 in addition to Lemmas 2 and 3; and we prove the uniqueness part of Proposition 1 and Theorem 6 for the homogeneous success function (i.e., under Assumption 1).

B.1 Proof of Theorem 1 Based on Lemmas 2 and 3

Proof. For a single battle in a dynamic contest, we denote by $V_\ell^{\ell'}$ the continuation value of player $\ell \in A, B$ if player ℓ' wins this battle. It is useful to prove the following intermediate result.

Lemma B1. *Consider a single-battle contest with $V_A^A > V_A^B$ and $V_B^B > V_B^A$. The equilibrium payoffs in the single-battle contest (U_A^*, U_B^*) satisfy*

$$U_A^* + U_B^* > \pi_1^* \max\{V_A^A + V_B^A, V_A^B + V_B^B\}. \tag{B1}$$

Proof. Without loss of generality, we assume that $\frac{V_A^A - V_A^B}{V_B^B - V_B^A} \geq 1$. Note that the value of winning is $\Delta_A = V_A^A - V_A^B$ for player A and $\Delta_B = V_B^B - V_B^A$ for player B. We have that

$$\begin{aligned} U_A^* + U_B^* &= V_A^B + (V_A^A - V_A^B)\pi_A^* + V_B^A + (V_B^B - V_B^A)\pi_B^* \\ &= (V_A^A + V_B^A)\pi_A^* + (V_A^B + V_B^B)\pi_B^* + (1 - \pi_A^* - \pi_B^*)\,(V_A^B + V_B^A). \end{aligned}$$

Note that $\pi_A^* + \pi_B^* = p_A^* + p_B^* - \frac{x_A^*}{\Delta_A} - \frac{x_B^*}{\Delta_B} \leq 1$. Further, since $\frac{V_A^A - V_A^B}{V_B^B - V_B^A} \geq 1$, $V_A^A + V_B^A \geq V_A^B + V_B^B$. As a result, $U_A^* + U_B^* > (V_A^A + V_B^A)\pi_A^* = \max\{V_A^A + V_B^A, V_A^B + V_B^B\}\pi_A^*$. Since $\Delta_A \geq \Delta_B$, by Lemma 3(ii), $\pi_A^* \geq \pi_1^*$. This completes the proof. □

Lemma B1 means that the expected aggregate payoff in a battle is higher than a factor π_1^* times the maximum of aggregate continuation payoff across both possible outcomes. Applying Lemma B1 repeatedly along the shortest path to a terminal node (which has aggregate payoff v) implies that the initial aggregate payoff is no less than $(\pi_1^*)^{L(\mathcal{M})}v$. Therefore, the expected total effort is less than $\left[1 - (\pi_1^*)^{L(\mathcal{M})}\right] v$. □

B.2 Proof for the Existence Part of Proposition 1 Based on Lemmas 2 and 3

Proof. **Equilibrium characterization.** The value of winning a battle (or the incentive to win), $\Delta(i; N, v)$, is given by the difference between the continuation value after winning the battle and that after losing it, i.e.,

$$\Delta(i; N, v) := V(i + 1; N, v) - V(i - 1; N, v).$$

The opponent's winning value is thus $\Delta(-i; N, v) := V(-i + 1; N, v) - V(-i - 1; N, v)$.

Recall a player ℓ's single-battle gain ratio function π_ℓ^*, as defined in (1). For tug-of-war games, we define a player's gain ratio function in the state i battle as $\pi(i) := \phi\big(\Delta(i)/\Delta(-i)\big)$. The equilibrium conditions are

$$V(N) = v,\ V(-N) = 0, V(i) = V(i - 1) + \pi(i)[V(i + 1) - V(i - 1)] \tag{B2}$$

for $-(N - 1) \leq i \leq N - 1$.

Equilibrium existence. Our goal is to construct a mapping from a set of values $(v_i)_{-N\leq i\leq N}$ to a new set of values, such that the fixed point of this mapping satisfies (B2), and then prove that a fixed point exists for this mapping.

First, we define the following function $\Pi^*(\Delta',\Delta):[0,v]^2\to[0,v]$, which augments the equilibrium payoff function for a single battle in a natural way:

$$\Pi^*(\Delta',\Delta) := \begin{cases} p(x^*(\Delta',\Delta),x^*(\Delta,\Delta'))\Delta' - x^*(\Delta',\Delta), & \text{if } \Delta',\Delta>0,\\ \pi_0(\Delta')\Delta', & \text{if } \Delta=0. \end{cases} \tag{B3}$$

By Lemma 3(i), $\Pi^*(\Delta',\Delta)$ is continuous on $[0,v]^2$. Moreover, it can be verified that the function $\mathbf{\Pi}^*(\Delta',\Delta):=(\Pi^*(\Delta',\Delta),\Pi^*(\Delta,\Delta'))$ is continuous on $[0,v]^2$.

Next, we proceed to define a mapping from $\mathcal{V}:=\{\boldsymbol{v}:=(v_i)_{i=-(N-1)}^{N-1}:0\leq v_{-N+1}\leq v_{-N+2}\leq\ldots\leq v_{N-1}\leq v \text{ and } v_i+v_{-i}\leq v, \text{ for all } -(N-1)\leq i\leq N-1\}$ to itself. Let $v_{-N}=0$, $v_N=v$, and $\mathbf{\Phi}:\mathcal{V}\to\mathcal{V}$. $\boldsymbol{v}'=\mathbf{\Phi}(\boldsymbol{v})$ is recursively given by the following, with initial values $v'_N=v$ and $v'_{-N}=0$:

$$\begin{pmatrix} v'_i \\ v'_{-i}\end{pmatrix} = \begin{pmatrix} v_{i-1} \\ v'_{-i-1}\end{pmatrix} + \mathbf{\Pi}^*(v'_{i+1}-v_{i-1}, v_{-i+1}-v'_{-i-1}), \text{ for } i=N-1,\ldots,1, \tag{B4}$$

$$v'_0 = v'_{-1} + \Pi^*(v'_1-v'_{-1}, v'_1-v'_{-1}). \tag{B5}$$

It is easy to see that the any fixed point of $\mathbf{\Phi}$ corresponds to the value function in a symmetric MPE of the tug-of-war, and vice versa.

We now verify that $\mathbf{\Phi}$ is well-defined. First, for $i=N-1$, it is clear that $v'_{i+1}-v_{i-1}\in[0,v]$ and $v_{-i+1}-v'_{-i-1}\in[0,v]$. Suppose that $v'_{i+1}-v_{i-1}\in[0,v]$ and $v_{-i+1}-v'_{-i-1}\in[0,v]$ hold for $i=i'\in\{1,\ldots,N-1\}$, we show below that they also hold for $i=i'-1$. Since $0\leq\Pi^*(\Delta',\Delta)\leq\Delta'$ for all $(\Delta',\Delta)\in[0,v]^2$, it follows from (B4) that $v'_{i'+1}\geq v'_{i'}\geq v_{i'-1}$. This implies that $v\geq v'_{i'}\geq v_{i'-2}$, and $v'_{i+1}-v_{i-1}\in[0,v]$ holds for $i=i'-1$. Again, since $\Pi^*(\Delta',\Delta)\leq\Delta'$ for all $(\Delta',\Delta)\in[0,v]^2$, it follows from (B4) that $v'_{-i'}\leq v_{-i'+1}\leq v_{-i'+2}$, so $v_{-i+1}-v'_{-i-1}\in[0,v]$ holds for $i=i'-1$. Finally, when $i=1$, (B4) implies that $v'_2\geq v'_1\geq v_0\geq v'_{-1}$, so $v'_1-v'_{-1}\in[0,v]$.

Then we verify that $\Phi(\boldsymbol{v})\in\mathcal{V}$. From (B4), it is easy to see that $v'_i\leq v'_{i+1}$ and $v'_{-i}\geq v'_{-i-1}$ for $i=N-1,\ldots,1$. We have shown above that $v'_1\geq v_0\geq v'_{-1}$. Therefore, $v\geq v'_{N-1}\geq\ldots\geq v'_1\geq v_0\geq v'_{-1}\geq\ldots\geq v'_{-N+1}\geq 0$. By (B5), $v'_1\geq v'_0\geq v'_{-1}$, and thus $1\geq v'_{N-1}\geq\ldots\geq v'_1\geq v'_0\geq v'_{-1}\geq\ldots\geq v'_{-N+1}\geq 0$. Note that $\Pi^*(\Delta',\Delta)+\Pi^*(\Delta,\Delta')\leq\max\{\Delta',\Delta\}$. By (B5), for

$1 \leq i \leq (N-1)$,

$$\begin{aligned} v'_i + v'_{-i} &\leq v_{i-1} + v'_{-i-1} + \max\{v'_{i+1} - v_{i-1}, v_{-i+1} - v'_{-i-1}\} \\ &= \max\{v'_{i+1} + v'_{-i-1}, v_{-i+1} + v_{i-1}\} \leq \max\{v'_{i+1} + v'_{-i-1}, v\}. \end{aligned}$$

Applying the inequality iteratively yields that $v'_i + v'_{-i} \leq \max\{v'_N + v'_{-N}, v\} = v$. Finally, by (B5), $2v'_0 \leq 2v'_{-1} + \max\{v'_1 - v'_{-1}, v'_1 - v'_{-1}\} = v'_{-1} + v'_1 \leq v$.

Since $\mathcal{V}$ is non-empty, convex, and compact, and $\boldsymbol{\Phi}$ is continuous due to the continuity of $\boldsymbol{\Pi}^*$, by Brouwer's fixed-point theorem, there exists $\boldsymbol{v}^* \in \mathcal{V}$ such that $\boldsymbol{v}^* = \boldsymbol{\Phi}(\boldsymbol{v}^*)$.

It only remains to show that $0 < v^*_{-N+1} < \ldots < v^*_{N-1} < v$. Suppose that there exists $-N+1 \leq i' \leq N-1$ such that $v^*_{i'-1} = v^*_{i'} < v^*_{i'+1}$. Then $v^*_{i'} = v^*_{i'-1} + \Pi^*(v^*_{i'+1} - v^*_{i'-1}, v^*_{-i'+1} - v^*_{-i'-1}) > v^*_{i'-1}$, a contradiction. Therefore, if $v^*_i = v^*_{i+1}$ for some i, then $v^*_i = v^*_j$ for all $j > i$. Suppose that i' is the smallest integer satisfying $v^*_{i'} = v^*_{i'+1}$, then

$$0 = v^*_{-N} < v^*_{-N+1} < \ldots < v^*_{i'} = \ldots = v^*_N = v. \tag{B6}$$

If $i' > 0$, from $v^*_{i'} = v^*_{i-1} + \Pi^*(v^*_{i'+1} - v^*_{i'-1}, v^*_{-i'+1} - v^*_{-i'-1}) = v^*_{i'+1}$, it follows that $v^*_{-i'+1} = v^*_{-i'-1}$, contradicting (B6). If $i' \leq 0$, $v^*_{i'} + v^*_{-i'} = 2v$, contradicting $v^*_{-i} + v^*_i \leq v$. $\square$

B.3 Proof for the Uniqueness Part of Proposition 1 under Assumption 1

Proof. See the proof of Theorem 6 in Appendix B.12. $\square$

B.4 Proof of Proposition 2 Based on Lemmas 2 and 3

Proof. It is useful to establish the following intermediate results first.

Lemma B2 (Self-reinforcement of Wins in Tug-of-War)**.** *In the symmetric MPE of the tug-of-war with margin N, the following holds for $-(N-2) \leq i \leq N-2$:*

$$\frac{\Delta(i+1)}{\Delta(-i-1)} = \underbrace{\frac{1-\pi(-(i+1))}{\pi(i+1)}}_{>1} \underbrace{\frac{1-\pi(i)}{\pi(-i)}}_{>1} \frac{\Delta(i)}{\Delta(-i)} > \frac{\Delta(i)}{\Delta(-i)}. \tag{B7}$$

Proof. For $-(N-1) \leq i \leq N-1$, the equilibrium condition (B2) can be rearranged in the following two ways:

$$V(i) - V(i-1) = \pi(i)\left[V(i+1) - V(i-1)\right], \tag{B8}$$

$$V(i+1) - V(i) = \big[1 - \pi(i)\big]\,\big[V(i+1) - V(i-1)\big]. \tag{B9}$$

Replacing i with $-i$ in (B8) yields

$$V(-i) - V(-i-1) = \pi(-i)\,\big[V(-i+1) - V(-i-1)\big], \tag{B10}$$

and dividing (B9) by (B10) yields

$$\frac{V(i+1) - V(i)}{V(-i) - V(-i-1)} = \frac{1 - \pi(i)}{\pi(-i)} \frac{V(i+1) - V(i-1)}{V(-i+1) - V(-i-1)}. \tag{B11}$$

Replacing i with $-(i+1)$ in (B11), we have that

$$\frac{V(-i) - V(-i-1)}{V(i+1) - V(i)} = \frac{1 - \pi(-i-1)}{\pi(i+1)} \frac{V(-i) - V(-i-2)}{V(i+2) - V(i)} \text{ for } -(N-2) \le i \le N-2. \tag{B12}$$

Finally, multiplying (B11) and (B12) and simple rearrangement give (B7). $\square$

Lemma B3. *There exists a positive integer N' such that for all $v > 0$, $i \ge N'$ and $N > i$, $\pi(i; N, v) > \widetilde{\pi} > \frac{1}{2}$.*

Proof. By Lemma B2, $\frac{\Delta(i)}{\Delta(-i)}$ strictly increases in i with $\frac{\Delta(0)}{\Delta(-0)} = 1$. For all i such that $\frac{\Delta(i)}{\Delta(-i)} \le R'$, by Lemma 3(iii), $\frac{1-\pi(i)}{\pi(-i)} \ge 1 + d_{R'}$. This and (B7) imply that $\frac{\Delta(i)}{\Delta(-i)} \ge (1 + d_{R'})^i$ for all $i \ge 0$ satisfying $\frac{\Delta(i)}{\Delta(-i)} \le R'$. Therefore, there exists a positive integer N' (independent of N and v) such that $\frac{\Delta(i)}{\Delta(-i)} \ge R'$ for all $i \ge N'$. We know from Lemma 3(ii) that $\pi(i) \ge \widetilde{\pi}$ for $i \ge N'$. $\square$

Dividing (B8) by (B9), we have $\frac{V(i)-V(i-1)}{V(i+1)-V(i)} = \frac{\pi(i)}{1-\pi(i)}$, which can be rewritten as $V(i+1) - V(i) = \frac{1-\pi(i)}{\pi(i)}[V(i) - V(i-1)]$. Therefore,

$$v - V(i) = \sum_{j=i}^{N-1}[V(j+1) - V(j)] = [V(-N+1) - V(-N)] \sum_{j=i}^{N-1} \left[\prod_{k=-N+1}^{j} \frac{1 - \pi(k)}{\pi(k)} \right],$$

which yields that

$$\frac{v - V(i+1)}{v - V(i)} = \frac{\sum_{j=i+1}^{N-1} \left[\prod_{k=-N+1}^{j} \frac{1-\pi(k)}{\pi(k)}\right]}{\sum_{j=i}^{N-1} \left[\prod_{k=-N+1}^{j} \frac{1-\pi(k)}{\pi(k)}\right]} = \frac{\sum_{j=i+1}^{N-1} \left[\Pi_{k=i+1}^{j} \frac{1-\pi(k)}{\pi(k)}\right]}{1 + \sum_{j=i+1}^{N-1} \left[\Pi_{k=i+1}^{j} \frac{1-\pi(k)}{\pi(k)}\right]}. \tag{B13}$$

By Lemma B3, for $i > N'$, we have that

$$\frac{\sum_{j=i+1}^{N-1}\left[\prod_{k=i+1}^{j}\frac{1-\pi(k)}{\pi(k)}\right]}{1+\sum_{j=i+1}^{N-1}\left[\prod_{k=i+1}^{j}\frac{1-\pi(k)}{\pi(k)}\right]} \leq \frac{\sum_{j=1}^{+\infty}(\frac{1-\tilde{\pi}}{\tilde{\pi}})^j}{1+\sum_{j=1}^{+\infty}(\frac{1-\tilde{\pi}}{\tilde{\pi}})^j} = \frac{1-\tilde{\pi}}{\tilde{\pi}}. \tag{B14}$$

Consequently,

$$\frac{v-V(N'+i)}{v} = \frac{v-V(N')}{v}\prod_{j=N'}^{N'+i-1}\frac{v-V(j+1)}{v-V(j)} \leq \frac{v-V(N')}{v}\left(\frac{1-\tilde{\pi}}{\tilde{\pi}}\right)^i, \tag{B15}$$

which implies that $\frac{v-V(N'+i)}{v} \to 0$ as $i \to +\infty$ holds uniformly for N, v and over all symmetric MPE. Notice that $V(N'+i; N, v) \leq vQ(N'+i; N, v)$. It then follows that

$$Q(N'+i; N, v) \geq \frac{V(N'+i)}{v} \to 1$$

uniformly for any v, N and over all symmetric MPE. □

B.5 Proof of Theorem 2 Based on Lemmas 2 and 3

Proof. Using (B13) with $i = -1$, we have that

$$\frac{v-V(0)}{v} < \frac{v-V(0)}{v-V(-1)} = \frac{\sum_{j=0}^{N-1}\left[\prod_{k=0}^{j}\frac{1-\pi(k)}{\pi(k)}\right]}{1+\sum_{j=0}^{N-1}\left[\prod_{k=0}^{j}\frac{1-\pi(k)}{\pi(k)}\right]}. \tag{B16}$$

By Lemma B3 and Lemma 3(ii), the following estimation holds.

$$\begin{aligned}\sum_{j=0}^{N-1}\left[\prod_{k=0}^{j}\frac{1-\pi(k)}{\pi(k)}\right] &\leq \sum_{j=0}^{N'-1}\left(\prod_{k=0}^{j}\frac{1-\pi_1^*}{\pi_1^*}\right) + \prod_{k=0}^{N'-1}\left(\frac{1-\pi_1^*}{\pi_1^*}\right)\sum_{j=N'}^{+\infty}\left(\prod_{k=N'}^{j}\frac{1-\tilde{\pi}}{\tilde{\pi}}\right) \\ &\leq \sum_{j=0}^{N'-1}\left(\prod_{k=0}^{j}\frac{1-\pi_1^*}{\pi_1^*}\right) + \left(\frac{\frac{1-\tilde{\pi}}{\tilde{\pi}}}{1-\frac{1-\tilde{\pi}}{\tilde{\pi}}}\right)\prod_{k=0}^{N'-1}\left(\frac{1-\pi_1^*}{\pi_1^*}\right).\end{aligned} \tag{B17}$$

Plug (B17) into (B16), we have that

$$\frac{V(0)}{v} \geq \left[1+\sum_{j=0}^{N'-1}\left(\prod_{k=0}^{j}\frac{1-\pi_1^*}{\pi_1^*}\right) + \left(\frac{\frac{1-\tilde{\pi}}{\tilde{\pi}}}{1-\frac{1-\tilde{\pi}}{\tilde{\pi}}}\right)\prod_{k=0}^{N'-1}\left(\frac{1-\pi_1^*}{\pi_1^*}\right)\right]^{-1} =: \alpha,$$

which finishes the proof. $\square$

B.6 Proof of Theorem 3 Based on Lemmas 2, 3, and A2

Proof. To prove this result, we show that there exists $\widetilde{\alpha} > 0$ depending only on the success function such that for every sequence $\{M_k\}_{k=1}^K$ of exchangeable contests with M_{k+1} being the N_{k+1}-extension of M_k,

$$\frac{1}{K}\sum_{k=1}^{K} V_{0;k} \geq \widetilde{\alpha} v.$$

Then, when $\lim_{k\to+\infty} V_{0;k}$ exists, it coincides with $\lim_{K\to+\infty} \frac{1}{K}\sum_{k=1}^{K} V_{0;k}$, and thus is also bounded from below.

Consider a sequence of exchangeable contests $\{M_k\}_{k=1}^K$ where M_{k+1} is the N_{k+1}-extension of M_k. In contest M_k, the game ends once some player achieves N_k consecutive wins from the start. Let $V_{i;k}$ be the equilibrium continuation value along the *consecutive path* in contest M_k:

- $V_{i;k}$ $(i \geq 0)$: value of the player who currently has a streak of i consecutive wins;
- $V_{-i;k}$ $(i \geq 0)$: value of the player who currently has a streak of i consecutive losses.

Note that $V_{N_k;k} = v$ and $V_{-N_k;k} = 0$.

The extension structure implies the following along the consecutive path: at state $i \in \{1, \ldots, N_k - 1\}$ in M_k, if the current streak leader loses the next battle, the continuation becomes contest M_{k-1} at state $i-1$. Hence, for $-N_k + 1 \leq i \leq N_k - 1$, the stake and stake ratio are

$$\Delta_{i;k} := V_{i+1;k} - V_{i-1;k-1}, \quad \theta_{i,k} := \frac{\Delta_{i;k}}{\Delta_{-i,k}}.$$

We proceed to prove the desired result in the following steps.

Step 1 (useful bounds in a single-battle). We set $\theta^* > \max\{\widehat{\theta}, R'\}$. Let

$$\eta(\Delta_\ell, \Delta_{-\ell}) := \frac{1 - \pi^*(\Delta_\ell, \Delta_{-\ell})}{1 - \pi^*(\Delta_{-\ell}, \Delta_\ell)} \frac{\Delta_\ell}{\Delta_{-\ell}}.$$

We show next that

$$\eta(\Delta_\ell, \Delta_{-\ell}) \leq C := \max\left\{\frac{\theta^*}{d''_{\theta^*}}, \frac{1}{\min\{M, 1\}}\right\} \text{ if } \Delta_\ell/\Delta_{-\ell} \leq \theta^*, \tag{B18}$$

where $\widehat{\theta}$ and M are given in Lemma A2. On one hand, if $\Delta_\ell/\Delta_{-\ell} \in [\frac{1}{\theta^*}, \theta^*]$, by Lemma 3(iii),

$\eta(\Delta_\ell, \Delta_{-\ell}) \in [\frac{d''_{\theta^*}}{\theta^*}, \frac{\theta^*}{d''_{\theta^*}}]$. On the other hand, if $\Delta_\ell/\Delta_{-\ell} < 1/\theta^*$, $\Delta_{-\ell}/\Delta_\ell > \theta^* > \widehat{\theta}$, thus

$$\eta(\Delta_{-\ell}, \Delta_\ell) = \frac{p(x^*_{-\ell}, x^*_l)\Delta_\ell + x^*_l}{p(x^*_l, x^*_{-\ell})\Delta_{-\ell} + x^*_{-\ell}} \geq \min\left\{\frac{p(x^*_{-\ell}, x^*_l)\Delta_\ell}{p(x^*_l, x^*_{-\ell})\Delta_{-\ell}}, \frac{x^*_l}{x^*_{-\ell}}\right\} \geq \min\{M, 1\}.$$

which indicates that $\eta(\Delta_\ell, \Delta_{-\ell}) \leq \frac{1}{\min\{M,1\}}$.

It is also worth pointing out that, by Lemma 3(ii),

$$\frac{1 - \pi^*(\Delta_\ell, \Delta_{-\ell})}{\pi^*(\Delta_\ell, \Delta_{-\ell})} \leq \frac{1 - \widetilde{\pi}}{\widetilde{\pi}} \text{ if } \Delta_\ell/\Delta_{-\ell} > \theta^*. \tag{B19}$$

Step 2 (within-contest telescoping). Since $V_{N_k;k} = v$,

$$v - V_{1;k} = \sum_{i=1}^{N_k - 1} (V_{i+1;k} - V_{i;k}).$$

Summing over $k = 2, \ldots, K$ and using the base case $v - V_{1;1} = 0$ yields

$$Kv - \sum_{k=1}^{K} V_{1;k} = \sum_{(k,i)\in\mathcal{R}} (V_{i+1;k} - V_{i;k}), \tag{B20}$$

where

$$\mathcal{R} := \{(k, i) : 2 \leq k \leq K,\ 1 \leq i \leq N_k - 1\}.$$

Step 3 (split the RHS of (B20)). Split $\mathcal{R}$ into

$$\mathcal{R}_{\text{large}} := \{(k, i) \in \mathcal{R} : \theta_{i,k} > \theta^*\}, \qquad \mathcal{R}_{\text{small}} := \{(k, i) \in \mathcal{R} : \theta_{i,k} \leq \theta^*\}.$$

Then

$$\sum_{(k,i)\in\mathcal{R}} (V_{i+1;k} - V_{i;k}) = \sum_{(k,i)\in\mathcal{R}_{\text{large}}} (V_{i+1;k} - V_{i;k}) + \sum_{(k,i)\in\mathcal{R}_{\text{small}}} (V_{i+1;k} - V_{i;k}).$$

Step 4 (bound the large-θ part). If $(k, i) \in \mathcal{R}_{\text{large}}$, then by (B19),

$$V_{i+1;k} - V_{i;k} = \frac{1 - \pi^*(\Delta_{i;k}, \Delta_{-i;k})}{\pi^*(\Delta_{i;k}, \Delta_{-i;k})}(V_{i;k} - V_{i-1;k-1}) \leq \frac{1 - \widetilde{\pi}}{\widetilde{\pi}}(V_{i;k} - V_{i-1;k-1}).$$

Hence

$$\sum_{(k,i)\in\mathcal{R}_{\text{large}}} (V_{i+1;k} - V_{i;k}) \leq \frac{1 - \widetilde{\pi}}{\widetilde{\pi}} \sum_{(k,i)\in\mathcal{R}} (V_{i;k} - V_{i-1;k-1}) \leq \frac{1 - \widetilde{\pi}}{\widetilde{\pi}}\Big(Kv - \sum_{k=1}^{K} V_{0;k}\Big).$$

To see the reason for the last inequality, group the terms of $\sum_{(k,i)\in\mathcal{R}}(V_{i;k} - V_{i-1;k-1})$ by diagonals $d := k - i$. Along a fixed diagonal, the sum telescopes:

$$\sum_{\substack{(k,i)\in\mathcal{R}\\ k-i=d}} \big(V_{i;k} - V_{i-1;k-1}\big) = V_{i_{\max};\,d+i_{\max}} - V_{0;d},$$

where $i_{\max}$ is the largest i on that diagonal in $\mathcal{R}$. Because values are bounded above by the prize, $V_{i_{\max};\,d+i_{\max}} \leq 1$, so the diagonal sum is at most $v - V_{0;d}$. Summing over $d = 1, \ldots, K$ yields the inequality.

Step 5 (bound the small-θ part). If $(k,i) \in \mathcal{R}_{\text{small}}$, then $\theta_{i,k} \leq \theta^*$ and by (B18),

$$V_{i+1;k} - V_{i;k} = \eta(\theta_{i,k})(V_{1-i;k-1} - V_{-i;k}) \leq C(V_{1-i;k-1} - V_{-i;k}).$$

Therefore

$$\sum_{(k,i)\in\mathcal{R}_{\text{small}}} (V_{i+1;k} - V_{i;k}) \leq C \sum_{(k,i)\in\mathcal{R}} (V_{1-i;k-1} - V_{-i;k}) \leq C \sum_{k=1}^{K} V_{0;k}.$$

To see why the last inequality holds, again fix a diagonal $d := k - i$. For some d, a summand on that diagonal is

$$V_{1-j;\,d+j-1} - V_{-j;\,d+j} = V_{-(j-1);\,d+j-1} - V_{-j;\,d+j},$$

for $j = \{1, \ldots, j_{\max}\}$, where $j_{\max}$ is the largest number such that $(j+d, j) \in \mathcal{R}$. Summing over all such j on the diagonal telescopes to

$$V_{0;d} - V_{-j_{\max};\,d+j_{\max}} \leq V_{0;d},$$

since values are nonnegative. Summing over $d = 1, \ldots, K$ gives the desired inequality.

Step 6 (combine with (B20)). Plugging Steps 4–5 into (B20) gives

$$Kv - \sum_{k=1}^{K} V_{1;k} \leq C \sum_{k=1}^{K} V_{0;k} + \frac{1-\widetilde{\pi}}{\widetilde{\pi}}\Big(Kv - \sum_{k=1}^{K} V_{0;k}\Big).$$

Since $V_{1;k} \geq V_{0;k}$ for each k, we have $\sum V_{0;k} \leq \sum V_{1;k}$, so

$$Kv - \sum_{k=1}^{K} V_{1;k} \leq C \sum_{k=1}^{K} V_{1;k} + \frac{1-\widetilde{\pi}}{\widetilde{\pi}} Kv.$$

Rearrange:

$$\frac{1}{K}\sum_{k=1}^{K} V_{1;k} \geq \frac{(2\widetilde{\pi}-1)}{(C+1)\widetilde{\pi}}v. \tag{B21}$$

Step 7 (from $V_{1;k}$ to $V_{0;k}$). By Lemma 3(ii),

$$V_{0;k} = [1-\pi^*(\Delta_{0;k},\Delta_{0;k})]V_{-1;k} + \pi^*(\Delta_{0;k},\Delta_{0;k})V_{1;k} \geq \pi_1^* V_{1;k}.$$

Average and use (B21):

$$\frac{1}{K}\sum_{k=1}^{K} V_{0;k} \geq \frac{\pi_1^*(2\widetilde{\pi}-1)}{(C+1)\widetilde{\pi}}v.$$

The proof is completed by setting $\widetilde{\alpha} := \frac{\pi_1^*(2\widetilde{\pi}-1)}{(C+1)\widetilde{\pi}} > 0$. □

B.7 Existence and Uniqueness of Symmetric MPE in Consecutive-win Contests

Proposition B1. *Under Assumption 1, a unique symmetric MPE exists for the K-consecutive-win contest with $K \in \mathbb{N}_{++}$. In the symmetric MPE, $0 = \widehat{V}(-K) < \widehat{V}(-K+1) < \ldots < \widehat{V}(K-1) < \widehat{V}(K) = v$.*

B.7.1 Proof of the Existence Part of Proposition B1 Based on Lemmas 2 and 3

Proof. Our approach is to build a mapping whose fixed point is $(\widehat{V}(-1), \widehat{V}(1))$, and show that a fixed point exists for the mapping.

Recall the single-battle equilibrium net gain (over the losing continuation payoff) function $\mathbf{\Pi}^*(\Delta', \Delta) := (\Pi^*(\Delta', \Delta), \Pi^*(\Delta, \Delta'))$, whose components are defined by (B3) in the proof of Proposition 1. Define

$$\boldsymbol{\zeta}_{(v_1,v_2)}(v_a, v_b) := \begin{pmatrix} v_a \\ v_2 \end{pmatrix} + \mathbf{\Pi}^*(v_1 - v_a,\ v_b - v_2), \tag{B22}$$

which gives the equilibrium payoffs when the continuation values are (v_1, v_2) after player A wins the current battle and (v_a, v_b) after player B wins. By definition, we have that

$$\boldsymbol{\zeta}_{(v_1,v_2)}(v_a, v_b) \in [v_a, v_1] \times [v_2, v_b], \tag{B23}$$

which means each player's equilibrium payoff lies between his losing and winning continuation payoffs. Moreover,

$$\boldsymbol{\zeta}_{(v_1,v_2)}(v_a, v_b) \in (v_a, v_1) \times (v_2, v_b), \text{ if } v_1 > v_a \text{ and } v_b > v_2. \tag{B24}$$

Define the domain

$$\mathcal{X} := \{(u, w) \in [0, v]^2 : \; u + w \le v\},$$

and define the mapping $\boldsymbol{\xi} : \mathcal{X} \to \mathcal{X}$ by

$$\boldsymbol{\xi}(u, w) := \boldsymbol{\zeta}^{K-1}_{(w,u)}(0, v), \tag{B25}$$

where $\boldsymbol{\zeta}^m$ denotes the m-fold composition. Note that, if an equilibrium exists, $\left(\widehat{V}(-1), \widehat{V}(1)\right)$ is a fixed point of $\boldsymbol{\xi}(u, w)$, for the following reason. By definition of $\boldsymbol{\zeta}$, it is easy to see that the continuation values when B is one battle win away from winning overall is $\boldsymbol{\zeta}_{\left(\widehat{V}(1),\widehat{V}(-1)\right)}(0, v)$. Applying this relation repeatedly, $\boldsymbol{\zeta}^{K-1}_{\left(\widehat{V}(1),\widehat{V}(-1)\right)}(0, v)$ is the continuation payoffs when B is $K-1$ battle wins away from ultimately winning, or equivalently, has a winning streak of 1. By symmetry, we have that $\left(\widehat{V}(-1), \widehat{V}(1)\right) = \boldsymbol{\zeta}^{K-1}_{\left(\widehat{V}(1),\widehat{V}(-1)\right)}(0, v)$, which is exactly the fixed point equation for $\boldsymbol{\xi}(u, w)$.

We proceed to show that a fixed point exists for $\boldsymbol{\xi}$. We first prove that $\boldsymbol{\xi}$ maps $\mathcal{X}$ into itself and is continuous. Continuity follows immediately from continuity of $\boldsymbol{\Pi}^*$. To see $\boldsymbol{\xi}(\mathcal{X}) \subseteq \mathcal{X}$, it suffices to show that the sum of coordinates never exceeds v along the iteration. Fix $(u, w) \in \mathcal{X}$ and let $(a_0, b_0) = (0, v)$ and $(a_1, b_1) = \boldsymbol{\zeta}_{(w,u)}(a_0, b_0)$, etc. Write $(a', b') = \boldsymbol{\zeta}_{(w,u)}(a, b)$. Suppose $(a, b) \in \mathcal{X}$, since $\Pi^*(\Delta', \Delta) + \Pi^*(\Delta, \Delta') \le \max\{\Delta, \Delta'\}$, we have that

$$\begin{aligned} a' + b' &= a + u + \Pi^*(w - a, \; b - u) + \Pi^*(b - u, \; w - a) \\ &\le a + u + \max\{w - a, \; b - u\} = \max\{u + w, \; a + b\} \le v. \end{aligned}$$

Hence all iterates satisfy $a_j + b_j \le v$, and in particular $\boldsymbol{\xi}(u, w) = (a_{K-1}, b_{K-1}) \in \mathcal{X}$. Since $\mathcal{X}$ is nonempty, compact, and convex, Brouwer's fixed-point theorem yields $(u^*, w^*) \in \mathcal{X}$ such that $\boldsymbol{\xi}(u^*, w^*) = (u^*, w^*)$.

Next, we construct $\widehat{V}(\cdot)$ from the fixed point (u^*, w^*). Define the iterates

$$(a^*_0, b^*_0) := (0, v), \qquad (a^*_j, b^*_j) := \boldsymbol{\zeta}_{(w^*,u^*)}(a^*_{j-1}, b^*_{j-1}), \quad j = 1, \ldots, K-1. \tag{B26}$$

By the fixed-point condition, $(a^*_{K-1}, b^*_{K-1}) = (u^*, w^*)$.

By (B23), $0 = a^*_0 \le a^*_1 \le \cdots \le a^*_{K-1} \le w^*$, so $u^* = a^*_{K-1} \le w^*$. Since $\boldsymbol{\zeta}^{K-1}_{(0,0)}(0, v) \ne (0, 0)$, $(u^*, w^*) \ne (0, 0)$. In combination, $u^* \le w^*$, $(u^*, w^*) \ne (0, 0)$, and $u^* + w^* \le v$ imply that

$w^* > 0$ and $v > u^*$. Then repeatedly applying (B24) yields that

$$0 = a_0^* < a_1^* < \cdots < a_{K-1}^* = u^* < w^* = b_{K-1}^* < \cdots < b_1^* < b_0^* = v.$$

Then it is straightforward to verify that the following construction satisfies the equilibrium condition:

$$\widehat{V}(-K) = 0, \quad \widehat{V}(K) = v, \qquad \widehat{V}(-K+j) := a_j^*, \quad \widehat{V}(K-j) := b_j^* \quad (j = 1, \ldots, K-1),$$

and

$$\widehat{V}(0) := u^* + \Pi^*(w^* - u^*,\, w^* - u^*).$$

□

B.7.2 Proof of the Uniqueness Part of Proposition B1 Based on Assumption 1

Proof. We prove uniqueness under Assumption 1. The single-battle ($K = 1$) case follows from Malueg and Yates (2005). Suppose $K \geq 2$. Let $(u, w) := (\widehat{V}(-1), \widehat{V}(1))$ be induced by a symmetric MPE. As in the existence proof, define

$$(a_0, b_0) := (0, v), \qquad (a_j, b_j) := \boldsymbol{\zeta}_{(w,u)}(a_{j-1}, b_{j-1}), \quad j = 1, \ldots, K-1.$$

Then the fixed-point condition is

$$(a_{K-1}, b_{K-1}) = (u, w). \tag{B27}$$

Under Assumption 1, (B22) becomes

$$a_j = a_{j-1} + (w - a_{j-1})\phi\left(\frac{w - a_{j-1}}{b_{j-1} - u}\right), \tag{B28}$$

$$b_j = u + (b_{j-1} - u)\phi\left(\frac{b_{j-1} - u}{w - a_{j-1}}\right). \tag{B29}$$

Now define

$$r_j := \frac{b_j - u}{w - a_j}, \qquad j = 0, \ldots, K-1.$$

Using (B28)–(B29), we obtain

$$w - a_j = (w - a_{j-1})\Big[1 - \phi(1/r_{j-1})\Big]$$

and

$$b_j - u = (b_{j-1} - u)\phi(r_{j-1}),$$

so

$$r_j = \frac{r_{j-1}\phi(r_{j-1})}{1 - \phi(1/r_{j-1})} =: \psi(r_{j-1}).$$

Hence

$$r_j = \psi^j(r_0), \qquad r_0 = \frac{v-u}{w}.$$

By Lemma B5, ψ is strictly increasing, so ψ^{K-1} is strictly increasing as well. From (B27), we have

$$r_{K-1} = \frac{b_{K-1} - u}{w - a_{K-1}} = \frac{w-u}{w-u} = 1.$$

Therefore r_0 is uniquely determined as the unique solution to

$$\psi^{K-1}(r) = 1.$$

Denote this unique solution by ρ_K. Then

$$\frac{v-u}{w} = \rho_K, \qquad \text{so} \qquad u = v - \rho_K w. \tag{B30}$$

Next, iterate the recursion for $w - a_j$:

$$w - a_j = w \prod_{t=0}^{j-1} \Big[1 - \phi(1/r_t)\Big].$$

Since $r_t = \psi^t(\rho_K)$, the product

$$P_K := \prod_{t=0}^{K-2} \Big[1 - \phi\big(1/\psi^t(\rho_K)\big)\Big]$$

depends only on K and the success function. Using again (B27), namely $a_{K-1} = u$, we get

$$w - u = w - a_{K-1} = wP_K.$$

Substituting (B30) gives

$$(1 + \rho_K - P_K)w = v.$$

Since $0 < \phi(\theta) < 1$ for every $\theta > 0$, we have $0 < P_K < 1$, hence $1 + \rho_K - P_K > 0$. Therefore

$$w = \frac{v}{1 + \rho_K - P_K}, \qquad u = v - \rho_K w$$

are uniquely determined.

So $(u, w) = (\widehat{V}(-1), \widehat{V}(1))$ is unique. Once (u, w) is fixed, the whole sequence (a_j, b_j) is uniquely generated by

$$(a_j, b_j) = \boldsymbol{\zeta}^j_{(w,u)}(0, v),$$

and therefore the value function $\widehat{V}(\cdot)$ is unique as well. $\square$

B.8 Proof of Proposition 3 Based on Lemmas 2 and 3

Proof. Consider a symmetric MPE in the K-consecutive-win contest, with valuation profile

$$v = \widehat{V}(K) > \widehat{V}(K-1) > \cdots > \widehat{V}(1) > \widehat{V}(-1) > \cdots > \widehat{V}(-K) = 0.$$

Let $\widehat{p}(i)$ denote the probability that the player at state i wins the next battle, for $i = K-1, \ldots, 1-K$, with $\widehat{p}(i) + \widehat{p}(-i) = 1$. For convenience, define the equilibrium effort at state $i \geq 0$ by

$$\widehat{x}(i) := x^*\big(\widehat{V}(i+1) - \widehat{V}(-1),\ \widehat{V}(1) - \widehat{V}(-i-1)\big).$$

We now focus on the $i \geq 0$ case and turn to the $i \leq -1$ case at the end of the proof. By definition of the value function, for $i \geq 0$,

$$\widehat{V}(i) = \widehat{p}(i)\widehat{V}(i+1) + [1 - \widehat{p}(i)]\widehat{V}(-1) - \widehat{x}(i).$$

Rearranging yields that

$$\frac{\widehat{V}(i+1) - \widehat{V}(-1)}{\widehat{V}(i) - \widehat{V}(-1)} = \frac{\widehat{V}(i) - \widehat{V}(-1) + \widehat{x}(i)}{\widehat{p}(i)[\widehat{V}(i) - \widehat{V}(-1)]}. \tag{B31}$$

Iterating (B31) from $i = 1$ to $K - 1$, and using the fact that $\widehat{V}(K) = v$, gives

$$\frac{v - \widehat{V}(-1)}{\widehat{V}(1) - \widehat{V}(-1)} = \prod_{i=1}^{K-1} \frac{\widehat{V}(i) - \widehat{V}(-1) + \widehat{x}(i)}{\widehat{p}(i)\,[\widehat{V}(i) - \widehat{V}(-1)]}.$$

As will be shown in the proof of Theorem 4, both $\widehat{V}(1)/v$ and $\widehat{V}(-1)/v$ approach 0 as $K \to +\infty$. Therefore, the left-hand side diverges. This implies that the right-hand side also diverges. Notably,

$$1 < \frac{\widehat{V}(i) - \widehat{V}(-1) + \widehat{x}(i)}{\widehat{p}(i)\,[\widehat{V}(i) - \widehat{V}(-1)]} \leq \frac{\widehat{V}(i) - \widehat{V}(-1) + \widehat{p}(i)\left[\widehat{V}(i+1) - \widehat{V}(-1)\right]}{\widehat{p}(i)\,[\widehat{V}(i) - \widehat{V}(-1)]}$$
$$= \frac{1 + \frac{\widehat{p}(i)}{\widehat{\pi}(i)}}{\widehat{p}(i)} \leq \frac{1 + \frac{1}{\pi_1^*}}{\pi_1^*},$$

where the last inequality follows from $1 \geq \widehat{p}(i) \geq \widehat{\pi}(i) \geq \pi_1^*$ for $i \geq 0$, by Lemma 3(ii). Notice that for a series $\frac{1+\frac{1}{\pi_1^*}}{\pi_1^*} > a_{Ki} > 1$, with $1 \leq i \leq K-1$, $\prod_{i=1}^{K-1} a_{Ki} \to +\infty$ as $K \to +\infty$ is equivalent to $\sum_{i=1}^{K-1}\left(1 - \frac{1}{a_{Ki}}\right) \to +\infty$ as $K \to +\infty$. It follows that

$$\sum_{i=1}^{K-1}\left(1 - \frac{\widehat{p}(i)[\widehat{V}(i) - \widehat{V}(-1)]}{\widehat{V}(i) - \widehat{V}(-1) + \widehat{x}(i)}\right) = \sum_{i=1}^{K-1}\left(\frac{\widehat{p}(i)\widehat{x}(i)}{\widehat{V}(i) - \widehat{V}(-1) + \widehat{x}(i)} + \widehat{p}(-i)\right) \to +\infty. \quad \text{(B32)}$$

By (B31), Lemma 2(ii), and the fact that $\widehat{x}(-i) \leq \widehat{p}(-i)\widehat{\Delta}(-i)$, we have that

$$\frac{\widehat{p}(i)\widehat{x}(i)}{\widehat{V}(i) - \widehat{V}(-1) + \widehat{x}(i)} = \frac{\widehat{x}(i)}{\widehat{V}(i+1) - \widehat{V}(-1)} = \frac{\widehat{x}(i)}{\widehat{\Delta}(i)} \leq C\frac{\widehat{x}(-i)}{\widehat{\Delta}(-i)} \leq C\widehat{p}(-i).$$

Substituting this bound into (B32) implies

$$(C+1)\sum_{i=1}^{K-1}\widehat{p}(-i) \to +\infty, \qquad \text{hence} \qquad \sum_{i=1}^{K-1}[1 - \widehat{p}(i)] \to +\infty,$$

which, together with the fact that $\widehat{p}(i) \leq e^{-[1-\widehat{p}(i)]}$, yields that

$$\prod_{i=1}^{K-1}\widehat{p}(i) \to 0.$$

Finally, we turn to the probability of ultimately winning the contest from state i, $\widehat{Q}(i;K,v)$. It satisfies

$$\widehat{Q}(i;K,v) = \widehat{p}(i)\widehat{Q}(i+1;K,v) + \widehat{p}(-i)\widehat{Q}(-1;K,v).$$

Rearranging,

$$\widehat{Q}(i+1;K,v) - \widehat{Q}(-1;K,v) = \frac{\widehat{Q}(i;K,v) - \widehat{Q}(-1;K,v)}{\widehat{p}(i)}.$$

Iterating this relation yields

$$\widehat{Q}(K;K,v) - \widehat{Q}(-1;K,v) = \frac{\widehat{Q}(i;K,v) - \widehat{Q}(-1;K,v)}{\prod_{j=i}^{K-1} \widehat{p}(j)}.$$

Since $\widehat{Q}(K;K,v) = 1$ and $\widehat{Q}(-1;K,v) \geq 0$, the left-hand side is bounded, while for fixed i the denominator converges to zero as $K \to +\infty$. Therefore,

$$\widehat{Q}(i;K,v) - \widehat{Q}(-1;K,v) \to 0.$$

Since $\widehat{Q}(i;K,v) \geq \frac{1}{2} \geq \widehat{Q}(-1;K,v)$, $\lim_{K>i,K\to+\infty} \widehat{Q}(i;K,v) = \frac{1}{2}$ for any given $i \geq 0$. Then $\lim_{K>i,K\to+\infty} \widehat{Q}(-i;K,v) = \frac{1}{2}$ follows from the fact that $\widehat{Q}(-i;K,v) = 1 - \widehat{Q}(i;K,v)$. □

B.9 Proof of Theorem 4 Based on Lemmas 2 and 3

Proof. For a player in state i, define his value of winning the upcoming battle as

$$\widehat{\Delta}(i;K,v) := \begin{cases} \widehat{V}(i+1;K,v) - \widehat{V}(-1;K,v), & \text{if } K-1 \geq i \geq 0, \\ \widehat{V}(1;K,v) - \widehat{V}(i-1;K,v), & \text{if } -(K-1) \leq i < 0. \end{cases}$$

When no confusion arises, write $\widehat{\Delta}(i)$ for $\widehat{\Delta}(i;K,v)$ and let $\widehat{\pi}(i) := \pi^*(\widehat{\Delta}(i), \widehat{\Delta}(-i))$ denote the player's equilibrium gain ratio in the state i battle, as defined in (1). It is useful to establish the following intermediate result first.

Lemma B4 (Self-reinforcement of Wins in Consecutive-win Contests)**.** *In any symmetric MPE of a K-consecutive-win contest, the following holds for $0 \leq i \leq K-2$:*

$$\frac{\widehat{\Delta}(i+1)}{\widehat{\Delta}(-i-1)} = \underbrace{\frac{1-\widehat{\pi}(-i-1)}{\widehat{\pi}(i+1)}}_{>1} \frac{\widehat{\Delta}(i)}{\widehat{\Delta}(-i)}. \tag{B33}$$

Proof. The single-battle characterization implies the following equilibrium valuation equations:

$$\widehat{V}(j) = \begin{cases} \widehat{V}(-1) + \widehat{\pi}(j)\widehat{\Delta}(j), & \text{if } K-1 \geq j \geq 0, \\ \widehat{V}(j-1) + \widehat{\pi}(j)\widehat{\Delta}(j), & \text{if } -(K-1) \leq j < 0. \end{cases}$$

For $0 \le i \le K-2$, this gives

$$\begin{aligned}\frac{\widehat{\Delta}(i)}{\widehat{\Delta}(-i)} &= \frac{\widehat{V}(i+1)-\widehat{V}(-1)}{\widehat{V}(1)-\widehat{V}(-i-1)} \\ &= \frac{\widehat{\pi}(i+1)\widehat{\Delta}(i+1)}{\left[1-\widehat{\pi}(-i-1)\right]\widehat{\Delta}(-i-1)}.\end{aligned}$$

Rearranging yields the equality. The factor is greater than one because, in the corresponding single battle, the two players' gain ratios sum to less than one. $\square$

We show next that for all $R'' > 1$ there exists $K_{R''} \in \mathbb{N}_{++}$ such that $\frac{\widehat{\Delta}(i;K,v)}{\widehat{\Delta}(-i;K,v)} > R''$ for all $i \ge K_{R''}$, $K > i$ and $v > 0$. This is because, by Lemma 3(iii), for all i such that $\frac{\widehat{\Delta}(i;K,v)}{\widehat{\Delta}(-i;K,v)} \le R''$, $\frac{1-\widehat{\pi}(-i-1)}{\widehat{\pi}(i+1)} \ge 1 + d_{R''}$. Together with Lemma B4, this implies that $K_{R''} := \log_{1+d_{R''}} R''$ satisfies our requirement.

Therefore, $\frac{\widehat{\Delta}(K-1;K,v)}{\widehat{\Delta}(1-K;K,v)} = \frac{\widehat{V}(K;K,v)-\widehat{V}(-1;K,v)}{\widehat{V}(1;K,v)-\widehat{V}(-K;K,v)} = \frac{v-\widehat{V}(-1;K,v)}{\widehat{V}(1;K,v)} \to +\infty$ as $K \to +\infty$, which implies that $\frac{v}{\widehat{V}(1;K,v)} \to +\infty$. As a result, for all $\epsilon > 0$, there exists $K^\dagger$ such that when $K > K^\dagger$, $\widehat{V}(-1) < \widehat{V}(1) < \frac{\epsilon}{2}v$, which indicates that $\widehat{V}(0) < \epsilon v$. $\square$

B.10 Proof of Proposition 4 Based on Lemmas 2 and 3

Proof. Fix any $\epsilon > 0$. By Theorem 4, there exists $K^\dagger$ such that for all $K > K^\dagger$, the equilibrium value at the initial state satisfies $\widehat{V}(0;K,v) < \epsilon v$. By monotonicity of the value function (Proposition B1), any history $h \in H_\ell^-$ has $V_\ell^-(h) = \widehat{V}(-K^\dagger;K,v) < \widehat{V}(0;K,v) < \epsilon v$. This proves statement (i).

For statement (ii), by Proposition 3, $\widehat{Q}(K^\dagger;K,v) \to 1/2$ as $K \to \infty$. Choose $K' > K^\dagger$ such that for all $K > K'$, $\widehat{Q}(K^\dagger;K,v) < 1/2 + \epsilon/8$. Let E_A (resp. E_B) denote the event that H_A^- (resp. H_B^-) is visited. Since $K > K^\dagger$, $\Pr(E_A \cup E_B) = 1$, and hence

$$\Pr(E_A \cap E_B) = 1 - \Pr(E_A^c \cap E_B) - \Pr(E_B^c \cap E_A).$$

By symmetry, it suffices to show $\Pr(E_A^c \cap E_B) < \epsilon/2$.

Condition on H_B^- being visited first. From player A's perspective, the state is $K^\dagger$. Let s_t denote the state after t additional battles, and define the stopping time

$$\tau := \inf\{t \ge 0 : s_t = -K^\dagger \text{ or } s_t = K\}.$$

Let $r := \Pr(s_\tau = K)$, the probability of hitting K before $-K^\dagger$ when starting from $K^\dagger$. Note that $E_A^c \cap E_B$ can occur only if H_B^- is visited before H_A^-, so

$$\Pr(E_A^c \cap E_B) = \Pr\big(H_B^- \text{ is visited before } H_A^-\big)\, r \le r.$$

We proceed to bound r.

The Optional Sampling Theorem for the bounded martingale $\widehat{Q}(s_t; K, v)$ gives

$$\widehat{Q}(K^\dagger; K, v) = (1 - r)\widehat{Q}(-K^\dagger; K, v) + r.$$

Since $\widehat{Q}(-K^\dagger; K, v) = 1 - \widehat{Q}(K^\dagger; K, v)$,

$$r = \frac{2\widehat{Q}(K^\dagger; K, v) - 1}{\widehat{Q}(K^\dagger; K, v)} < \frac{\epsilon}{2}.$$

Hence, $\Pr(E_A^c \cap E_B) < \epsilon/2$, which proves statement (ii). □

B.11 Proof of Theorem 5 Based on Lemmas 2 and 3

Proof. We first bound the initial value of player A, denoted by V_A^*. For a terminal history $h \in H^\dagger$, let $\widetilde{H}(h) := \{h' : h' \text{ is a prefix of } h\}$ denote the set of all subhistories of h. If $\widetilde{H}(h) \cap H_A^- \neq \varnothing$, let h_A^- denote the *shortest* prefix (i.e., subhistory) of h that belongs to H_A^-. It holds that

$$V_A^* \le \sum_{\substack{h \in H^\dagger \text{ and} \\ \widetilde{H}(h) \cap H_A^- = \varnothing}} \Pr(h) v + \sum_{\substack{h \in H^\dagger \text{ and} \\ \widetilde{H}(h) \cap H_A^- \neq \varnothing}} \Pr(h) V_A(h_A^-),$$

where $V_A(h_A^-)$ denotes the continuation value of player A when a terminal history h first reaches H_A^-.

By Definition 4, the probability of not reaching H_A^- is at most ϵ, and every history in H_A^- gives player A continuation value at most ϵv. Hence

$$\sum_{\substack{h \in H^\dagger \text{ and} \\ \widetilde{H}(h) \cap H_A^- = \varnothing}} \Pr(h) \le \epsilon \text{ and } V_A(h_A^-) \le \epsilon v.$$

Therefore, $V_A^* \le 2\epsilon v$. The same argument applied to player B yields $V_B^* \le 2\epsilon v$, and the expected total effort is $v - V_A^* - V_B^* \ge (1 - 4\epsilon)\, v$. □

B.12 Proof of Theorem 6 under Assumption 1

Proof. Since the success function is homogeneous, we normalize the prize to $v = 1$ throughout this proof. We first prove the existence and uniqueness of the symmetric MPE.

Step 1: States and value functions. Let the state $i \in \{-N, -N+1, \ldots, N\}$ denote a player's lead in the number of net battle wins since the last reset. States $\pm N$ are absorbing: if $i = N$, the player has won the contest and receives 1, whereas if $i = -N$, the player receives 0.

To incorporate the reset lottery, it is convenient to distinguish two types of continuation values:

- $V(i)$ is the player's continuation value at a *decision node* in state i, i.e., immediately before the next battle when the reset lottery (if any) has already been realized.
- $\widetilde{V}(i)$ is the player's continuation value at an *intermediate node* in state i, i.e., immediately after the most recent battle outcome has updated the lead to i, but *before* the reset lottery is realized.

By definition of the reset rule, for every nonterminal $i \in \{-N+1, \ldots, N-1\}$,

$$\widetilde{V}(i) = p\,V(0) + (1-p)\,V(i). \tag{B34}$$

At terminal states, $\widetilde{V}(N) = V(N) = 1$ and $\widetilde{V}(-N) = V(-N) = 0$.

Step 2: Bellman equations using the single-battle characterization. By Lemma A1, the current battle has a unique equilibrium, and player A's equilibrium payoff from this battle equals the losing continuation value plus the equilibrium gain:

$$V(i) = \widetilde{V}(i-1) + \big(\widetilde{V}(i+1) - \widetilde{V}(i-1)\big)\,\phi\!\left(\frac{\widetilde{V}(i+1) - \widetilde{V}(i-1)}{\widetilde{V}(1-i) - \widetilde{V}(-i-1)}\right), \quad \text{for } i = 1-N, \ldots, N-1. \tag{B35}$$

Combining (B34) and (B35) yields a closed system for $\{\widetilde{V}(i)\}_{i=-N}^{N}$. We proceed to show that this system has a unique solution with a constructive approach.

Step 3: Normalization and recursive construction. We define the normalized values:

$$\widetilde{\Delta}(i) := \frac{\widetilde{V}(i) - \widetilde{V}(0)}{\widetilde{V}(1) - \widetilde{V}(-1)}, \qquad i = -N, \ldots, N. \tag{B36}$$

Then $\widetilde{\Delta}(0) = 0$ and $\widetilde{\Delta}(1) - \widetilde{\Delta}(-1) = 1$.

At state $i = 0$, we have $\widetilde{V}(0) = V(0)$ and (B35) becomes

$$\widetilde{V}(0) = \widetilde{V}(-1) + (\widetilde{V}(1) - \widetilde{V}(-1))\phi(1).$$

Rearranging and using the normalization (B36) gives the *initial conditions*

$$\widetilde{\Delta}(1) = 1 - \phi(1), \qquad \widetilde{\Delta}(-1) = -\phi(1). \tag{B37}$$

For each $i \in \{1, \ldots, N-1\}$ define the ratio

$$\theta_i := \frac{\widetilde{\Delta}(i+1) - \widetilde{\Delta}(i-1)}{\widetilde{\Delta}(1-i) - \widetilde{\Delta}(-i-1)} \;=\; \frac{\widetilde{V}(i+1) - \widetilde{V}(i-1)}{\widetilde{V}(1-i) - \widetilde{V}(-i-1)}. \tag{B38}$$

Subtract $\widetilde{V}(0)$ from both sides of (B34)–(B35), divide by $\widetilde{V}(1) - \widetilde{V}(-1)$, and use (B38). This yields, for $i = 1, \ldots, N-1$,

$$\begin{aligned}
\widetilde{\Delta}(i) &= (1-p)\Big[\widetilde{\Delta}(i-1) + \big(\widetilde{\Delta}(i+1) - \widetilde{\Delta}(i-1)\big)\phi(\theta_i)\Big],\\
\widetilde{\Delta}(-i) &= (1-p)\Big[\widetilde{\Delta}(-i-1) + \big(\widetilde{\Delta}(1-i) - \widetilde{\Delta}(-i-1)\big)\phi(1/\theta_i)\Big]\\
&= (1-p)\Big[\widetilde{\Delta}(1-i) - \big(\widetilde{\Delta}(1-i) - \widetilde{\Delta}(-i-1)\big)\big(1 - \phi(1/\theta_i)\big)\Big].
\end{aligned} \tag{B39}$$

Using (B39) and (B38), a direct rearrangement gives, for each $i = 1, \ldots, N-1$,

$$-\frac{\widetilde{\Delta}(i) - (1-p)\widetilde{\Delta}(i-1)}{\widetilde{\Delta}(-i) - (1-p)\widetilde{\Delta}(-i-1)} = \psi(\theta_i), \tag{B40}$$

where

$$\psi(\theta) := \frac{\theta\,\phi(\theta)}{1 - \phi(1/\theta)}, \qquad \theta > 0.$$

We establish the following result to facilitate the inductive computation of $\widetilde{\Delta}(i)$.

Lemma B5. *The function $\psi(\theta) := \frac{\theta\phi(\theta)}{1-\phi(1/\theta)}$ is strictly increasing in $\theta > 0$.*

Proof. Note that

$$\psi(\theta) = \frac{\theta\phi(\theta)}{1 - \phi(\frac{1}{\theta})} = \frac{\phi(\theta)}{\frac{\gamma(\theta)}{\theta} + \gamma'(\theta)},$$

where the second equality is because $1 - \gamma\big(\frac{1}{\theta}\big) = \gamma(\theta)$ and $\theta\gamma'(\theta) = \frac{1}{\theta}\gamma'\big(\frac{1}{\theta}\big)$. Since $\phi(\theta)$ is strictly increasing, and $\frac{\gamma(\theta)}{\theta} + \gamma'(\theta)$ is decreasing due to the concavity of $\gamma(\theta)$, it is clear that $\psi(\theta)$ is strictly increasing. $\square$

Induction/construction. Start from (B37) and $\widetilde{\Delta}(0) = 0$. Suppose that for some $k \in \{1, \ldots, N-1\}$ the values $\widetilde{\Delta}(j)$ have been constructed for all $j \in \{-k, \ldots, k\}$ and satisfy

$$\widetilde{\Delta}(k) > \widetilde{\Delta}(k-1) > \cdots > \widetilde{\Delta}(0) = 0 > \cdots > \widetilde{\Delta}(1-k) > \widetilde{\Delta}(-k).$$

Then

$$-\frac{\widetilde{\Delta}(k) - (1-p)\widetilde{\Delta}(k-1)}{\widetilde{\Delta}(-k) - (1-p)\widetilde{\Delta}(1-k)} = \frac{\widetilde{\Delta}(k) - (1-p)\widetilde{\Delta}(k-1)}{\widetilde{\Delta}(1-k) - \widetilde{\Delta}(-k) - p\widetilde{\Delta}(1-k)},$$

the numerator of the right-hand side is strictly positive (because $\widetilde{\Delta}(k) > (1-p)\widetilde{\Delta}(k-1)$), and the denominator is strictly positive (because $\widetilde{\Delta}(1-k) > \widetilde{\Delta}(-k)$ and $\widetilde{\Delta}(1-k) \leq 0$). Hence θ_k is uniquely determined as

$$\theta_k = \psi^{-1}\left(-\frac{\widetilde{\Delta}(k) - (1-p)\widetilde{\Delta}(k-1)}{\widetilde{\Delta}(-k) - (1-p)\widetilde{\Delta}(1-k)}\right). \tag{B41}$$

Given θ_k, solve (B39) for $\widetilde{\Delta}(k+1)$ and $\widetilde{\Delta}(-k-1)$:

$$\widetilde{\Delta}(k+1) = \widetilde{\Delta}(k-1) + \underbrace{\frac{\widetilde{\Delta}(k)/(1-p) - \widetilde{\Delta}(k-1)}{\phi(\theta_k)}}_{\substack{>\widetilde{\Delta}(k)/(1-p)-\widetilde{\Delta}(k-1) \\ \text{because } 0<\phi(\theta)<1 \text{ for all } \theta>0}} > \frac{\widetilde{\Delta}(k)}{1-p}, \tag{B42}$$

$$\begin{aligned}\widetilde{\Delta}(-k-1) &= \frac{\widetilde{\Delta}(-k)}{(1-p)(1-\phi(1/\theta_k))} + \frac{\phi(1/\theta_k)[-\widetilde{\Delta}(1-k)]}{1-\phi(1/\theta_k)} \\ &< \frac{\widetilde{\Delta}(-k)}{(1-p)(1-\phi(1/\theta_k))} < \frac{\widetilde{\Delta}(-k)}{1-p}.\end{aligned} \tag{B43}$$

Because $0 < \phi(\theta) < 1$ for all $\theta > 0$ under Assumption 1, the the strict monotonicity of $\widetilde{\Delta}(\cdot)$ extends to $\pm(k+1)$. This completes the inductive construction and shows that, for each fixed (N, p), there is a unique normalized sequence $\{\widetilde{\Delta}(i)\}_{i=-N}^{N}$ satisfying (B37)–(B39).

Step 4: Recovering $\widetilde{V}$ and concluding uniqueness of the symmetric MPE. Given $\{\widetilde{\Delta}(i)\}_{i=-N}^{N}$, the boundary conditions $\widetilde{V}(-N) = 0$ and $\widetilde{V}(N) = 1$ pin down the affine scaling in (B36) uniquely. Indeed, since $\widetilde{\Delta}(N) > \widetilde{\Delta}(-N)$, set

$$\widetilde{V}(i) = \frac{\widetilde{\Delta}(i) - \widetilde{\Delta}(-N)}{\widetilde{\Delta}(N) - \widetilde{\Delta}(-N)}.$$

Then $\widetilde{V}(-N) = 0$, $\widetilde{V}(N) = 1$, and the series $\widetilde{V}(i)$ solves the equilibrium equations (B34) and (B35) by construction.

Uniqueness follows because any symmetric MPE induces (via (B36)) a normalized sequence $\{\widetilde{\Delta}(i)\}_{i=-N}^{N}$ satisfying the same system (B37)–(B39). The inductive construction above shows this normalized sequence is unique, and therefore the associated $(\widetilde{V}, V)$ and equilibrium efforts are unique as well.

Next, we establish the transient dominance property for large N when $p \in (0,1)$. Fix $p \in (0,1)$ and $\epsilon > 0$. For this part, it is convenient to extend the normalized sequence $\{\widetilde{\Delta}(i)\}_{i=-N}^{N}$ constructed above to an *infinite* sequence $\{\widetilde{\Delta}(i)\}_{i=-\infty}^{+\infty}$ by iterating the recursion (B41)–(B43) for $k = 1, 2, \dots$ (starting from the initial conditions (B37)). For any finite N, the normalized sequence in the margin-N game is just the truncation of this infinite sequence on $\{-N, \dots, N\}$.

For $k \geq 1$ define the ratio

$$R_k := -\frac{\widetilde{\Delta}(k)}{\widetilde{\Delta}(-k)} \ > \ 0.$$

We first show that R_k is strictly increasing and diverges to $+\infty$; we then use this to verify the two conditions in Definition 4.

Step 5: Monotonicity of R_k. Since $1 > \phi(\theta) + \phi(1/\theta)$ for $\theta > 0$, we have $\psi(\theta) < \theta$ and $\psi(1/\theta) < 1/\theta$, hence $1/\psi(1/\theta) > \theta$. Together with (B42)–(B43), we know that for every $k \geq 1$,

$$\frac{\widetilde{\Delta}(k+1) - \frac{\widetilde{\Delta}(k)}{1-p}}{\frac{\widetilde{\Delta}(-k)}{1-p} - \widetilde{\Delta}(-k-1)} = \frac{1}{\psi(1/\theta_k)} > \theta_k > \psi(\theta_k) = \frac{\widetilde{\Delta}(k) - (1-p)\widetilde{\Delta}(k-1)}{-\widetilde{\Delta}(-k) + (1-p)\widetilde{\Delta}(1-k)}. \tag{B44}$$

We prove that $R_{k+1} > R_k$ by induction on k. For $k = 1$, since $\widetilde{\Delta}(0) = 0$ the rightmost fraction in (B44) equals $-\widetilde{\Delta}(1)/\widetilde{\Delta}(-1) = R_1$, so (B44) implies

$$\frac{\widetilde{\Delta}(2) - \frac{\widetilde{\Delta}(1)}{1-p}}{\frac{\widetilde{\Delta}(-1)}{1-p} - \widetilde{\Delta}(-2)} > R_1.$$

Writing $\widetilde{\Delta}(2) = \frac{\widetilde{\Delta}(1)}{1-p} + \left(\widetilde{\Delta}(2) - \frac{\widetilde{\Delta}(1)}{1-p}\right)$ and $-\widetilde{\Delta}(-2) = \frac{-\widetilde{\Delta}(-1)}{1-p} + \left(\frac{\widetilde{\Delta}(-1)}{1-p} - \widetilde{\Delta}(-2)\right)$, this inequality implies $R_2 = -\widetilde{\Delta}(2)/\widetilde{\Delta}(-2) > R_1$.

Now fix $k \geq 2$ and suppose $R_k > R_{k-1}$. Let $c := \widetilde{\Delta}(k)$, $d := -\widetilde{\Delta}(-k)$, $c' := (1-p)\widetilde{\Delta}(k-1)$, and $d' := (1-p)(-\widetilde{\Delta}(1-k))$. Then $c/d = R_k$ and $c'/d' = R_{k-1}$. The induction hypothesis $c/d > c'/d'$ implies

$$\frac{c - c'}{d - d'} > \frac{c}{d},$$

i.e., the rightmost fraction in (B44) satisfies $\psi(\theta_k) > R_k$. Combining with (B44) yields

$$\frac{\widetilde{\Delta}(k+1) - \frac{\widetilde{\Delta}(k)}{1-p}}{\frac{\widetilde{\Delta}(-k)}{1-p} - \widetilde{\Delta}(-k-1)} > R_k.$$

Using the same "ratio-of-sums" argument as in the base case, we conclude $R_{k+1} = -\widetilde{\Delta}(k+1)/\widetilde{\Delta}(-k-1) > R_k$. Hence $\{R_k\}$ is strictly increasing.

Step 6: Divergence of R_k. Since R_k is increasing, either $R_k \to +\infty$ or $R_k \to M < +\infty$. Suppose toward a contradiction that $R_k \to M < +\infty$. Then $R_k \leq M + 1$ for all large k.

First, θ_k is uniformly bounded from above. By (B41) and the monotonicity of ψ^{-1},

$$\theta_k = \psi^{-1}\left(-\frac{\widetilde{\Delta}(k) - (1-p)\widetilde{\Delta}(k-1)}{\widetilde{\Delta}(-k) - (1-p)\widetilde{\Delta}(1-k)}\right) \leq \psi^{-1}\left(-\frac{\widetilde{\Delta}(k)}{p\,\widetilde{\Delta}(-k)}\right) \leq \psi^{-1}\left(\frac{M+1}{p}\right), \quad \text{(B45)}$$

where the first inequality is because

$$-\frac{\widetilde{\Delta}(k) - (1-p)\widetilde{\Delta}(k-1)}{\widetilde{\Delta}(-k) - (1-p)\widetilde{\Delta}(1-k)} = \frac{\widetilde{\Delta}(k) - (1-p)\widetilde{\Delta}(k-1)}{(1-p)(\widetilde{\Delta}(1-k) - \widetilde{\Delta}(-k)) - p\,\widetilde{\Delta}(-k)} \leq -\frac{\widetilde{\Delta}(k)}{p\,\widetilde{\Delta}(-k)}. \quad \text{(B46)}$$

On the other hand, Step 5 gives $\theta_k > \psi(\theta_k) > R_k$, so $\liminf_{k\to\infty} \theta_k \geq M$. Let $\theta_- := \liminf_{k\to\infty} \theta_k \in [M, \psi^{-1}((M+1)/p)]$.

Taking lim inf in (B44) yields

$$\liminf_{k\to\infty} \frac{\widetilde{\Delta}(k+1) - \frac{\widetilde{\Delta}(k)}{1-p}}{\frac{\widetilde{\Delta}(-k)}{1-p} - \widetilde{\Delta}(-k-1)} = \liminf_{k\to\infty} \frac{1}{\psi(1/\theta_k)} = \frac{1}{\psi(1/\theta_-)}. \quad \text{(B47)}$$

Since $\psi(1/\theta) < 1/\theta$ for $\theta > 0$, we have $1/\psi(1/\theta_-) > \theta_- \geq M$, and hence

$$\liminf_{k\to\infty} \frac{\widetilde{\Delta}(k+1) - \frac{\widetilde{\Delta}(k)}{1-p}}{\frac{\widetilde{\Delta}(-k)}{1-p} - \widetilde{\Delta}(-k-1)} > M.$$

Next, note that R_{k+1} can be written as a convex combination:

$$\frac{\widetilde{\Delta}(k+1)}{-\widetilde{\Delta}(-k-1)} = \left(\frac{\widetilde{\Delta}(k+1) - \frac{\widetilde{\Delta}(k)}{1-p}}{\frac{\widetilde{\Delta}(-k)}{1-p} - \widetilde{\Delta}(-k-1)}\right) r_k + \left(-\frac{\widetilde{\Delta}(k)}{\widetilde{\Delta}(-k)}\right)(1 - r_k), \quad \text{(B48)}$$

where $r_k := \frac{\frac{\widetilde{\Delta}(-k)}{1-p} - \widetilde{\Delta}(-k-1)}{-\widetilde{\Delta}(-k-1)} \in (0,1)$. Moreover, using (B39) we have

$$\frac{\widetilde{\Delta}(-k)}{1-p} - \widetilde{\Delta}(-k-1) = \big(\widetilde{\Delta}(1-k) - \widetilde{\Delta}(-k-1)\big)\,\phi(1/\theta_k),$$

and since $\widetilde{\Delta}(1-k) > \widetilde{\Delta}(-k) > (1-p)\widetilde{\Delta}(-k-1)$,

$$r_k = \frac{\big(\widetilde{\Delta}(1-k) - \widetilde{\Delta}(-k-1)\big)\,\phi(1/\theta_k)}{-\widetilde{\Delta}(-k-1)} \geq p\,\phi(1/\theta_k) \geq p\,\phi\!\left(\frac{1}{\psi^{-1}((M+1)/p)}\right) =: \widehat{r} > 0, \tag{B49}$$

where the last inequality uses (B45) and the monotonicity of ϕ.

Taking lim inf in (B48) and using (B47)–(B49) gives

$$\begin{aligned}\liminf_{k\to\infty} R_{k+1} &\geq \widehat{r}\left(\liminf_{k\to\infty} \frac{\widetilde{\Delta}(k+1) - \frac{\widetilde{\Delta}(k)}{1-p}}{\frac{\widetilde{\Delta}(-k)}{1-p} - \widetilde{\Delta}(-k-1)}\right) + (1-\widehat{r})\left(\liminf_{k\to\infty} R_k\right)\\ &> \widehat{r}M + (1-\widehat{r})M = M,\end{aligned}$$

contradicting $R_k \to M$. Hence $R_k \to +\infty$.

Step 7: Constructing H_A^- and H_B^- and verifying Definition 4. Step 6 implies $\lim_{k\to\infty} \frac{-\widetilde{\Delta}(-k)}{\widetilde{\Delta}(k)} = 0$. Choose k large enough so that $\frac{-\widetilde{\Delta}(-k)}{\widetilde{\Delta}(k)} \leq \epsilon$, and then choose $m \in \mathbb{N}_{++}$ so that $(1-\frac{p}{2})^{m-1} \leq \epsilon$. Set $N^\dagger := k+m$, and consider $N \geq N^\dagger$.

Let H_A^- (resp. H_B^-) collect the intermediate states (i.e., right after a battle outcome and before the reset lottery) where player B (resp. A) has a lead of at least k battle wins since the last reset. Since $\widetilde{V}(\cdot)$ is increasing, it suffices to bound $\widetilde{V}(-k)$. Using the scaling in Step 4 and the monotonicity of $-\frac{\widetilde{\Delta}(i)}{\widetilde{\Delta}(-i)}$, for any $i \geq k$,

$$\frac{1-\widetilde{V}(i)}{\widetilde{V}(-i)} = \frac{\widetilde{\Delta}(N) - \widetilde{\Delta}(i)}{\widetilde{\Delta}(-i) - \widetilde{\Delta}(-N)} \geq -\frac{\widetilde{\Delta}(i)}{\widetilde{\Delta}(-i)} \geq -\frac{\widetilde{\Delta}(k)}{\widetilde{\Delta}(-k)}.$$

Taking $i = k$ and using $1 - \widetilde{V}(k) \leq 1$ yields

$$\widetilde{V}(-k) \leq \frac{-\widetilde{\Delta}(-k)}{\widetilde{\Delta}(k)} \leq \epsilon.$$

This verifies Condition (i) in Definition 4 (recall $v = 1$ under our normalization).

For Condition (ii), since one of H_A^- or H_B^- must be reached before the game can end, it suffices to bound the probability that the realized history *fails* to reach, say, $H_A^- = \{h :$

$i(h) \leq -k\}$, starting from a situation in which player A already has a lead of k, which lies in H_B^-. We denote this conditional probability by $\widetilde{P}$. From $i = k$ to termination at $+N$ the contest must advance through at least $m-1$ further post-battle reset lotteries. At each such lottery, a reset occurs with probability p and sends the game back to state 0. Conditional on a reset, symmetry implies that the next time the process hits the set $\{\pm k\}$, player B is the one leading (i.e., the history reaches H_A^-) with probability 1/2. Hence, at each of these $m-1$ lotteries, the conditional probability that we *do not* end up reaching H_A^- via a reset is at most $1 - \frac{p}{2}$. This means that the probability that H_A^- is not reached after *any* of these lotteries is at most $\left(1 - \frac{p}{2}\right)^{m-1}$. Therefore,

$$1 - \widetilde{P} \leq H_A^- \text{ is not reached after } \textit{any} \text{ of these } m-1 \text{ lotteries } \leq \left(1 - \frac{p}{2}\right)^{m-1} \leq \epsilon.$$

This verifies Condition (ii) in Definition 4 and completes the proof. □

B.13 Proof of Lemma 1 Based on Lemmas 2 and 3

To accommodate the case where the success function may not be homogeneous, we reformulate this lemma as follows. For an unbalanced contest $\mathcal{M}$ with final prize $v' > 0$, let $V_+^{\mathcal{M}}(v')$ denote the advantaged player's expected equilibrium payoff and $V_-^{\mathcal{M}}(v')$ denote the disadvantaged player's expected equilibrium payoff.

The following definition formalizes the condition of sufficiently unbalanced subcontests, and the lemma that follows provides two such instances.

Definition 5. A sequence of subcontests $\{\mathcal{M}_K\}_{K=1}^{+\infty}$ is said to *become infinitely unbalanced* if $\frac{v' - V_+^{\mathcal{M}_K}(v')}{V_-^{\mathcal{M}_K}(v')}$ uniformly converges to $+\infty$ as $K \to +\infty$ for all $v' > 0$.

Lemma 1′. *When the success function admits properties in Lemma 2 and Lemma 3, both the sequence of $\mathcal{M}(K,1)$ contests and the sequence of tug-of-war with margin $K+1$ and initiated from state K become infinitely unbalanced.*

Proof. $\mathcal{M}(K,1)$ **contest.** Consider a $\mathcal{M}(K,1)$ contest with prize $v' > 0$. Let $V_+^K := V_+^{\mathcal{M}(K,1)}(v')$ and $V_-^K := V_-^{\mathcal{M}(K,1)}(v')$. The following holds for the first battle.

$$\begin{cases} V_+^K = V_+^{K-1} + \pi^*\left(v' - V_+^{K-1}, V_-^{K-1}\right)\left(v' - V_+^{K-1}\right), \\ V_-^K = \pi^*\left(V_-^{K-1}, v' - V_+^{K-1}\right) V_-^{K-1}. \end{cases}$$

Subtracting the first equation from v' and dividing the result by the second equation yields

$$\frac{v' - V_+^K}{V_-^K} = \underbrace{\frac{1 - \pi^*\left(v' - V_+^{K-1}, V_-^{K-1}\right)}{\pi^*\left(V_-^{K-1}, v' - V_+^{K-1}\right)}}_{>1} \frac{v' - V_+^{K-1}}{V_-^{K-1}}.$$

For any large $R'' > 1$, by Lemma 3(iii), if $\frac{v'-V_+^{K-1}}{V_-^{K-1}} < R''$, then $\frac{1-\pi^*\left(v'-V_+^{K-1}, V_-^{K-1}\right)}{\pi^*\left(V_-^{K-1}, v'-V_+^{K-1}\right)} > 1 + d_{R''}$, and the above equation implies that $\frac{v'-V_+^K}{V_-^K}$ grows faster than geometrically with a ratio $1 + d_{R''}$. Since $\frac{v'-V_+^1}{V_-^1} \geq 1$, $\frac{v'-V_+^K}{V_-^K} > R''$ for all $K > \log_{1+d_{R''}} R'' + 1$, implying uniform convergence across all $v' > 0$.

Tug-of-war contest with margin $K + 1$ and initiated from state K. By (B11) we have

$$\begin{aligned}\frac{v' - V_+^{\mathcal{M}_K}(v')}{V_-^{\mathcal{M}_K}(v')} &= \frac{V(K+1; K+1, v') - V(K; K+1, v')}{V(-K; K+1, v') - V(-K-1; K+1, v')} \\ &= \frac{1 - \pi(K; K+1, v')}{\pi(-K; K+1, v')} \frac{\Delta(K; K+1, v')}{\Delta(-K; K+1, v')} \geq \frac{\Delta(K; K+1, v')}{\Delta(-K; K+1, v')}.\end{aligned}$$

It is easy to see from the proof of Lemma B3 that $\frac{\Delta(K;K+1,v')}{\Delta(-K;K+1,v')}$ uniformly converges to $+\infty$ as $K \to +\infty$ across $v' > 0$. The uniform convergence of $\frac{v'-V_+^{\mathcal{M}_K}(v')}{V_-^{\mathcal{M}_K}(v')}$ follows immediately. $\square$

B.14 Proof of Theorem 7 Based on Lemmas 2 and 3

Proof. We first establish the following lemma, which will be useful for establishing the "transient" part of the transient dominance property.

Lemma B6. *If the success function admits properties in Lemma 2 and Lemma 3, there exists $\lambda(\cdot) > 0$ defined on $\mathbb{N}_{++}$ such that for any contest with any prize $v' > 0$, if player $\ell \in \{A, B\}$ wins the contest after winning the first $L \in \mathbb{N}_{++}$ battles consecutively, then player ℓ's equilibrium payoff is greater than $\lambda(L)v'$.*

Proof. Without loss of generality, we focus on player A and suppose that player A wins the contest after winning the first L battles. Consider the history $\underbrace{(A, \ldots, A)}_{L \text{ times}}$, and let V_A^n denote player A's continuation value at history $\underbrace{(A, \ldots, A)}_{n \text{ times}}$ for $n \in \{0, 1, \ldots, L\}$. Clearly, V_A^0 is player A's equilibrium payoff and $V_A^L = v'$.

We prove by induction that $V_A^n \geq \alpha_n v'$ for some $\alpha_n > 0$ that is independent of the contest form and prize value. This trivially holds for $n = L$. Suppose that $V_A^n \geq \alpha_n v'$ for some $1 \leq n \leq L$ with $\alpha_L = 1$. Consider the nth battle—i.e., the battle that follows the history $\underbrace{(A, \ldots, A)}_{n-1 \text{ times}}$. Fix some $d \in (0,1)$, and let Δ_ℓ^n denote player ℓ's incentive to win in this battle. If $\Delta_A^n > V_A^n d \geq \alpha_n d v'$, then $\Delta_A / \Delta_B \geq \alpha_n d$, and by Lemma 3(ii), $V_A^{n-1} \geq \pi^*_{\alpha_n d} V_A^n$. On the other hand, if $\Delta_A^n \leq V_A^n d$, we know that for player A, losing the battle leads to a continuation value that is at least $(1-d)V_A^n$, so $V_A^{n-1} \geq (1-d)V_A^n \geq (1-d)\alpha_n v'$. In any case,

$$V_A^{n-1} \geq \alpha_{n-1} := \min\{\pi^*_{\alpha_n d}, (1-d)\alpha_n\} v',$$

which completes the proof. $\square$

We now proceed to prove the theorem. For expositional ease, we first prove the $q = 1$ case (so the subcontest in each round always happens), and then show how the proof can be extended to the general $q \in (0, 1]$ case in a straightforward manner.

By Lemma 1′, there exists $K^\star$ such that $\frac{v' - V_+(\mathcal{M}_{K^\star}; v')}{V_-(\mathcal{M}_{K^\star}; v')} \geq \frac{1}{\epsilon}$ for all $v' > 0$. By Lemma B6, the winning probability of the disadvantaged player in $\mathcal{M}_{K^\star}$ with any final prize $v' > 0$ is at least $\lambda(L_{K^\star}) > 0$. Let $N^\star$ satisfy $[1 - \lambda(L_{K^\star})]^{N^\star} \leq \epsilon$.

We show next that the $N^\star$-round iterated incumbency contest with component subcontest $\mathcal{M}_{K^\star}$ satisfies the transient dominance property. Then by Theorem 5, expected total effort in the contest is at least $(1 - 4\epsilon)v$.

Let H_ℓ^- be the histories where the opponent $-\ell \in \{A, B\}$ just became the incumbent for a new round. Let $W_+^{\mathcal{M}}(n; N, v)$ denote the incumbent's continuation value at the beginning of round $n \geq 1$ and $W_-^{\mathcal{M}}(n; N, v)$ denote that of the laggard.

Then player ℓ's continuation value at a history in H_ℓ^- is $W_-^{\mathcal{M}_{K^\star}}(n; N^\star, v)$ for some $n \in \{1, \ldots, N\}$ and the opponent's continuation value is $W_+^{\mathcal{M}_{K^\star}}(n; N^\star, v)$. To establish the transient dominance property, we need to show that (i) $\frac{v}{W_-^{\mathcal{M}_{K^\star}}(n; N^\star, v)} \geq \frac{1}{\epsilon}$ for all $n \in \{1, \ldots, N\}$ and (ii) the probability that the initial incumbent loses its incumbency is at least $1 - \epsilon$. Point (ii) follows immediately from the construction of $N^\star$ because the probability that the initial incumbent never loses his incumbency is less than $[1 - \lambda(L_{K^\star})]^{N^\star} \leq \epsilon$.

We proceed to prove (i). Let $v_n := W_+^{\mathcal{M}_{K^\star}}(n+1; N^\star, v) - W_-^{\mathcal{M}_{K^\star}}(n+1; N^\star, v)$ for $n = 1, \ldots, N$, with the understanding that $W_+^{\mathcal{M}_{K^\star}}(N+1; N^\star, v) = v$ and $W_-^{\mathcal{M}_{K^\star}}(N+1; N^\star, v) = 0$. It follows from the structure of the iterated incumbency contest that

$$W_+^{\mathcal{M}_{K^\star}}(n; N^\star, v) = W_-^{\mathcal{M}_{K^\star}}(n+1; N^\star, v) + V_+^{\mathcal{M}_{K^\star}}(v_n),$$
$$W_-^{\mathcal{M}_{K^\star}}(n; N^\star, v) = W_-^{\mathcal{M}_{K^\star}}(n+1; N^\star, v) + V_-^{\mathcal{M}_{K^\star}}(v_n),$$

where the first equation can be rewritten as

$$v - W_+^{\mathcal{M}_{K^\star}}(n; N^\star, v) = v - W_+^{\mathcal{M}_{K^\star}}(n+1; N^\star, v) + [v_n - V_+^{\mathcal{M}_{K^\star}}(v_n)].$$

Consequently,

$$v - W_+^{\mathcal{M}_{K^\star}}(n; N^\star, v) = \sum_{j=n}^{N}[v_n - V_+^{\mathcal{M}_{K^\star}}(v_n)],$$

$$W_-^{\mathcal{M}_{K^\star}}(n; N^\star, v) = \sum_{j=n}^{N} V_-^{\mathcal{M}_{K^\star}}(v_n).$$

By construction, $\frac{v_n - V_+^{\mathcal{M}_{K^\star}}(v_n)}{V_-^{\mathcal{M}_{K^\star}}(v_n)} \geq \frac{1}{\epsilon}$. It follows that

$$\sum_{j=n}^{N}[v_n - V_+^{\mathcal{M}_{K^\star}}(v_n)] \geq \frac{1}{\epsilon}\sum_{j=n}^{N} V_-^{\mathcal{M}_{K^\star}}(v_n),$$

which implies that $\frac{v - W_+^{\mathcal{M}_{K^\star}}(n;N^\star,v)}{W_-^{\mathcal{M}_{K^\star}}(n;N^\star,v)} \geq \frac{1}{\epsilon}$, and hence $\frac{v}{W_-^{\mathcal{M}_{K^\star}}(n;N^\star,v)} \geq \frac{1}{\epsilon}$. This finishes the proof for the $q = 1$ case.

For the $q \in (0, 1]$ case, notice that the round-n competition is the combination of the exogenous shock and some subcontest $\mathcal{M}$. The incumbent and the laggard's expected payoffs from such a competition with prize v' are, respectively,

$$\widetilde{V}_+^{\mathcal{M}}(v') = (1-q)v' + qV_+^{\mathcal{M}}(v') \text{ and } \widetilde{V}_-^{\mathcal{M}}(v') = qV_-^{\mathcal{M}}(v').$$

Therefore,

$$\frac{v' - \widetilde{V}_+^{\mathcal{M}}(v')}{\widetilde{V}_-^{\mathcal{M}}(v')} = \frac{v' - V_+^{\mathcal{M}}(v')}{V_-^{\mathcal{M}}(v')}.$$

Then it is clear that the same proof for the $q = 1$ case applies with $K^\star$ taken to satisfy $\frac{v' - V_+(\mathcal{M}_{K^\star}; v')}{V_-(\mathcal{M}_{K^\star}; v')} \geq \frac{1}{\epsilon}$ for all $v' > 0$ and $N^\star$ taken to satisfy $\{1 - q + q[1 - \lambda(L_{K^\star})]\}^{N^\star} \leq \epsilon$. $\square$