On the Optimal Design of All-Pay Auctions*

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Abstract

We consider the optimal design of complete-information all-pay auctions with multiple heterogeneous players when a designer can manipulate contestants' relative competitiveness by imposing identity-dependent treatments. Two types of instruments are considered: (i) multiplicative bias that assigns individualized weight to each contender' effective effort entry and (ii) additive headstarts that directly add to it. We show that both instruments will be used in the optimum in general. Moreover, the contest designer is able to induce almost every allocation of the prize while achieving full rent extraction. As a result, she can implement the first-best for a large class of design objectives.

Keywords: All-pay Auction; Bias; Headstarts; Favoritism.

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1 Introduction

A wide variety of competitive activities resemble a contest. Politicians campaign for votes; interest groups lobby for policy influence; firms race for technological breakthroughs that can be patented; and students engage in academic efforts to secure seats at premium colleges. In all of these scenarios, contenders expend costly effort to vie for limited prizes, while the competitive outlays are nonrefundable regardless of the outcome.

A voluminous economics literature has examined contestants' strategic behavior and the optimal design of contests. An all-pay auction—which fully rewards superior effort—is an intuitive framework to model the prize allocation mechanism. It awards the prize to the highest bidder with certainty: In its simplest form, a contestant wins the contest with probability one if his effort x_i exceeds those of the others, i.e.,

$$p_i(\boldsymbol{x}) = 1, \text{ if } x_i > x_j, \forall j \neq i,$$

for a given set of effort entries $\boldsymbol{x} := (x_1, \ldots, x_n)$.

In this paper, we explore the optimal design of all-pay auctions when a contest designer is able to award identity-dependent preferential treatments on contestants to manipulate the competitive balance of the playing field. The economics literature has long espoused the strategic use of preferential treatments tailored to individual characteristics to incentivize effort supply: A contest designer can strategically favor or handicap contestants to bias the competition to promote her own interests (e.g., Siegel, 2014; Szech, 2015).[1] Numerous examples have been documented in the literature, which evidences the prevalence of this practice.[2]

Two instruments are broadly adopted in the literature to model the biases imposed on contestants' effort entries: (i) multiplicative bias and (ii) additive headstarts. The former—e.g., Fu (2006) and Epstein, Mealem, and Nitzan (2011)—places a fixed weight on one's effort, while the latter—e.g., Kirkegaard (2012) and Pastine and Pastine (2012)—directly adds to it. In a biased all-pay auction, each contestant's effort is adjusted by the biases and converted into a score; the highest scoring contestant wins the prize.

[1]See also Mealem and Nitzan (2016); Chowdhury, Esteve-González, and Mukherjee (2019); and Fu and Wu (2019a) for thorough surveys of this strand of the literature.

[2]Consider, for instance, government policies that favor small and medium-sized enterprises (SMEs) in public procurement to support local entrepreneurship in various countries (Che and Gale, 2003; Epstein, Mealem, and Nitzan, 2011). Prestigious colleges often allocate bonus points to minority applicants when practicing affirmative action in admissions (Fu, 2006; Franke, 2012). In competitions for a vacant position, internal candidates are often ex ante preferred to external candidates to incentivize productive efforts (Chan, 1996). In a corporate succession race, the leading candidate is often awarded a key appointment—e.g., a president or chief operating officer—that allows him/her privileged access to corporate resources in carrying out assigned tasks (Fu and Wu, 2019b).

In this paper, we consider a multi-player all-pay auction and allow the designer to use both instruments to optimize toward a general design objective. Fu and Wu (2020) and Deng, Fu, and Wu (2021) develop an indirect optimization approach for the design of biased lottery contests. We adapt this approach to the setting of all-pay auctions. This allows us to (i) first characterize the feasibility frontier of the contest under a general objective function, and (ii) then demonstrate that an optimally designed all-pay auction, with the use of both multiplicative biases and headstart, can achieve the first best described by the feasibility frontier. The result also implies that a properly designed all-pay auction outperforms any possible contest mechanism that yield pure-strategy equilibria.[3]

Our paper extends the literature in three main dimensions. First, we construct a general objective function that encompasses a broad array of scenarios. The literature on contest design typically focuses on specific objective functions, with the majority to maximize total effort, such as Kirkegaard (2012); Li and Yu (2012); Franke, Kanzow, Leininger, and Schwartz (2014); and Franke, Leininger, and Wasser (2018). However, the pursuit of alternative objectives is not uncommon in practice. Consider, for instance, that a college presumably only cares about the academic quality of its admitted student body (see Fu, 2006). In a crowdsourcing competition for a technical solution, the buyer would only value the quality of the winning entry. More suspense regarding the outcome of a sporting event makes it easier to promote (see Chan, Courty, and Hao, 2008). Alternatively, in public procurement, a government could care about both domestic suppliers' efforts as a buyer and their welfare as a social planner (see Epstein et al., 2011). We construct an objective function that encompasses all of these concerns.

Second, our analysis departs from the usual two-player setting and allows for an arbitrary number of contestants. Equilibrium analysis of all-pay auctions with three or more players poses a technical challenge when contestants are heterogeneous and biases can be imposed. Fu and Wu (2020) develop an alternative technique that bypasses the analytical difficulty in generalized lottery contests. In this paper, we revive the equilibrium characterization result of Baye, Kovenock, and De Vries (1996), which further allows us to adapt the approach of Fu and Wu (2020) to all-pay auctions.

Third, thanks to the above-mentioned optimization approach, our analysis allows the designer to choose an arbitrary combination of multiplicative biases and headstarts. In the majority of prior studies of optimally biased contests, the designer is endowed with a single instrument (e.g., Franke et al., 2014). We demonstrate that the optimum, in general, requires that the two instruments be imposed together. Notable exceptions include Kirkegaard (2012); Franke et al. (2018); and Zhu (2021). However, all of these studies focus on spe-

[3]Franke, Leininger, and Wasser (2018) show that an all-pay auction, with a proper combination of multiplicative biases and headstarts, can achieve the first best. However, they only consider the maximization of total effort.

cific objective functions. Kirkegaard (2012) and Franke et al. (2018) consider total effort maximization, while Zhu (2021) also considers maximization of the maximum effort.

The rest of the paper is organized as follows. Section 2 sets up the contest model and describes the objective function for contest design. Section 3 conducts the analysis and discusses its implications, and Section 4 concludes.

2 The Model

There are $n \geq 2$ risk-neutral contestants competing for a prize. The prize has a value $v_i > 0$ for each contestant $i \in \mathcal{N} \equiv \{1, \ldots, n\}$—with $v_1 \geq \ldots \geq v_n$—which is commonly known. To win the prize, contestants simultaneously commit to their efforts $x_i \geq 0$. One's bid incurs a unity marginal effort cost.

Fixing a set of effort entries $\boldsymbol{x} \equiv (x_1, \ldots, x_n) \geq (0, \ldots, 0)$, let us denote by $p_i(x_i, \boldsymbol{x}_{-i})$ a contestant i's probability of winning the contest, where $\boldsymbol{x}_{-i} \equiv (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ is the effort profile of his rivals. Contestant i's probability of winning the contest—i.e., the contest success function (CSF)—is given by

$$p_i(x_i, \boldsymbol{x}_{-i}) = \begin{cases} 1 & \text{if } \alpha_i x_i + \beta_i > \max_{j \neq i} \left\{\alpha_j x_j + \beta_j\right\}, \\ \frac{1}{m} & \text{if } \alpha_i x_i + \beta_i \text{ is among the } m \text{ highest of } \left\{\alpha_j x_j + \beta_j\right\}_{j=1}^{n} \text{ with a tie,} \\ 0 & \text{if } \alpha_i x_i + \beta_i < \max_{j \neq i} \left\{\alpha_j x_j + \beta_j\right\}, \end{cases} \quad (1)$$

where $\alpha_i \geq 0$ and $\beta_i \geq 0$ are the *multiplicative bias* and *additive headstarts* the designer imposes on each contestant $i \in \mathcal{N}$, respectively. A contestant i wins the contest if his effective output or score—i.e., $\alpha_i x_i + \beta_i$—exceeds those of all the others, and his expected payoff can be written as

$$\pi_i(x_i, \boldsymbol{x}_{-i}) := p_i(x_i, \boldsymbol{x}_{-i}) \cdot v_i - x_i, \text{for all } i \in \mathcal{N}.$$

Both instruments are popularly adopted in the literature to model preferential treatments. For multiplicative bias, see Fu (2006); Franke (2012); Franke et al. (2014); and Epstein et al. (2011). For headstarts, see Clark and Riis (2000); Konrad (2002); Siegel (2009, 2014); Li and Yu (2012); and Seel and Wasser (2014). Kirkegaard (2012); Franke et al. (2018); and Zhu (2021) allow for both.

Contest Design and Contest Objective It is well known that a complete-information all-pay auction, in general, does not have pure-strategy equilibria. For ease of exposition, denote contestant i's expected effort and that conditional on winning the contest—i.e., $\mathbb{E}(x_i)$

and $\mathbb{E}(x_i|\alpha_i x_i+\beta_i > \max_{j\neq i}\{\alpha_j x_j+\beta_j\})$—by x_i^e and x_i^w, respectively. It follows immediately that $x_i^w \geq x_i^e$. Let $\boldsymbol{x}^e := (x_1^e,\ldots,x_n^e)$ and $\boldsymbol{x}^w := (x_1^w,\ldots,x_n^w)$.

Prior to the competition, the designer, anticipating contestants' equilibrium plays, chooses and commits to a contest rule $(\boldsymbol{\alpha},\boldsymbol{\beta}) \geq (\mathbf{0},\mathbf{0})$ to maximize an objective function, which we denote by $\Lambda(\cdot)$. We assume that $\Lambda(\cdot)$ is a function of the profile of expected effort $\boldsymbol{x}^e \equiv (x_1^e,\ldots,x_n^e)$; the profile of expected effort conditional on winning the contest $\boldsymbol{x}^w \equiv (x_1^w,\ldots,x_n^w)$; the profile of equilibrium winning probabilities $\boldsymbol{p} \equiv (p_1,\ldots,p_n)$; and the profile of contestants' prize valuations $\boldsymbol{v} \equiv (v_1,\ldots,v_n)$. The following assumption is imposed on the objective function $\Lambda(\cdot)$ throughout the paper.

Assumption 1 *Fixing $\boldsymbol{p} \equiv (p_1,\ldots,p_n)$ and $\boldsymbol{v} \equiv (v_1,\ldots,v_n)$, $\Lambda(\boldsymbol{x}^e,\boldsymbol{x}^w,\boldsymbol{p},\boldsymbol{v})$ is weakly increasing in x_i^e and x_i^w for all $i \in \mathcal{N}$.*

Assumption 1 specifies a mild regularity condition, and the objective function $\Lambda(\cdot)$ encompasses a broad array of scenarios for contest design. It can be satisfied by many popularly studied objective functions in the literature. Several examples are provided below to demonstrate the versatility of $\Lambda(\cdot)$.

Example 1 ***(Objective Functions)*** *The following objective functions satisfy Assumption 1:*

(i) Total effort: $\Lambda = \sum_{i=1}^n \mathbb{E}(x_i) \equiv \sum_{i=1}^n x_i^e$;

(ii) Expected winner's effort: $\Lambda = \sum_{i=1}^n \left[p_i \cdot \mathbb{E}\left(x_i \,|\, \alpha_i x_i + \beta_i > \max_{j\neq i}\{\alpha_j x_j + \beta_j\}\right)\right] \equiv \sum_{i=1}^n p_i x_i^w$;

(iii) Closeness concerns: $\Lambda = \sum_{i=1}^n x_i^e - \gamma \sum_{i=1}^n \left(p_i - \frac{\sum_{i=1}^n p_i}{n}\right)^2$, *with* $\gamma \geq 0$;

(iv) Welfare concerns: $\Lambda = \tau \sum_{i=1}^n \pi_i + (1-\tau) \sum_{i=1}^n x_i^e$, *where* $\pi_i := p_i v_i - x_i^e$ *and* $\tau \in [0, \frac{1}{2}]$.

Total effort is the most widely assumed objective for contest design in the literature. Maximization of the expected winner's effort is common in the auction literature (e.g., Moldovanu and Sela, 2006), and has recently gained increasing attention in studies of contests (e.g., Baye and Hoppe, 2003; Serena, 2017; Fu and Wu, 2019b, 2020). The third objective depicts a typical scenario in the administration of sporting events: Spectators often not only appreciate contenders' efforts, but also demand more suspense about the eventual winner (see Chan et al., 2008; Fort and Quirk, 1995; Szymanski, 2003; and Runkel, 2006, among others). The term $\sum_{i=1}^n \left[p_i - (\sum_{i=1}^n p_i)/n\right]^2$ is the variance of the equilibrium winning probability distribution, which measures the predictability of the competitive event. The objective function thus reflects the designer's preference for a closer race when $\gamma > 0$. Finally, a contest designer may care about contestants' welfare (e.g., Epstein et al., 2011). Note that the profile of contestants' prize valuations enters the objective for contest design in this case. Further, the welfare concern alludes to a preference for allocative efficiency (or the expected prize valuation of the winner): Ceteris paribus, contestants' aggregate welfare improves when the prize is given to the one with the highest valuation. To satisfy Assumption 1, however, the welfare concern must remain moderate, i.e., $\tau \leq \frac{1}{2}$.

3 Analysis and Results

The literature on the optimal design of complete-information all-pay auctions typically assumes two players and employs a direct brute-force approach: They first solve for the unique equilibrium bidding strategy for any given contest rule $(\boldsymbol{\alpha}, \boldsymbol{\beta})$, insert the solution into the objective function, then search for the optimal rule (e.g., Epstein et al., 2011; Li and Yu, 2012; Zhu, 2021). This approach relies on the equilibrium characterization and cannot be applied to the multi-player setting ($n \geq 3$) because a complete equilibrium characterization of a biased multi-player all-pay auction is technically challenging and remains an open question in the literature.[4]

To overcome the aforementioned difficulty, the literature usually takes an indirect constructive approach. For instance, Franke et al. (2014) investigate the effort-maximizing multiplicative bias. They first establish an upper bound and a lower bound for the contest performance, then show that the two bounds coincide. Similarly, Franke et al. (2018) search for the optimal combinations of multiplicative biases and headstarts that maximize the expected total effort. Again, they construct a contest rule to achieve the maximum expected total effort (revenue), which corresponds to the highest prize valuation among the contestants. Their constructions are effective when total effort (revenue) is concerned, and may lose its bite when alternative objectives are pursued for contest design.

Our analysis borrows from the indirect approach proposed by Fu and Wu (2020), which can be summarized as follows. Instead of focusing on contestants' equilibrium effort profile under a contest rule, we take a detour and focus on the equilibrium winning probability distribution. Specifically, we show that almost every equilibrium winning probability distribution can be induced by some contest rule. We then demonstrate that we can further modify the contest rule to induce a desirable winning probability distribution in equilibrium while fully dissipating the rent, which implements the first-best. The detail will be revealed in Theorem 1 and its sketch proof.

[4] See Siegel (2009, 2014) and Franke et al. (2018) for important results on equilibrium characterization.

In the rest of the section, we first carry out the analysis to characterize the main result. We then discuss the implications of our results in relation to the literature.

3.1 Optimal All-pay Auction

Before we proceed to the formal analysis, it is useful to state the following.

Definition 1 ***(Feasible Effort Profile)*** *An effort profile $\boldsymbol{x}^e \equiv (x_1^e, \ldots, x_n^e)$ is feasible for a distribution of winning probabilities $\boldsymbol{p} \in \Delta^{n-1}$ if there exists a contest rule $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \geq (\mathbf{0}, \mathbf{0})$ to induce an equilibrium that generates the profile of expected effort $\boldsymbol{x}^e$ and leads to $\boldsymbol{p}$.*

Baye, Kovenock, and De Vries (1993, 1996) show that there may exist a continuum of mixed-strategy equilibria in an all-pay auction with three or more players. In such a scenario, we select the equilibrium that is the most favorable to the contest designer.

Suppose $\boldsymbol{\beta} = \mathbf{0}$. Let $\hat{v}_i := \alpha_i v_i$, $\hat{\boldsymbol{v}} := (\hat{v}_1, \ldots, \hat{v}_2)$, and denote by $\hat{x}_i$ player i's bidding strategy in an unbiased contest, i.e., $\alpha_i = \alpha_j > 0$ for all $i, j \in \mathcal{N}$. The following result by Franke et al. (2014) allows us to transform the biased all-pay auction with zero headstarts—i.e., with $\alpha_i \neq \alpha_j$ for some $i, j \in \mathcal{N}$—into a standard unbiased all-pay auction.

Lemma 1 *Consider a biased all-pay auction contest with zero headstarts. For every equilibrium $\boldsymbol{x}$ under $\langle \boldsymbol{v}, \boldsymbol{\alpha} \rangle$, there exists an equilibrium $\hat{\boldsymbol{x}}$ under $\langle \hat{\boldsymbol{v}}, \hat{\boldsymbol{\alpha}} \rangle := \langle (\alpha_1 v_1, \ldots, \alpha_n v_n), (1, \ldots, 1) \rangle$ such that $\mathbb{E}(\hat{x}_i) = \alpha_i \mathbb{E}(x_i) \equiv \alpha_i x_i^e$ for all $i \in \mathcal{N}$. Moreover, the two equilibria $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ lead to the same profile of winning probabilities $\boldsymbol{p}$.*

Lemma 1 unveils the strategic equivalence between the biased all-pay auction and the transformed unbiased counterpart, which revives the equilibria characterization result of Baye et al. (1996) in our setting. We obtain the following key result.

Theorem 1 *Consider all-pay auctions and fix an arbitrary $\boldsymbol{p} \in \Delta^{n-1}$ such that $p_i \neq 1$ for all $i \in \mathcal{N}$.*[5] *Then $\boldsymbol{x}^e \equiv (x_1^e, \ldots, x_n^e) = (p_1 v_1, \ldots, p_n v_n)$ is feasible for $\boldsymbol{p}$.*

Theorem 1 states that for (almost) every equilibrium winning probability $\boldsymbol{p}$, there always exists a contest rule $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ that induces $\boldsymbol{p}$ and a profile of expected efforts $\boldsymbol{x}^e \equiv (p_1 v_1, \ldots, p_n v_n)$. Obviously, each player's participation constraint binds under this contest rule, indicating that the maximum expected effort is achieved.

The result is proven by construction. A sketch proof is laid out in the main text: It elucidates the different roles played by the multiplicative biases and headstarts in this context

[5] The condition $p_i \neq 1$ contained in Theorem 1 is due to the restriction that tie-breaking must be symmetric in the CSF specified in (1), and can be dropped if we allow the contest designer to choose the tie-breaking rule, as in Szech (2015) and Franke et al. (2018).

and also helps us understand the comparison between all-pay auctions and noisy contests, which we further elaborate on in Section 3.2.1. For ease of exposition, let us consider the case in which $p_1 > p_2 \geq \ldots \geq p_n > 0$.[6] The sketch proof proceeds in the following two steps:

Step 1 (Introducing Multiplicative Bias): Fix an arbitrary equilibrium winning probability distribution $\boldsymbol{p} \in \Delta^{n-1}$ with $p_i \neq 1$ for all $i \in \mathcal{N}$, and set headstarts to zero. We can construct a set of multiplicative biases $\boldsymbol{\alpha}^* \equiv (\alpha_1^*, \ldots, \alpha_n^*)$ that satisfies $\alpha_1^* v_1 > \alpha_2^* v_2 = \ldots = \alpha_n^* v_n > 0$, such that there exists a mixed-strategy equilibrium that leads to the given distribution of winning probabilities $\boldsymbol{p} \equiv (p_1, \ldots, p_n)$. By our construction, contestants 2 to n each earn an expected payoff of zero, and player 1 receives a positive expected payoff of size $(\alpha_1^* v_1 - \alpha_2^* v_2)/\alpha_1^*$.

Step 2 (Introducing Additive Headstarts): We add headstarts to the contest rule to further incentivize player 1 without disturbing the equilibrium incentives of the other players. Consider the following set of contest rules $(\boldsymbol{\alpha}^\dagger, \boldsymbol{\beta}^\dagger)$:

$$\left(\alpha_i^\dagger, \beta_i^\dagger\right) := \begin{cases} \left(\alpha_1^*, 0\right) & \text{for } i = 1, \\ \left(\alpha_i^*, \alpha_1^* v_1 - \alpha_2^* v_2\right) & \text{for } i \in \{2, \ldots, n\}. \end{cases}$$

In words, we give the same headstarts to all players except for player 1. Compared with the equilibrium constructed in Step 1, contestant 1's equilibrium effort distribution is shifted upward by $(\alpha_1^* v_1 - \alpha_2^* v_2)/\alpha_1^*$, whereas all other players' equilibrium strategies remain unchanged. The additional effort supply from contestant 1 completely offsets the preferential treatment awarded to other players through the headstart, and the size of the headstarts is chosen to fully deplete the surplus left to contestant 1 in the equilibrium, i.e., earning zero expected payoff in the contest. This, in turn, implies that $x_i^e = p_i v_i$ for all $i \in \mathcal{N}$ in equilibrium under the contest rule $(\boldsymbol{\alpha}^\dagger, \boldsymbol{\beta}^\dagger)$.

We subsequently elaborate on the implications of this result.

3.2 Discussions

In what follows, we first take a closer look at Theorem 1 and further elaborate on its implications. We then discuss the respective roles of multiplicative bias and additive headstarts in the contest design.

[6]Step 2 is unnecessary for the proof of Theorem 1 in the case $p_1 = p_2 \geq \ldots \geq p_n > 0$. In other words, headstarts are not used if the contest designer aims to induce a distribution of winning probabilities in which the highest equilibrium winning probability is equal to the second highest one.

3.2.1 All-pay Auction Achieves the First-best

Franke et al. (2018) show that a proper combination of multiplicative biases and additive headstarts can achieve the first-best result when the designer aims to maximize expected total effort. Theorem 1 implies that their result extends to a large class of objective functions as described by Assumption 1. To see this, note that a contestant $i \in \mathcal{N}$ can always guarantee himself a payoff of at least zero by investing zero effort. As a result, in every equilibrium of every contest (i.e., with an arbitrary CSF), the expected payoff of contestant i must be nonnegative, i.e., $x_i^e \leq p_i v_i$. By Theorem 1, with an appropriately designed all-pay auction, (almost) every prize allocation—that induces $x_i^e = p_i v_i$ for each contestant $i \in \mathcal{N}$—can be implemented. This implies immediately that all-pay auctions dominate *any* other contest mechanism—e.g., the generalized lottery contest specified in (2)—in terms of the resultant effort x_i^e.

Further, $x_i^w \geq x_i^e$ always holds in an all-pay auction. In contrast, a contest yields $x_i^w = x_i^e$ in its pure-strategy equilibria. These observations, altogether, indicate that a properly designed all-pay auction outperforms any other contest mechanism that induces a pure-strategy equilibrium for any design objective described by Assumption 1.

Remark 1 ***(All-pay auction achieves the first-best result)*** *Suppose that Assumption 1 is satisfied. Then an optimal all-pay auction generates a higher payoff for the contest designer than any other form of contest that induces a pure-strategy equilibrium (e.g., a generalized lottery contest).*

A handful of studies examine the comparison between all-pay auctions and Tullock contests—e.g., Fang (2002); Epstein et al. (2011); Franke et al. (2014); and Franke et al. (2018). Our analysis sheds light on this literature: It accommodates a broader design objective and establishes the dominance of all-pay auctions over a larger class of contest mechanisms, i.e., *any* contest that induces pure-strategy bidding.

3.2.2 Multiplicative Biases vs. Additive Headstarts

The literature typically focuses on contest design with a single instrument, either multiplicative biases or headstarts. Kirkegaard (2012); Franke et al. (2018); and Zhu (2021) show that in a revenue-maximizing all-pay auction, it is generally optimal to employ both. The following can directly be inferred from Theorem 1 and its proof.

Remark 2 *The two steps in the sketch proof of Theorem 1 imply that in general, the optimum requires a combination of multiplicative biases (Step 1) and additive headstarts (Step 2) for a general contest objective described by Assumption 1.*

A proper combination of the two instruments allows the contest to achieve the frontier of feasible effort profile. However, the same does not hold in generalized lottery contests with ratio-form contest success functions. Consider a contest in which one's winning probability is given by

$$p_i(x_i, \boldsymbol{x}_{-i}) = \begin{cases} \dfrac{\alpha_i f(x_i) + \beta_i}{\sum_{j=1}^{n} [\alpha_i f(x_j) + \beta_i]} & \text{if } \sum_{j=1}^{n} [\alpha_i f(x_j) + \beta_i] > 0, \\ \dfrac{1}{n} & \text{if } \sum_{j=1}^{n} [\alpha_i f(x_j) + \beta_i] = 0, \end{cases} \tag{2}$$

where $f(\cdot)$ is twice differentiable, with $f(0) = 0$, $f'(x) > 0$, and $f''(x) \leq 0$ for all $x > 0$. Fu and Wu (2020) establish the following result.

Remark 3 ***(Fu and Wu, 2020, Theorem 2)*** *Suppose that Assumption 1 is satisfied. The optimum can always be achieved by choosing multiplicative biases* $\boldsymbol{\alpha}$ *only and setting headstarts* $\boldsymbol{\beta}$ *to zero.*[7]

The contrast between Remarks 2 and 3 demonstrates that headstarts play *different* roles in all-pay auctions and generalized lottery contests. By Remark 3, headstarts would not be required in optimizing generalized lottery contests. As shown by Fu and Wu (2020), for any contest rule that involves positive headstarts, one can always construct an alternative rule with zero headstarts that induces the same winning probability distribution and strictly higher effort. However, an all-pay auction would invoke headstarts in the optimum. The sketch proof of Theorem 1 reveals the logic: In the first step, we resort to multiplicative biases $\boldsymbol{\alpha} \equiv (\alpha_1, \ldots, \alpha_n)$ to induce the desirable winning probability distribution. We can then further incentivize the contestant with the highest winning probability by giving additive headstarts to his opponents, as in the second step. This occurs because of the perfectly discriminatory nature of all-pay auctions: The headstarts awarded to underdogs simply force the favorite to shift up the distribution of his effort, which perfectly offsets the headstarts and preserves all contestants' winning odds; this is impossible in a noisy contest that leads to a pure-strategy equilibrium, given the probabilistic nature of the winner-selection mechanism (2).

4 Concluding Remark

In this paper, we consider the optimal design of complete-information all-pay auctions with general contest objectives. We apply the indirect approach suggested by Fu and Wu

[7] It should be noted that we do not allow for negative headstarts. Drugov and Ryvkin (2017) allow for negative headstarts and show that a deviation from zero headstarts can locally improve the performance of the contest, depending on the sign of the third derivative of the effort cost function.

(2020) and Deng, Fu, and Wu (2021) and characterize the general properties of the optimal contest. In particular, we show that both instruments will be used in the optimum in general. Further, an optimally designed all-pay auction can implement the first-best outcome for a large class of objectives.

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Appendix: Proofs

Proof of Theorem 1

Proof. Denote the equilibrium winning probability distribution we would like to induce by $\boldsymbol{p}^* \equiv (p_1^*, \ldots, p_n^*) \in \Delta^{n-1}$. Without loss of generality, let us assume $p_1^* \geq \ldots \geq p_n^*$. We apply the equilibrium characterization in Theorem 2 in Baye et al. (1996) to prove the result for the case $p_1^* > p_2^* \geq \ldots \geq p_n$. The case $p_1^* = p_2^* \geq \ldots \geq p_n$ can be proved in a similar way by invoking Theorem 1 in Baye et al. (1996).

Step I: We show that fixing an arbitrary $\boldsymbol{p}^* \equiv (p_1^*, \ldots, p_n^*) \in \Delta^{n-1}$, we can construct a set of multiplicative biases $\boldsymbol{\alpha}^*$ to induce $\boldsymbol{p}^*$. To proceed, we set $\boldsymbol{\beta} = \mathbf{0}$ and choose $\boldsymbol{\alpha} \equiv (\alpha_1, \ldots, \alpha_n)$ such that $\hat{v}_1 > \hat{v}_2 = \cdots = \hat{v}_n > 0$, where $\hat{v}_i = \alpha_i v_i$ for all $i \in \mathcal{N}$. Let $G_i(\hat{x}_i)$ denote the cumulative distribution function (CDF) of player i's equilibrium bidding strategy. By Theorem 2 in Baye et al. (1996), there exists a continuum of equilibria of the unbiased all-pay auction with valuations $\hat{\boldsymbol{v}}$ and $\hat{\boldsymbol{\alpha}} \equiv (1, \ldots, 1)$, which is fully characterized by a set of cutoffs $\boldsymbol{b} \equiv (b_1, \ldots, b_n)$ that satisfy $0 = b_1 = b_2 \leq \ldots \leq b_n \leq \hat{v}_2$. In equilibrium, player i stays inactive with some probability and bids continuously over $(b_i, \hat{v}_2]$ with complementary probability. For notational convenience, let $c_i := \frac{\hat{v}_1 - \hat{v}_2 + b_i}{\hat{v}_1}$ for all $i \in \mathcal{N}$. The following CDFs constitute a mixed-strategy equilibrium of the unbiased all-pay auction:

$$
\begin{aligned}
\forall x \in \big[b_n, \hat{v}_2\big]: \quad & G_1(x) = \frac{x}{\hat{v}_2}\left[\frac{\hat{v}_1 - \hat{v}_2 + x}{\hat{v}_1}\right]^{\frac{2-n}{n-1}};\\
& G_i(x) = \left[\frac{\hat{v}_1 - \hat{v}_2 + x}{\hat{v}_1}\right]^{\frac{1}{n-1}}, \ i \in \big\{2, 3, \ldots, n\big\};\\
\forall x \in \big[b_j, b_{j+1}\big), j \in \{3, \ldots, n-1\}: \quad & G_1(x) = \frac{x}{\hat{v}_2}\left[\frac{\hat{v}_1 - \hat{v}_2 + x}{\hat{v}_1}\right]^{\frac{2-j}{j-1}} \prod_{k>j} c_k^{-\frac{1}{(k-1)(k-2)}};\\
& G_i(x) = \left[\frac{\hat{v}_1 - \hat{v}_2 + x}{\hat{v}_1}\right]^{\frac{1}{j-1}} \prod_{k>j} c_k^{-\frac{1}{(k-1)(k-2)}}, \ i \in \big\{2, \ldots, j\big\};\\
& G_k(x) = c_k^{\frac{1}{k-1}} \prod_{s>k} c_s^{-\frac{1}{(s-1)(s-2)}}, \ k \in \big\{j+1, \ldots, n\big\};\\
\forall x \in \big[0, b_3\big]: \quad & G_1(x) = \frac{x}{\hat{v}_2} \prod_{k>2} c_k^{-\frac{1}{(k-1)(k-2)}};\\
& G_2(x) = \left[\frac{\hat{v}_1 - \hat{v}_2 + x}{\hat{v}_1}\right] \prod_{k>2} c_k^{-\frac{1}{(k-1)(k-2)}};\\
& G_k(x) = c_k^{\frac{1}{k-1}} \prod_{s>k} c_s^{-\frac{1}{(s-1)(s-2)}}, \ k \in \big\{3, \ldots, n\big\}.
\end{aligned}
$$

According to the above equilibrium strategy, we can calculate contestant i's expected effort $\mathbb{E}(\hat{x}_i)$. For notational convenience, define $\mu := \frac{\hat{v}_2}{\hat{v}_1} < 1$ and let $b_{n+1} := \hat{v}_2$. The expected effort of player 1 can then be derived as

$$\begin{aligned}
\mathbb{E}(\hat{x}_1) = & \int_0^{\hat{v}_2} x dG_1(x) \\
= & \hat{v}_2 - \sum_{j=2}^{n} \left[\int_{b_j}^{b_{j+1}} G_1(x) dx \right] \\
= & \hat{v}_2 - \sum_{j=2}^{n} \left[\int_{b_j}^{b_{j+1}} \frac{x}{\hat{v}_2} \left(\frac{\hat{v}_1 - \hat{v}_2 + x}{\hat{v}_1} \right)^{\frac{2-j}{j-1}} \prod_{k>j} c_k^{-\frac{1}{(k-1)(k-2)}} dx \right] \\
= & \hat{v}_2 - \sum_{j=2}^{n} \left[\int_{c_j}^{c_{j+1}} \frac{\hat{v}_1^2}{\hat{v}_2} (y - 1 + \mu) y^{\frac{2-j}{j-1}} \prod_{k>j} c_k^{-\frac{1}{(k-1)(k-2)}} dy \right] \\
= & \hat{v}_2 - \frac{\hat{v}_1^2}{\hat{v}_2} \sum_{j=2}^{n} \left\{ \left[\frac{j-1}{j} \left(c_{j+1}^{\frac{j}{j-1}} - c_j^{\frac{j}{j-1}} \right) - (1-\mu)(j-1) \left(c_{j+1}^{\frac{1}{j-1}} - c_j^{\frac{1}{j-1}} \right) \right] \prod_{k>j} c_k^{-\frac{1}{(k-1)(k-2)}} \right\}.
\end{aligned} \tag{3}$$

Similarly, for contestant $i \in \{2, \ldots, n\}$, we have that

$$\begin{aligned}
\mathbb{E}(\hat{x}_i) = & \int_{b_i}^{\hat{v}_2} x dG_i(x) \\
= & \hat{v}_2 - b_i G_i(b_i) - \sum_{j=i}^{n} \left[\int_{b_j}^{b_{j+1}} G_i(x) dx \right] \\
= & \hat{v}_2 - b_i G_i(b_i) - \sum_{j=i}^{n} \left[\int_{b_j}^{b_{j+1}} \left(\frac{\hat{v}_1 - \hat{v}_2 + x}{\hat{v}_1} \right)^{\frac{1}{j-1}} \prod_{k>j} c_k^{-\frac{1}{(k-1)(k-2)}} dx \right] \\
= & \hat{v}_2 - b_i G_i(b_i) - \sum_{j=i}^{n} \left[\hat{v}_1 \int_{c_j}^{c_{j+1}} y^{\frac{1}{j-1}} \prod_{k>j} c_k^{-\frac{1}{(k-1)(k-2)}} dy \right] \\
= & \hat{v}_2 - b_i G_i(b_i) - \sum_{j=i}^{n} \left[\hat{v}_1 \frac{j-1}{j} \left(c_{j+1}^{\frac{j}{j-1}} - c_j^{\frac{j}{j-1}} \right) \prod_{k>j} c_k^{-\frac{1}{(k-1)(k-2)}} \right] \\
= & \hat{v}_2 - \hat{v}_1 (c_i - 1 + \mu) c_i^{\frac{1}{i-1}} \prod_{k>i} c_k^{-\frac{1}{(k-1)(k-2)}} - \sum_{j=i}^{n} \left[\hat{v}_1 \frac{j-1}{j} \left(c_{j+1}^{\frac{j}{j-1}} - c_j^{\frac{j}{j-1}} \right) \prod_{k>j} c_k^{-\frac{1}{(k-1)(k-2)}} \right].
\end{aligned} \tag{4}$$

By Theorem 2 in Baye et al. (1996), player 1 earns an expected payoff of $\hat{v}_1 - \hat{v}_2$, while every other player receives an expected payoff of zero in the transformed unbiased all-pay auction with valuations $\hat{\boldsymbol{v}} \equiv (\hat{v}_1, \ldots, \hat{v}_n)$, i.e.,

$$p_1\hat{v}_1 - \mathbb{E}(\hat{x}_1) = \hat{v}_1 - \hat{v}_2, \tag{5}$$

$$p_i\hat{v}_i = \mathbb{E}(\hat{x}_i), i \in \{2, \ldots, n\}. \tag{6}$$

Combining (3) and (5), we can obtain p_1 as a function of μ and $\boldsymbol{c} \equiv (c_1, \ldots, c_n)$:

$$\begin{aligned} p_1(\mu, \boldsymbol{c}) &= \frac{\mathbb{E}(\hat{x}_1)}{\hat{v}_1} + \frac{\hat{v}_1 - \hat{v}_2}{\hat{v}_1} \\ &= 1 - \frac{1}{\mu}\sum_{j=2}^{n}\left\{\left[\frac{j-1}{j}\left(c_{j+1}^{\frac{j}{j-1}} - c_j^{\frac{j}{j-1}}\right) - (1-\mu)(j-1)\left(c_{j+1}^{\frac{1}{j-1}} - c_j^{\frac{1}{j-1}}\right)\right]\prod_{k>j} c_k^{-\frac{1}{(k-1)(k-2)}}\right\}. \end{aligned} \tag{7}$$

Similarly, combining (4) and (6), for $i \in \{2, \ldots, n\}$, we have that

$$\begin{aligned} p_i(\mu, \boldsymbol{c}) &= \frac{\mathbb{E}(\hat{x}_i)}{\hat{v}_2} \\ &= 1 - \frac{1}{\mu}\left(c_i - 1 + \mu\right) c_i^{\frac{1}{i-1}} \prod_{k>i} c_k^{-\frac{1}{(k-1)(k-2)}} - \sum_{j=i}^{n}\left[\frac{1}{\mu}\frac{j-1}{j}\left(c_{j+1}^{\frac{j}{j-1}} - c_j^{\frac{j}{j-1}}\right)\prod_{k>j} c_k^{-\frac{1}{(k-1)(k-2)}}\right]. \end{aligned} \tag{8}$$

A closer look at (7) and (8) yields the following:

Lemma 2 *For all $i \geq 2$, $p_i(\mu, \boldsymbol{c})$ is strictly increasing in μ if $c_i < 1$, and is constant if $c_i = 1$. Moreover, $p_1(\mu, \boldsymbol{c})$ is strictly decreasing in μ.*

Proof. For $i = 2$, it is straightforward to see that $p_2(\mu, \boldsymbol{c})$ is strictly increasing in μ from (8). For $i \geq 3$, $p_i(\mu, \boldsymbol{c})$ can be rewritten as

$$\begin{aligned} p_i(\mu, \boldsymbol{c}) = 1 - c_i^{\frac{1}{i-1}} \prod_{k>i} c_k^{-\frac{1}{(k-1)(k-2)}} \\ - \frac{1}{\mu}\left\{\sum_{j=i}^{n}\left[\frac{j-1}{j}\left(c_{j+1}^{\frac{j}{j-1}} - c_j^{\frac{j}{j-1}}\right)\prod_{k>j} c_k^{-\frac{1}{(k-1)(k-2)}}\right] - (1-c_i)c_i^{\frac{1}{i-1}}\prod_{k>i} c_k^{-\frac{1}{(k-1)(k-2)}}\right\} \end{aligned}$$

To prove that $p_i(\mu, \boldsymbol{c})$ is increasing in μ for $i \geq 3$, it suffices to show that

$$\sum_{j=i}^{n}\left[\frac{j-1}{j}\left(c_{j+1}^{\frac{j}{j-1}} - c_j^{\frac{j}{j-1}}\right)\prod_{k>j} c_k^{-\frac{1}{(k-1)(k-2)}}\right] - (1-c_i)c_i^{\frac{1}{i-1}}\prod_{k>i} c_k^{-\frac{1}{(k-1)(k-2)}} \geq 0. \tag{9}$$

Note that $c_{j+1} \geq c_j$. By Lagrange's Mean Value Theorem, for all $j \geq i$, there exists $d_j \in (c_j, c_{j+1})$ such that

$$\frac{j-1}{j}\left(c_{j+1}^{\frac{j}{j-1}} - c_j^{\frac{j}{j-1}}\right) = d_j^{\frac{1}{j-1}}\left(c_{j+1} - c_j\right). \tag{10}$$

Substituting (10) into the left-hand side of (9), we can obtain

$$\begin{aligned}
&\sum_{j=i}^{n}\left[\frac{j-1}{j}\left(c_{j+1}^{\frac{j}{j-1}} - c_j^{\frac{j}{j-1}}\right)\prod_{k>j} c_k^{-\frac{1}{(k-1)(k-2)}}\right] - (1-c_i)c_i^{\frac{1}{i-1}}\prod_{k>i} c_k^{-\frac{1}{(k-1)(k-2)}} \\
=&\sum_{j=i}^{n}\left[\left(c_{j+1} - c_j\right) d_j^{\frac{1}{j-1}}\prod_{k>j} c_k^{-\frac{1}{(k-1)(k-2)}}\right] - (1-c_i)c_i^{\frac{1}{i-1}}\prod_{k>i} c_k^{-\frac{1}{(k-1)(k-2)}} \\
\geq&\sum_{j=i}^{n}\left[\left(c_{j+1} - c_j\right) c_j^{\frac{1}{j-1}}\prod_{k>i} c_k^{-\frac{1}{(k-1)(k-2)}}\prod_{j\geq k>i} c_k^{\frac{1}{(k-1)(k-2)}}\right] - (1-c_i)c_i^{\frac{1}{i-1}}\prod_{k>i} c_k^{-\frac{1}{(k-1)(k-2)}} &(11)\\
\geq&\sum_{j=i}^{n}\left[\left(c_{j+1} - c_j\right) c_i^{\frac{1}{j-1}}\prod_{k>i} c_k^{-\frac{1}{(k-1)(k-2)}}\prod_{j\geq k>i} c_i^{\frac{1}{(k-1)(k-2)}}\right] - (1-c_i)c_i^{\frac{1}{i-1}}\prod_{k>i} c_k^{-\frac{1}{(k-1)(k-2)}} &(12)\\
=&\sum_{j=i}^{n}\left[\left(c_{j+1} - c_j\right) c_i^{\frac{1}{i-1}}\prod_{k>i} c_k^{-\frac{1}{(k-1)(k-2)}}\right] - (1-c_i)c_i^{\frac{1}{i-1}}\prod_{k>i} c_k^{-\frac{1}{(k-1)(k-2)}} = 0,
\end{aligned}$$

where the first inequality follows from $d_j \geq c_j$; the second inequality follows from $c_k \geq c_i$ for $k > i$; the second equality follows from

$$\prod_{j\geq k>i} c_i^{\frac{1}{(k-1)(k-2)}} = c_i^{\sum_{j\geq k>i}\frac{1}{(k-1)(k-2)}} = c_i^{\sum_{j\geq k>i}\left(\frac{1}{k-2}-\frac{1}{k-1}\right)} = c_i^{\frac{1}{i-1}-\frac{1}{j-1}};$$

and the last equality follows from $\sum_{j=i}^{n}(c_{j+1} - c_j) = c_{n+1} - c_i = 1 - c_i$. Note that the inequalities in (11) and (12) are strict if $c_i < 1$ and the left-hand side of (9) is equal to zero if $c_i = 1$. Therefore, for $i \geq 3$, $p_i(\mu, \boldsymbol{c})$ is strictly increasing in μ if $c_i < 1$ and is constant if $c_i = 1$. Recall that $p_2(\mu, \boldsymbol{c})$ is strictly increasing in μ; together with the fact that $p_1(\mu, \boldsymbol{c}) = 1 - \sum_{j=2}^{n} p_j(\mu, \boldsymbol{c})$, we have that $p_1(\mu, \boldsymbol{c})$ is strictly decreasing in μ. This completes the proof. ■

Lemma 3 *For any $i \geq 3$, $p_i(\mu, \boldsymbol{c})$ is strictly decreasing in c_i.*

Proof. $p_i(\mu, \boldsymbol{c})$ in (8) can be rewritten as

$$p_i(\mu,\boldsymbol{c}) = \left\{ 1 - \sum_{j=i+1}^{n} \left[\frac{1}{\mu}\frac{j-1}{j} \left(c_{j+1}^{\frac{j}{j-1}} - c_j^{\frac{j}{j-1}} \right) \prod_{k>j} c_k^{-\frac{1}{(k-1)(k-2)}} \right] - \frac{1}{\mu}\frac{i-1}{i} c_{i+1}^{\frac{i}{i-1}} \prod_{k>i} c_k^{-\frac{1}{(k-1)(k-2)}} \right\}$$
$$+ \frac{1}{\mu} \prod_{k>i} c_k^{-\frac{1}{(k-1)(k-2)}} \left[\frac{i-1}{i} c_i^{\frac{i}{i-1}} - (c_i - 1 + \mu) c_i^{\frac{1}{i-1}} \right].$$

Therefore, it suffices to show that

$$h(c_i) := \frac{i-1}{i} c_i^{\frac{i}{i-1}} - (c_i - 1 + \mu) c_i^{\frac{1}{i-1}}$$

is decreasing in c_i. Simple algebra yields that

$$h'(c_i) = -\frac{1}{i-1} \left[c_i - (1-\mu) \right] c_i^{\frac{2-i}{i-1}} \leq 0,$$

where the inequality follows from $c_i \geq c_2 \equiv 1 - \mu$ for $i \geq 3$. This concludes the proof. ∎

We are now ready to prove the statement we made at the beginning of Step I. It suffices to construct $\mu \in (0, 1]$ and $\boldsymbol{c} \equiv (c_1, \ldots, c_n)$, with $1 - \mu = c_1 = c_2 \leq \ldots \leq c_n \leq 1$, such that $p_i^* = p_i(\mu, \boldsymbol{c})$ for all $i \in \mathcal{N}$. Note that $p_i(\mu, \boldsymbol{c})$ is a function of μ and $(c_i, \ldots, c_n)$, and is independent of $(c_1, \ldots, c_{i-1})$ for $i \geq 3$. With slight abuse of notation, we write $p_i(\mu, \boldsymbol{c})$ in (8) as $p_i(\mu, c_i, \ldots, c_n)$ in what follows.

Fix $\boldsymbol{p}^* \equiv (p_1^*, \ldots, p_n^*)$. We define a set of functions $\left\{\tilde{c}_i(\mu)\right\}_{i=1}^n$ and a function $\psi(\mu)$ recursively as the following:

Step 0: Set $\psi(\mu) = 1$, and define $\tilde{c}_n(\mu)$ as

$$\tilde{c}_n(\mu) := \begin{cases} 1-\mu & \text{if } p_n(\mu, 1-\mu) < p_n^*, \\ \text{The unique solution to } p_n(\mu, c_n) = p_n^* & \text{otherwise.} \end{cases} \tag{13}$$

Lemma 3, together with the fact that $p_n(\mu, 1) = 0$, implies that $\tilde{c}_n(\mu)$ is well defined and $\tilde{c}_n(\mu) \in [1-\mu, 1]$. If $p_n(\mu, 1-\mu) < p_n^*$, define $\tilde{c}_i(\mu) = 1 - \mu$ for $i \geq 3$, update $\psi(\mu) = n$, and move to Step $n-2$. Otherwise, we proceed to Step 1.

Step $j \in \{1, \ldots, n-3\}$: Define $\tilde{c}_{n-j}(\mu)$ as

$$\tilde{c}_{n-j}(\mu) := \begin{cases} 1-\mu, \text{ if } p_{n-j}\left(\mu, 1-\mu, \tilde{c}_{n-j+1}(\mu), \ldots, \tilde{c}_n(\mu)\right) < p_{n-j}^*, \\ \text{The unique solution to } p_{n-j}\left(\mu, c_{n-j}, \tilde{c}_{n-j+1}(\mu), \ldots, \tilde{c}_n(\mu)\right) = p_{n-j}^*, \text{ otherwise.} \end{cases} \tag{14}$$

Lemma 3, together with the fact that $p_{n-j}\big(\mu, \tilde{c}_{n-j+1}(\mu), \tilde{c}_{n-j+1}(\mu), \ldots, \tilde{c}_n(\mu)\big) = p^*_{n-j+1} \le p^*_{n-j}$, implies that $\tilde{c}_{n-j}(\mu)$ is well defined and $\tilde{c}_{n-j}(\mu) \in \big[1-\mu, \tilde{c}_{n-j+1}(\mu)\big]$. If $p_{n-j}\big(\mu, 1-\mu, \tilde{c}_{n-j+1}(\mu), \ldots, \tilde{c}_n(\mu)\big) < p^*_{n-j}$, define $\tilde{c}_i(\mu) = 1-\mu$ for $i \in \{3, \ldots, n-j\}$, update $\psi(\mu) = n-j$, and move to Step $n-2$. Otherwise, we proceed to Step $j+1$.

Step $n-2$: Set $\tilde{c}_1(\mu) = \tilde{c}_2(\mu) = 1-\mu$.

Let $\tilde{\boldsymbol{c}}(\mu) := \big(\tilde{c}_1(\mu), \ldots, \tilde{c}_n(\mu)\big)$. Fixing μ, we can calculate $\tilde{\boldsymbol{c}}(\mu)$ and $\psi(\mu)$ through the steps above. To complete the proof, it suffices to show that there exists $\mu \in (0,1]$ such that $p_1\big(\mu, \tilde{\boldsymbol{c}}(\mu)\big) = p^*_1$ and $\psi(\mu) = 1$.

We first show that there exists a solution to $p_1\big(\mu, \tilde{\boldsymbol{c}}(\mu)\big) = p^*_1$. It can be verified that $\tilde{\boldsymbol{c}}(\mu)$ is continuous on the interval $\mu \in (0,1]$. Moreover, it follows from Equation (13) and the construction in Step 0 that $\tilde{\boldsymbol{c}}(\mu) = (1-\mu, \ldots, 1-\mu)$ when μ is sufficiently small; together with Equation (7), we have that $\lim_{\mu \searrow 0} p_1\big(\mu, \tilde{\boldsymbol{c}}(\mu)\big) = 1 > p^*_1$. Therefore, it suffices to show that $p_1\big(1, \tilde{\boldsymbol{c}}(1)\big) \le p^*_1$. We consider two cases:

Case I: $\psi(1) = 1$. Then $\tilde{c}_2 = 1-\mu = 0$, and thus $p_1\big(1, \tilde{\boldsymbol{c}}(1)\big) = p_2\big(1, \tilde{c}_2(1), \ldots, \tilde{c}_n(1)\big)$ by (7) and (8). Moreover, we have that $p_j\big(1, \tilde{c}_j(1), \ldots, \tilde{c}_n(1)\big) = p^*_j$ for all $j \ge 3$. Therefore, we have that

$$p_1\big(1, \tilde{\boldsymbol{c}}(1)\big) = p_2\big(1, \tilde{c}_2(1), \ldots, \tilde{c}_n(1)\big) = \frac{p^*_1 + p^*_2}{2} \le p^*_1.$$

Case II: $\psi(1) \ne 1$. For notational convenience, let $\kappa := \psi(1) \ge 3$. By (7), (8), and the definition of $\psi(\cdot)$, $p_j\big(1, \tilde{c}_j(1), \ldots, \tilde{c}_n(1)\big) = p^*_j$ for all $j \ge \kappa+1$ and $p_1\big(1, \tilde{\boldsymbol{c}}(1)\big) = \ldots = p_\kappa\big(1, \tilde{c}_\kappa(1), \ldots, \tilde{c}_n(1)\big)$. By the same argument as in Case I, we have that

$$p_1\big(1, \tilde{\boldsymbol{c}}(1)\big) = \frac{\sum_{i=1}^{\kappa} p^*_i}{\kappa} \le p^*_1.$$

Denote the solution to $p_1\big(\mu, \tilde{\boldsymbol{c}}(\mu)\big) = p^*_1$ by μ^*. It remains to show that $\kappa^* := \psi(\mu^*) = 1$. Suppose to the contrary that $\kappa^* \ge 3$. Then

$$p_j\big(\mu^*, \tilde{c}_j(\mu^*), \ldots, \tilde{c}_n(\mu^*)\big) = p^*_j \text{ for all } j \ge \kappa^* + 1,$$

and

$$p_2\big(\mu^*, \tilde{c}_2(\mu^*), \ldots, \tilde{c}_n(\mu^*)\big) = \ldots = p_{\kappa^*}\big(\mu^*, \tilde{c}_{\kappa^*}(\mu^*), \ldots, \tilde{c}_n(\mu^*)\big) < p^*_{\kappa^*},$$

by (8) and the definition of $\psi(\cdot)$. Therefore, we have that

$$\begin{aligned}
p_1\big(\mu^*, \tilde{\boldsymbol{c}}(\mu^*)\big) &= 1 - \sum_{i=2}^{\kappa^*} p_i\big(\mu^*, \tilde{c}_i(\mu^*), \ldots, \tilde{c}_n(\mu^*)\big) - \sum_{i=\kappa^*+1}^{n} p_i\big(\mu^*, \tilde{c}_i(\mu^*), \ldots, \tilde{c}_n(\mu^*)\big) \\
&> 1 - (\kappa^* - 1)p^*_{\kappa^*} - \sum_{i=\kappa^*+1}^{n} p^*_i \\
&\geq 1 - \sum_{i=2}^{\kappa^*} p^*_i - \sum_{i=\kappa^*+1}^{n} p^*_i = p^*_1,
\end{aligned}$$

which is a contradiction against $p_1\big(\mu^*, \tilde{\boldsymbol{c}}(\mu^*)\big) = p^*_1$. Therefore, $p_i\big(\mu^*, \tilde{\boldsymbol{c}}(\mu^*)\big) = p^*_i$ for all $i \in \mathcal{N}$ and $\psi(\mu^*) = 1$.

Step II: We now add headstarts to the contest rule and adjust the weight we have constructed in Step I to satisfy $\mathbb{E}(x_i) = p^*_i v_i$ for all $i \in \mathcal{N}$. Denote the set of multiplicative biases that induces $\boldsymbol{p}^*$ by $\boldsymbol{\alpha}^* \equiv (\alpha^*_1, \ldots, \alpha^*_n)$ and let $\hat{v}^*_i := \alpha^*_i v_i$ for all $i \in \mathcal{N}$. Recall that $\hat{v}^*_1 > \hat{v}^*_2 = \ldots = \hat{v}^*_n$. Moreover, we have that $p^*_1 \hat{v}^*_1 - \mathbb{E}(\hat{x}_1) = \hat{v}^*_1 - \hat{v}^*_2$, and $p^*_i \hat{v}_i = \mathbb{E}(\hat{x}_i)$ for $i \in \{2, \ldots, n\}$. Therefore,

$$\begin{aligned}
\mathbb{E}(x_1) &= \frac{\mathbb{E}(\hat{x}_1)}{\alpha^*_1} = p^*_1 \frac{\hat{v}^*_1}{\alpha^*_1} - \frac{\hat{v}^*_1 - \hat{v}^*_2}{\alpha^*_1} = p^*_1 v_1 - \frac{\alpha^*_1 v_1 - \alpha^*_2 v_2}{\alpha^*_1} < p^*_1 v_1, \\
\mathbb{E}(x_i) &= \frac{\mathbb{E}(\hat{x}_i)}{\alpha^*_i} = p^*_i \frac{\hat{v}^*_i}{\alpha^*_i} = p^*_i v_i, \text{ for } i \in \{2, \ldots, n\}.
\end{aligned}$$

Consider the following contest bias rule $(\boldsymbol{\alpha}^\dagger, \boldsymbol{\beta}^\dagger)$:

$$(\alpha^\dagger_i, \beta^\dagger_i) := \begin{cases} (\alpha^*_1, 0) & \text{for } i = 1, \\ (\alpha^*_i, \alpha^*_1 v_1 - \alpha^*_2 v_2) & \text{for } i \in \{2, \ldots, n\}. \end{cases}$$

It can be verified that a mixed-strategy equilibrium exists in the all-pay auction under the contest rule $(\boldsymbol{\alpha}^\dagger, \boldsymbol{\beta}^\dagger)$, in which player 1 randomizes according to CDF $G_1\left(\alpha^*_1 x_1 - (\alpha^*_1 v_1 - \alpha^*_2 v_2)\right)$ and player $i = 2, \ldots, n$ randomizes according to CDF $G_i\left(\alpha^*_i x_i\right)$. Moreover, the expected effort for player $i \in \mathcal{N}$ is $p^*_i v_i$. This concludes the proof. ■