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# "Reverse" nested lottery contests[☆]

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ABSTRACT

This paper proposes a multi-prize "reverse" nested lottery contest model, which can be viewed as the "mirror image" of the conventional nested lottery contest of Clark and Riis (1996a). The reverse-lottery contest model determines winners by selecting losers based on contestants' one-shot effort through a hypothetical sequence of lotteries. We provide a microfoundation for the reverse-lottery contest from a perspective of (simultaneous) noisy performance ranking and establish that the model is underpinned by a unique performance evaluation rule. We further demonstrate that the noisy-ranking model can be interpreted intuitively as a "worst-shot" contest, in which contestants' performances are evaluated based on their most severe mistakes. The reverse-lottery contest model thus depicts a great variety of widely observed competitive activities of this nature. A handy closed-form solution for a symmetric equilibrium of the reverse-lottery contest is obtained. We show that the winner-take-all principle continues to hold in reverse-lottery contests. Moreover, we find that a reverse-lottery contest elicits more effort than a conventional lottery contest whenever the prizes available to contestants are relatively scarce.


## 1. Introduction

Varieties of economic activities are generically viewed as contests, which include college admissions, rent seeking, R&D races, legal battles, political campaigns, etc. The phenomenal pervasiveness of contests inspires numerous analyses of contenders' strategic behavior under various circumstances, which in turn demands sensible and tractable models of diverse evolving characteristics.

One long-lasting concern in contest modeling is how to formulate the mechanisms that translate contestants' effort outlays into the probabilities of each contestant's receiving each prize. A large chunk of the literature is anchored to a winner-take-all lottery contest model, which is built on the well-known ratio-form *Contest Success Function* (CSF).[3] With $I$ contestants, the likelihood that a contestant $i$ will win the single prize is given by

$$p_i(\mathbf{x}) = \frac{g_i(x_i)}{\sum_{j=1}^{I} g_j(x_j)}, \tag{1}$$

where $x_j$ is the effort contributed by an arbitrary contestant $j$ and the *impact function* $g_j(\cdot)$ increases with one's own effort.[4] Based on the single-winner lottery model, Clark and Riis (1996a) propose a multi-winner nested lottery contest model to allow a block of prizes to be distributed in a one-shot contest, which has become one of the most prominent frameworks for modeling multi-prize imperfectly discriminatory contests.[5] To give away $L \leq I$ prizes $\mathbf{V} = (V_1, \ldots, V_L)$, the contest implements a sequence of $L$ independent *hypothetical* lotteries, with each picking one recipient. All

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3 The Tullock CSF (Tullock, 1980), which has been widely adopted and analyzed in the literature, is a special case of the ratio-form CSF.

4 Skaperdas (1996) and Clark and Riis (1997) axiomatize this model. Clark and Riis (1996b) point out the stochastic equivalence of a random choice model based on McFadden (1973, 1974) and a winner-take-all lottery contest. This equivalence result is rediscovered by Jia (2008).

5 Multi-prize lottery contest models have been applied in Clark and Riis (1996a, 1998), Amegashie (2000), Yates and Heckelman (2001), Szymanski and Valletti (2005), Fu and Lu (2009, 2012a,b), Schweinzer and Segev (2012), etc.

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contestants commit to their one-shot efforts. The first winner is chosen among all contestants by a lottery defined by CSF (1). He is awarded the top prize $V_1$ and is removed from the pool of contestants eligible for other prizes. The recipient of the second prize, $V_2$, is subsequently picked by a lottery among the remaining contestants. This procedure repeats until the last prize $V_L$ is given away.[6]

The nested lottery contest model depicts a mechanism that identifies prize recipients directly among eligible candidates (Clark and Riis, 1996a): One wins a prize when being picked by a hypothetical lottery in a series of successive draws. Many real-world competitions, however, implement an opposite procedure in their decision processes. Consider, for example, the bids to host the 2012 Olympics. Moscow was identified as the least competitive candidate and eliminated first. New York was eliminated in the second round, and Madrid and Paris in the third and fourth rounds, respectively. London was the only candidate left. The contest identifies losers directly and eliminates them successively: One wins a prize only when he is not excluded.[7] It should be noted that the competition among the five cities is a simultaneous contest with one-shot effort. Cities submit and present their proposals before IOC members' voting, and are not allowed to revise their bids (i.e., bidding proposals and promotional campaign messages) during the elimination process. Candidates exert a one-shot effort throughout the entire process despite the multi-stage selection procedure.

To provide an account of such a decision process based on one-shot effort of contestants, we propose a "reverse" nested lottery contest, which selects winners indirectly by excluding losers directly using a series of hypothetical lotteries. A model of the reverse selection procedure requires a function that specifies the probability of one's being identified as a loser given the effort of contestants. For this purpose, we first propose a ratio-form *Contest Elimination Function* (CEF). The likelihood of a constant $i$'s being selected as a loser amongst $I$ contenders also takes a ratio form, which is formally expressed as

$$q_i(\mathbf{x}) = \frac{[g_i(x_i)]^{-1}}{\sum_{j=1}^{I}[g_j(x_j)]^{-1}}, \tag{2}$$

where the *impact function* $g_i(\cdot)$ increases with one's own effort. As a result, *ceteris paribus*, one reduces the likelihood of his losing by stepping up his effort.

Analogous to Clark and Riis (1996a), the reverse nested lottery contest literally employs a sequence of independent lotteries to give away $L$ prizes. Contestants place their one-shot efforts, and each lottery picks one loser by the CEF (2). This contestant is eliminated from the pool of candidates eligible for subsequent draws. The next loser is picked from the remaining pool by CEF (2). A contestant is given the bottom prize, $V_L$, if and only if he is picked by the $(I-L+1)$th lottery and is then eliminated from the slate. The contestant who is picked in the next lottery receives the second-to-bottom prize, $V_{L-1}$. Based on the sunk bids (e.g., Olympics host-city candidates' proposals and their promotional campaign efforts), the selection procedure repeats for $I-1$ rounds to determine the ranks of all contestants, and the last survivor is awarded the top prize $V_1$.

The reverse nested lottery contest (which will also be called the "reverse-lottery contest" for simplicity in the remainder of the paper), to a large extent, can be viewed as the mirror image of Clark and Riis (1996a), who also model multi-prize contests with one-shot effort. By their hypothetically sequential selection procedures, the latter contest successively selects winners, and contestants step up their bids to be picked in early draws. In contrast, the reverse-lottery contest successively selects losers, and contestants are compelled to exert effort to survive more rounds of elimination and thereby attain higher ranks.

In this paper, we explore three main aspects of the reverse nested lottery contest model.

First, we provide microfoundations for the reverse-lottery contest model and its underpinning CEF by considering a noisy performance ranking model, which is a generalized variant of Hirshleifer and Riley (1992).[8] We establish a unique stochastic equivalence between the reverse-lottery contest and the alternative noisy-ranking contest.

Second, we also show that the noisy-ranking model can be interpreted as a "worst-shot" contest in which contestants are evaluated by the most severe mistakes they make in performing a given task. In other words, the performance evaluation rule based on worst shots embodies the "wooden barrel" principle: The shortest plank determines the amount of water held in a wooden barrel.

Third, with homogeneous contestants, we establish the conditions for the unique symmetric pure-strategy equilibrium in a reverse-lottery contest and provide a handy close-form solution to the equilibrium. The link between the reverse-lottery contest model and the conventional nested lottery contest of Clark and Riis (1996a) is further explicated. We then discuss several potential applications of and implications for our equilibrium results.

This study complements the existing literature on contests in several respects. The reverse-lottery contest model depicts a different competitive environment from those abstracted by a conventional lottery-contest model. As demonstrated by Baye and Hoppe (2003) and Fu and Lu (2012b), a conventional lottery contest is underpinned by a best-shot contest in which a contestant's performance is evaluated by the most favorable shock to his performance, e.g., the research tournament model of Fullerton and McAfee (1999). The two types of contest models must be matched to different contexts.

The rest of this paper proceeds as follows. In Section 2, we formally set up the reverse-lottery contest model and derive its stochastic microfoundation. Section 3 reveals the economic implications of the reverse-lottery contest through an equivalent worst-shot contest. Section 4 presents the symmetric equilibrium solution to a reverse-lottery contest and discusses two main applications. A concluding remark is provided in Section 5.

## 2. The reverse nested lottery contest model and its microfoundation

### 2.1. Reverse nested lottery contest model

A reverse nested lottery contest involves $I$ ($\geq 2$) contestants and gives away $I$ (weakly) decreasing nonnegative prizes ($V_k$, $k = 1, 2, \ldots, I$), with $V_1 \geq \cdots \geq V_I$ and $V_1 > V_I$.[9] The ordered set $\mathbf{I} \triangleq \{1, 2, \ldots, I\}$ refers to contestants participating in the contest. A vector $\mathbf{x} = (x_1, x_2, \ldots, x_I)$ denotes the one-shot effort entries of the entire set of contestants. There is a set of *impact functions*

[6] Fu and Lu (2012b) uncover a (simultaneous) noisy-ranking mechanism that underpins both the single-lottery CSF (1) and the *hypothetical* sequential-lottery mechanism of Clark and Riis (1996a). The ranking mechanism unifies the single-winner and multi-winner lottery models under the same umbrella.

[7] A similar procedure is used in many beauty contests and TV talent shows: Judges pick the winners in a competition by identifying and eliminating losers.

[8] In the noisy ranking model, each contestant's observable output is the sum of a deterministic component, which increases with his effort, and a random noise term. The contest ranks contestant's observable outputs in a descending order, i.e., one attains a better rank by contributing larger output.

[9] With $V_k = 0, \forall k \in \{2, \ldots, I\}$, the setting boils down to a winner-take-all contest.

$\mathbf{g} = (g_1(\cdot), g_2(\cdot), \ldots, g_I(\cdot))$, with $g_i(0) = 0$ and $g_i'(\cdot) > 0$, $\forall i \in \mathbf{I}$. The ordered set $\mathbf{g}$ depicts the technology of the contest. For a given $\mathbf{x}$, a stochastic prize allocation plan is specified as follows.

The prize allocation procedure begins with a lottery among all contestants. The lottery takes $[g_i(x_i)]^{-1}$ as a contestant $i$'s entry. Define $[g_j(x_j = 0)]^{-1}$ as infinity for any $j \in \mathbf{I}$. Let $\#(j|x_j = 0)$ be the count of zero effort entries in $\mathbf{x}$. Each contestant $i$ is picked by the lottery with a probability $\frac{[g_i(x_i)]^{-1}}{\sum_{j=1}^{I}[g_j(x_j)]^{-1}}$ if $x_i \neq 0$, while he would be picked with a probability $\frac{1}{\#(j|x_j=0)}$ otherwise. In particular, when there is a tie on $g_i(x_i)$ across contestants, every contestant is picked up with an equal probability.[10, 11] The selected contestant receives the smallest prize $V_I$ and is eliminated from the pool eligible for remaining prizes. Let $\Omega_k$, with $k = I-1, I-2, \ldots, 2, 1$, denote the set of remaining contestants eligible for all remaining prizes $V_{k'}$, with $k' \leq k$.

A prize $V_{I-1}$ is allocated by a lottery among all remaining contestants $\Omega_{I-1}$. The lottery continues to take $[g_i(x_i)]^{-1}$ as an eligible contestant $i$'s entry. He receives $V_{I-1}$ with a probability $\frac{[g_i(x_i)]^{-1}}{\sum_{j\in\Omega_{I-1}}[g_j(x_j)]^{-1}}$ if $x_i \neq 0$, while he receives it with probability $\frac{1}{\#(j|x_j=0, j\in\Omega_{I-1})}$ otherwise, where $\#(j|x_j = 0, j \in \Omega_{I-1})$ is the count of zero effort entries in $\mathbf{x}_{\Omega_{I-1}}$.[12] The recipient of prize $V_{I-1}$ is eliminated from the pool for the subsequent draw, which is to give away prize $V_{I-2}$ by picking one recipient from the pool of eligible contestants $\Omega_{I-2}$.

The process continues for $I-1$ rounds. By the procedure sketched in the preceding paragraph, each prize $V_k$, with $k \in \{2, \ldots, I\}$, is given away to one contestant through the $(I+1-k)$th lottery. The allocation of the top prize $V_1$ is determined automatically when the recipient of $V_2$ is picked in the last draw.

We call $\frac{[g_i(x_i)]^{-1}}{\sum_{j\in\Omega_k}[g_j(x_j)]^{-1}}$, $i \in \Omega_k$, a ratio-form *Contest Elimination Function* (CEF), which differentiates it from the widely used concept of *Contest Success Function* (CSF). Let the sequence $\{i_k\}_{k=1}^{I}$, with $i_k \in \mathbf{I}$, denote a prize allocation outcome, i.e., contestant $i_k$ receives prize $V_k$ (i.e., being picked in the $(I-k+1)$th draw). For given $\mathbf{x} > \mathbf{0}$, the likelihood of this allocation outcome can be expressed as

$$p(\{i_k\}_{k=1}^{I}) = \Pi_{k=1}^{I}\frac{[g_{i_k}(x_{i_k})]^{-1}}{\sum_{k'=1}^{k}[g_{i_{k'}}(x_{i_{k'}})]^{-1}}. \tag{3}$$

As aforementioned, this reverse nested lottery contest model resembles a mirror image of a conventional nested-lottery contest (Clark and Riis, 1996a). It picks losers successively by the "single-loser" CEF (2).

### 2.2. Microfoundation: a noisy-ranking perspective

We consider the following noisy performance ranking contest, which would pave a foundation for the reverse nested lottery contest.

There are $I \geq 2$ contestants, indexed by $i \in \mathbf{I} \triangleq \{1, 2, \ldots, I\}$. They simultaneously submit their one-shot effort entries $\mathbf{x} = \{x_1, \ldots, x_I\}$, to compete for one of $I$ prizes. The $I$ prizes are ordered by their values, with $V_1 \geq V_2 \geq \cdots \geq V_I$ and $V_1 > V_I$. A contestant $i$ supplies effort $x_i$, which allows him to produce a composite output $y_i$. It can be described by

$$\log y_i = \log g_i(x_i) + \varepsilon_i, \quad \forall i \in \mathbf{I}. \tag{4}$$

According to (4), the perceivable output $\log y_i$ of contestant $i$ includes two parts: the deterministic output function $\log g_i(x_i)$, which is strictly increasing in $x_i$, and an additive noise term $\varepsilon_i$, which is randomly drawn from a Type-I extreme value (minimum) distribution with cumulative distribution function (c.d.f.)

$$F(\varepsilon) = 1 - e^{-e^{\varepsilon}}, \quad \varepsilon \in (-\infty, +\infty). \tag{5}$$

Contestants are ranked by their perceivable outputs in descending order. Prizes are accordingly allocated by their ranks. Ties are broken randomly.

This noisy-ranking (minimum) model is similar to the main setting of Fu and Lu (2012b).[13] They differ from each other only in the underlying distributions of their additive noise terms. The current model assumes a Type-I extreme value (minimum) distribution, while Fu and Lu (2012b) assume a Type-I extreme value (maximum) distribution.[14] Fu and Lu (2012b) demonstrate that the noisy-ranking (maximum) model is stochastically equivalent to the conventional nested lottery contest built on the ratio-form CSF (1).[15]

The following Theorems 1–3 can be obtained by establishing a statistical link between the current model to the noisy-ranking model of Fu and Lu (2012b). For brevity, all the derivations are relegated to Appendix A.

**Theorem 1.** *Consider a given $\mathbf{x} > \mathbf{0}$ such that $g_i(x_i) > 0$, $\forall i \in \mathbf{I}$. The probability that contestant i achieves the lowest performance and is ranked at the bottom in model (4) is*

$$p(i|\mathbf{x}) = \frac{[g_i(x_i)]^{-1}}{\sum_{j\in\mathbf{I}}[g_j(x_j)]^{-1}}, \quad \forall i \in \mathbf{I}. \tag{6}$$

**Proof.** See Appendix A. ■

The probability expression (6) depicts the probability of one's performance ($\log y_i$) being ranked the lowest in model (4), which is given by the ratio of the inverse of one's deterministic output, $[g_i(x_i)]^{-1}$, to the sum $\sum_{j\in\mathbf{I}}[g_j(x_j)]^{-1}$. Apparently, expression (6) coincides with the ratio-form CEF (2).

We then consider the probabilistic distribution of all possible complete rankings for a given set of effort entries $\mathbf{x} > \mathbf{0}$. Let the sequence $\{i_k\}_{k=1}^{I}$ denote a complete ranking of all $I$ contestants in model (4), where $i_k$ is the contestant in the $k$th position, i.e., contestant $i_k$ has the $k$th highest observed output.

**Theorem 2.** *In a noisy-ranking (minimum) contest model (4), for any given effort entries $\mathbf{x} > \mathbf{0}$ such that $g_i(x_i) > 0$, $\forall i \in \mathbf{I}$, the likelihood of any complete ranking outcome $\{i_k\}_{k=1}^{I}$ can be expressed as*

$$p(\{i_k\}_{k=1}^{I}) = \Pi_{k=1}^{I}\frac{[g_{i_k}(x_{i_k})]^{-1}}{\sum_{k'=1}^{k}[g_{i_{k'}}(x_{i_{k'}})]^{-1}}. \tag{7}$$

10 Note that when $x_i \neq 0$ and $\#(j|x_j = 0) \geq 1$, we have $\frac{[g_i(x_i)]^{-1}}{\sum_{j=1}^{I}[g_j(x_j)]^{-1}} = 0$ since $[g_j(x_j = 0)]^{-1}$ is defined as infinity. Those who contribute zero effort have an equal chance of being chosen as the loser in the lottery.

11 Similar to a standard lottery contest with ratio-form success function, contestants with equal positive "output," i.e., $g_i(x_i) = g_j(x_j) \neq 0$, will be chosen as the loser with equal chance, provided that all other contestants make positive effort.

12 Because $[g_j(x_j = 0)]^{-1}$ is defined as infinity, when $x_i \neq 0$ and $\#(j|x_j = 0, j \in \Omega_{I-1}) \geq 1$, we have $\frac{[g_i(x_i)]^{-1}}{\sum_{j\in\Omega_{I-1}}[g_j(x_j)]^{-1}} = 0$. That is, a contestant will not be selected as a loser if he places positive bids while some of other contestants make zero effort.

13 The term "(minimum)" refers to the fact that the noise term follows a Type-I extreme value (minimum) distribution.

14 They belong to the same family of extreme value distributions (Gumbel distributions), but lead to contrasting implications: The former depicts the asymptotic distribution for the minimum of a large number of identically distributed random variables, while the latter depicts that for the maximum.

15 Similarly, the term "(maximum)" is used to indicate that the additive noise term follows a Type-I extreme value (maximum) distribution.

**Proof.** See Appendix A. ■

(7) gives the ex ante likelihood of any complete ranking, which is simply the cumulative product of all the terms included in the series $\{[g_{i_k}(x_{i_k})]^{-1}/\sum_{k'=1}^{k}[g_{i_{k'}}(x_{i_{k'}})]^{-1}\}_{k=1}^{I}$. When the noise terms follow a Type-I extreme value (minimum) distribution, $[g_{i_k}(x_{i_k})]^{-1}/\sum_{k'=1}^{k}[g_{i_{k'}}(x_{i_{k'}})]^{-1}$ gives the conditional probability of contestant $i_k$'s output being the lowest among the top $k$ highest ones, given that contestants $\{i_k\}_{k'=1}^{k}$ are the top $k$ highest ones.[16]

The result of (7) reveals the link of the noisy-ranking (minimum) contest model to the reverse-lottery contest model. Recall that in the reverse-lottery contest model, the ex ante likelihood of a prize allocation outcome $p(\{i_k\}_{k=1}^{I})$ is given by (3), which coincides with (7) in the noisy-ranking (minimum) contest model (4). Therefore, as implied by Theorems 1–2, the hypothetical sequential lottery process applied in the reverse-lottery contest model in fact reflects a statistical property of a hidden simultaneous noisy-ranking rule implied in the noisy-ranking contest model. The probabilistic prize allocation mechanism, as described by (3), does not necessarily rely on a sequentially implemented selection procedure.

**Theorem 3.** *When $I \geq 3$, the noisy-ranking (minimum) contest model (4) is stochastically equivalent to the reverse-lottery-contest model $R(\mathbf{I}, \mathbf{g}(\cdot), \mathbf{V})$ if and only if $\varepsilon_i$ follows a Type-I extreme value (minimum) distribution with c.d.f. of $F(\varepsilon) = 1 - e^{-e^{\varepsilon+b}}$, with $\varepsilon \in (-\infty, +\infty)$ and $b \in \mathbb{R}$.*[17]

**Proof.** See Appendix A. ■

Theorem 3 establishes the unique stochastic equivalence of the noisy-ranking (minimum) contest model (4) with the family of multi-prize reverse lottery contests built on ratio-form CEF (2). The theorem provides a microeconomic underpinning for the reverse-lottery contest: The reverse nested lottery contest model is underpinned by a unique noisy-ranking system.

## 3. "Worst-shot" contests

We set up an alternative but equivalent worst-shot contest, which intuitively interprets the statistical nature of the Type-I extreme value (minimum) distribution and reveals the economic implications and practical relevance of our contest models from a different perspective. For this purpose, we transform the noisy-ranking (minimum) contest model (4) into a direct variant, which is formulated originally by Hirshleifer and Riley (1992).[18] Let contestants be ranked by $y_i$ with

$$y_i = g_i(x_i) \times \tilde{\varepsilon}_i \tag{8}$$

in descending order. Here, $\tilde{\varepsilon}_i$ is defined as $\tilde{\varepsilon}_i = \exp \varepsilon_i$. With $\varepsilon_i$ drawn from a Type-I extreme value (minimum) distribution, $y_i$ follows a Weibull (minimum) distribution with c.d.f.

$$\Pr(y_i \leq y) = 1 - e^{\frac{-y}{g_i(x_i)}}. \tag{9}$$

We further define a uniform *decreasing* transformation of $y_i$ with

$$m_i = F_0^{-1}(e^{-y_i}), \quad i = 1, 2, \ldots, I,$$

where $F_0(\cdot) : \mathbb{R}_+ \to [0, 1]$ is a cumulative distribution function of a continuous random variable. The random variable $m_i$ is independently distributed. Because $m_i$ is *decreasing* in $y_i$, ranking contestants by $y_i$ in descending order is equivalent to ranking them by $m_i$ in ascending order. That is, the lower the realized value of $m_i$, the higher the contestant $i$'s rank. The c.d.f. of $m_i$ is derived as follows:

$$\begin{aligned}\Pr(m_i \leq m) &= \Pr(F_0^{-1}(e^{-y_i}) \leq m)\\ &= \Pr(e^{-y_i} \leq F_0(m))\\ &= \Pr(y_i \geq -\log F_0(m))\\ &= 1 - [1 - \exp(\log F_0(m)/g_i(x_i))]\\ &= \exp(\log F_0(m)/g_i(x_i))\\ &= F_0(m)^{\frac{1}{g_i(x_i)}}.\end{aligned} \tag{10}$$

If we interpret $F_0(\cdot)$ as the distribution of the magnitude of a mistake whenever it happens, expression (10) can then be interpreted as the c.d.f. of the highest-order statistic of $[g_i(x_i)]^{-1}$ random mistakes. The above statistical transformation thus alludes intuitively to a worst-shot contest, in which each contestant is evaluated based on his least favorable performance, i.e., his biggest mistake. Apparently, extra effort decreases $[g_i(x_i)]^{-1}$ (i.e., the number of mistakes) and causes a downward shift of probability mass in the distribution of one's perceived defect, which reflects the conventional wisdom that "practice (more effort) makes perfect (fewer mistakes)". Contestants are then incentivized to improve the reliability of their performance.

By the statistical transformation from noisy-ranking (minimum) contest model (4)–(10), and the above definition of a worst-shot contest, the following is immediate.

**Theorem 4.** *The worst-shot contest with an arbitrary distribution of the underlying mistake distribution $F_0(\cdot)$ is stochastically equivalent to the noisy-ranking contest model with a noise term following the Type-I extreme value (minimum) distribution. Hence, a worst-shot contest is also stochastically equivalent to a reverse-lottery contest.*

Worst-shot contests resemble common real-world competitive situations, in which contenders' performance is judged by how closely they meet predefined standards, and one's worst mistake matters the most in his ultimate performance measure—e.g., competitions in acrobatics, diving, and gymnastics. Alternatively, a worker's performance is often required to closely follow codes of practice in the workplace, while a product's rating often depends on how well it meets existing technological standards.[19]

The statistical nature and economic logic of the noisy-ranking (minimum) contest model, as well as those of the reverse-lottery contest, therefore unfold in this transparent and lucid alternative environment of worst-shot contests. This statistical foundation explicates the proper scope of application for a reverse-lottery-contest model.

## 4. Symmetric equilibrium

### 4.1. Symmetric equilibrium in reverse-lottery contest

In this part, we characterize the bidding equilibrium in a reverse-lottery contest, and depict the conditions that guarantee equilibrium existence. For the sake of tractability, we focus on a symmetric setting. Contestants are homogeneous in the sense that $g_i(x) = g(x), \forall i \in \mathbf{I}$, and they have the same linear cost function $c(x) = x$. We focus on the symmetric pure-strategy equilibrium of the game, in which all contestants exert the same amount of effort

16 It should be emphasized that both Theorems 1 and 2 are derived from the noisy-ranking (minimum) contest model (4) where players' efforts are made simultaneously and they are ranked by their perceivable outputs. In contrast, the reverse nested lottery contest literally employs a sequence of (hypothetical) independent lotteries to give away multiple prizes.

17 Note that $R(\mathbf{I}, \mathbf{g}(\cdot), \mathbf{V})$ denotes a reverse nested lottery contest with $\mathbf{I} = \{1, 2, \ldots, I\}$ contestants, output functions $\mathbf{g}(\cdot) = (g_1(x_1), g_2(x_2), \ldots, g_I(x_I))$, and prizes $\mathbf{V} = (V_1, V_2, \ldots, V_I)$.

18 A two-player version of this multiplicative-noise ranking contest is further discussed by Konrad (2009), as well as Fu and Lu (2012b).

19 The quality of engineering projects, such as road and utility construction, is measured against preset standards, and industrial-equipment manufacturers go to great lengths to reduce the failure rates of their products, which are the key determinant of a manufacturer's market success.

$x^*$.[20] To provide a sensible solution to the reverse-lottery contest, we impose a mild regularity condition on contest technologies. We assume that the impact function is log-concave, which implies that $\frac{g(\cdot)}{g'(\cdot)}$ is increasing in its argument.

We first establish the conditions for the existence of such an equilibrium. For this purpose, we focus on the most widely adopted class of impact functions $g(x) = x^r$, due to the technical difficulties that typically arise from a general impact function. Skaperdas (1996), however, shows that the CSF with an impact function $x^r$ is the only additive CSF that is homogeneous of degree zero.[21] We first consider a single-prize (i.e., winner-take-all) reverse-lottery contest.

**Theorem 5.** *In a winner-take-all reverse-lottery contest model with $g(x) = x^r$, a (unique) symmetric pure-strategy equilibrium (SPSE), exists if and only if the discriminatory parameter $r$ falls below a cutoff $r^{pc}$, i.e., $r \leq r^{pc}$, where*

$$r^{pc} = \frac{1}{-1 + \sum_{t=0}^{I-1} \frac{1}{I-t}}. \tag{11}$$

**Proof.** See Appendix B. ■

Theorem 5 provides a sufficient and necessary condition for the existence of a SPSE. Analogous to its counterpart in a conventional lottery contest, the SPSE requires a moderate discriminatory parameter $r$. The upper bound $r^{pc}$ for $r$ simply ensures that players' *participation constraints* are satisfied. That is, each player receives a nonnegative expected payoff whenever all of them exert equilibrium effort in the SPSE. The bound $r^{pc}$ depends on the number of contestants. When $I$ increases, $\sum_{t=0}^{I-1} \frac{1}{I-t}$ must increase, which lowers $r^{pc}$ and makes the SPSE less likely.

As shown by Perez-Castrillo and Verdier (1992), in a conventional lottery contest with a winner-take-all prize structure and $g(x) = x^r$, a unique SPSE exists if and only if the discriminatory parameter $r$ falls below a cutoff $r^{pc\prime}$, i.e., $r \leq r^{pc\prime}$, where

$$r^{pc\prime} = \frac{1}{1 - 1/I}.\text{[22]} \tag{12}$$

The conventional wisdom holds that a more discriminatory contest, i.e., a larger $r$, leads to excessive rent dissipation. Hence, a moderate $r$ is required to rein in contestants' effort incentives, thereby maintaining the participation constraints in the game. The logic applies in both conventional and reverse lottery contests.

Comparing (11) with (12), for a given $r$, a SPSE is less likely to exist in the reverse lottery contest than in a conventional lottery contest because $r^{pc} < r^{pc\prime}$ for all $I \geq 3$.[23] Our subsequent analysis (Section 4.2) sheds further light on this observation. Corollary 1 shows that under a winner-take-all structure, contestants exert more effort in a reverse lottery contest than in a conventional lottery contest. Hence, contestants' participation constraints are more likely to break down in a reverse lottery contest than in a conventional lottery contest, which explains the observation of $r^{pc} < r^{pc\prime}$.[24]

We further consider a more general setting that allows for multiple decreasing positive prizes.

**Theorem 6.** *Consider a reverse lottery contest model with $g(x) = x^r$ and $L(\leq I)$ decreasing positive prizes, i.e., $V_1 \geq \cdots \geq V_L > 0$ and $V_j = 0$ if $j > L$. A symmetric pure-strategy equilibrium (SPSE) exists if the following conditions hold: (C1) $r \leq 1/(I-1)$; and (C2) $L \leq I - \underline{k} + 1$, where $\underline{k}$ is the minimal $k$ such that*[25]

$$\sum_{t=0}^{k} \frac{1}{I-t} > 1.$$

**Proof.** See Appendix B. ■

Theorem 6 states that in a more general multi-prize setting, a unique SPSE exists if the contest is not excessively discriminatory and the number of positive prizes is not excessively large. A few remarks are in order. First, Theorem 6 requires that $r \leq 1/(I-1)$, which is a much stronger restriction on the size of $r$ than that for the winner-take-all contest, i.e., $r \leq r^{pc}$: By (11), it is straightforward to observe $r^{pc} > 1/(I-1)$. Second, the result applies to a contest with an arbitrary number of contestants, and the restriction on prize structure is rather weak. Third, Theorem 6 provides a sufficient condition for the existence of the SPSE in a multi-prize setting. Neither (C1) nor (C2) is necessary for the existence of SPSE.

Denote by $x_r^*$ each individual contestant's equilibrium effort in a reverse-lottery contest. The SPSE requires that the equilibrium effort $x_r^*$ globally maximize a representative player's expected payoff, provided that others exert effort $x_r^*$. With impact function $g(x) = x^r$ and $I \leq 100$, our numerical simulations verify that for any sequence of weakly decreasing prizes, $x_r^*$ indeed constitutes a global maximizer as long as $r \leq r^{pc}$ holds, in which case neither (C1) nor (C2) is required.[26]

**Theorem 7.** *Suppose that a symmetric pure-strategy equilibrium exists for a reverse-lottery contest with (weakly) decreasing prizes $V_k, k = 1, 2, \ldots, I$ and increasing impact functions $g_i(x) = g(x), \forall i \in \mathbf{I}$. In a SPSE, each contestant makes an effort*

$$x_r^* = H^{-1}\left(\frac{1}{I}\sum_{k=1}^{I} V_k \tilde{c}_k\right), \tag{13}$$

*where*

$$H(x) \equiv \frac{g(x)}{g'(x)}; \qquad \tilde{c}_k = -1 + \sum_{j=0}^{I-k} \frac{1}{I-j}. \tag{14}$$

**Proof.** See Appendix B. ■

The solution of (13) depicts the symmetric pure-strategy equilibrium of the contest whenever such an equilibrium exists. Note that the log-concavity of $g(\cdot)$ ensures that this equilibrium is unique whenever it exists. When the designer has full flexibility to allocate a budget $\Gamma > 0$ among the $I$ nonnegative prizes, (13) implies that the optimal prize allocation (i.e., that maximizes total effort) must follow the winner-take-all principle, i.e., $V_1 = \Gamma$ and $V_k = 0, \forall k \geq 2$.[27]

### 4.2. Reverse lottery contest versus conventional lottery contest

The results of Theorem 5 allow us to compare contestants' equilibrium behaviors under different competitive environments.

20 We can allow for a more generally specified effort cost function $c(x)$. However, this setting is strategically equivalent to an alternative contest, in which contestants bear linear effort cost, while having an impact function $h(\cdot) = g(c^{-1}(\cdot))$.

21 Contests with an impact function $x^r$ are well-known as generalized Tullock contests.

22 Similar to its counterpart in the reverse lottery contest, the bound $r^{pc\prime}$ decreases as $I$ increases, i.e., an increase in $I$ lowers $r^{pc}$ and makes the SPSE less likely.

23 The two types of the contests (i.e., the conventional and reverse nested lottery contests) converge if and only if $I = 2$.

24 In Section 4.2, more general results (see Theorems 8 and 9) are derived and the intuition behind the results is also discussed.

25 One can verify that roughly $\underline{k} \approx 0.632 \cdot I$.

26 Note that $r^{pc} > \frac{1}{I-1}$.

27 The winner-take-all principle has been identified in a number of studies in different settings, such as Moldovanu and Sela (2001), Clark and Riis (1998), Fu and Lu (2009, 2012a).

In this part, we explore whether contestants are better incentivized in a reverse-lottery contest than in a conventional lottery contest. Intuitions underlying the difference in their performance will be illustrated by the natures of the two contests.

To facilitate the comparison, let us restate the existing equilibrium solutions to conventional lottery contests. Denote by $x_c^*$ a contestant's equilibrium effort in a conventional lottery contest.

**Lemma 1** (*Fu and Lu, 2009*). *In a symmetric pure-strategy equilibrium for a conventional lottery contest of Clark and Riis (1996a) with prizes $V_k$, $k = 1, 2, \ldots, I$, and impact functions $g_i(\cdot) = g(\cdot)$, $\forall i \in \mathbf{I}$, each contestant makes an effort*

$$x_c^* = H^{-1}\left(\frac{1}{I}\sum_{k=1}^{I} V_k c_k\right), \tag{15}$$

*with*

$$H(x) \equiv \frac{g(x)}{g'(x)} \quad \text{and} \quad c_k = 1 - \sum_{i=0}^{k-1}\frac{1}{I-i}. \tag{16}$$

By (13) and (15), the equilibrium effort in either contest can be expressed as an increasing function of a weighted sum of the entire set of prizes. A weight $\tilde{c}_k$ is attached to each prize $V_k$ in a reverse-lottery contest, while $c_k$ is assigned to $V_k$ in a conventional lottery contest. Such weight indicates the role played by prize $V_k$ in eliciting effort, and measures the intensity of incentives provided by the prize. One key fact deserves to be noted before the formal analysis of effort comparison.

**Remark 1.** $\tilde{c}_k = -c_{I+1-k}$.

Remark 1 highlights a "symmetry" between the two contest models, which further confirms the mirror-image relation between the conventional lottery contest and the reverse lottery contest. The statistical underpinning of this mirror-image relation is illustrated in more detail by the analysis we present in Appendix A.

Further, we compare $x_c^*$ with $x_r^*$. The symmetry highlighted by Remark 1, which results from the "mirror-image" relation, largely facilitates the analysis. We focus on contests with $I \geq 3$ players.[28]

The comparison begins with a simple case with $L \in \{1, \ldots, I-1\}$ homogeneous strictly positive prizes, i.e., $V_1 = V_2 = \cdots = V_L = V > 0$. We obtain the following.

**Theorem 8.** (i) *A reverse-lottery contest elicits strictly more (strictly less) effort than a conventional lottery contest if and only if the number of prizes L is strictly less (strictly more) than $\frac{I}{2}$, i.e., $x_r^* \gtrless x_c^*$ if and only if $L \lessgtr \frac{I}{2}$.*
(ii) *When the number of contestants, I, is an even number, the two contests elicit the same amount of effort if $L = \frac{I}{2}$, i.e., $x_r^* = x_c^*$ if $L = \frac{I}{2}$.*

**Proof.** See Appendix B. ■

With homogeneous prizes, the ranking between $x_c^*$ and $x_r^*$ depends entirely on the number of prizes available to contestants. A reverse-lottery contest better incentivizes effort supply when prizes are relatively scarce, i.e., when no more than half of contenders can eventually be rewarded. However, the prediction is reversed when prizes are abundant.

This result immediately sheds light on the widely observed case of winner-take-all contests. By Corollary 1, a winner-take-all prize structure also maximizes total effort in reverse-lottery contests. Theorem 8 leads to the following.

[28] As we pointed out earlier, the two types of the contests (i.e., the conventional and reverse nested lottery contests) converge if and only if $I = 2$.

**Corollary 1.** *Under a winner-take-all prize structure, contestants exert more effort in a reverse-lottery contest than in a conventional lottery contest.*

The logic behind the results in Theorem 8 can be interpreted in light of the difference between the two hypothetical sequential lottery procedures, which results from their different statistical foundations. In a conventional lottery contest with $L$ homogeneous prizes and $I$ contestants, one strives to improve the likelihood of being picked within the first $L$ lotteries (to get a prize $V$). Only the outcomes of the first $L$ lotteries affect his payoffs. By contrast, a contestant in a reverse-lottery contest strives to survive $I - L$ rounds of elimination to win a prize $V$. His payoff is affected by the outcomes of the $I - L$ draws. When $L < I/2$, we have $I - L > L$. A contestant expects a higher marginal benefit from his effort in a reverse-lottery contest than his counterpart in a conventional one: One's effort in the former could remain "effective" for more rounds of lotteries than in the latter. Higher marginal benefits thus compel contestants to contribute more effort. The same logic applies to the opposite case of $L > I/2$ and $I - L < L$. In this case, one's effort "expires" sooner in a reverse-lottery contest than in a conventional lottery contest, which leads to $x_r^* < x_c^*$. In the knife-edge case of $L = I/2$, the two types of contests must be equally effective because $I - L = L$.

In a sequential-elimination contest, Rosen (1986) shows that an extra large prize in the final stage are required to incentivize stable (constant or increasing) effort supply along the hierarchical ladder of the contest. Our setup differs critically from that of Rosen: Rosen analyzes an $n$-stage contest in which surviving contestants repeatedly exert effort when advancing toward the finale; we instead study a static contest in which contestants make one-shot effort. Despite the different setups, our result to some extent can be related to that of Rosen: Under the hypothetical sequential loser-selection procedure of the reverse lottery contest, the top prize incentivizes contestants more than other prizes as players have to survive $I - 1$ (hypothetical) rounds of elimination to attain it.

A more subtle comparison results when prizes are heterogeneous. The comparison reaches a definitive conclusion when the number of prizes ($L$) is relatively small compared to the number of contestants ($I$).

**Theorem 9.** *Consider a contest with heterogeneous decreasing prizes $V_1 \geq V_2 \geq \cdots \geq V_L > 0$ and $V_1 > V_L$. A reverse-lottery contest elicits more effort than its conventional lottery counterpart whenever $L \leq \frac{I}{2}$, i.e. $x_r^* > x_c^*$ if $L \leq \frac{I}{2}$.*

**Proof.** See Appendix B. ■

Theorem 9 states that, with heterogeneous prizes, a reverse-lottery contest continues to dominate a conventional lottery contest when prizes are relatively scarce, i.e., when $L \leq I/2$. Furthermore, it implies that a conventional lottery contest could elicit more effort only if prizes are abundant, i.e., when $L > I/2$.

The logic that underpins Theorem 8 continues to shed light on Theorem 9. As aforementioned, one's effort in a reverse-lottery contest remains effective for more rounds of lotteries when prizes are scarce, which leads to a higher marginal benefit for his bid than its counterpart in a conventional lottery contest. Heterogeneous prizes amplify the additional incentives in a reverse-lottery contest. With homogeneous prizes, no additional benefit can be expected once one survives the first $I - L$ draws, while with heterogeneous prizes, one would be compelled to survive more rounds to receive larger prizes. The more rewarding the top prizes, the stronger this incentive effect. However, heterogeneous prizes cannot create this incentive effect in a conventional lottery contest, in which top prizes are given away in early lotteries and the marginal benefit for one's effort vanishes as the process continues.

Because of this logic, a reverse-lottery contest tends to incentivize contestants better under heterogeneous prize. The effort comparison under heterogeneous prizes depends sensitively on the prevailing prize structure and the number of contestants, which prevents us from drawing a more definitive conclusion. However, the condition of Theorem 9 is clearly not necessary for the dominance of reverse-lottery contests. With strictly decreasing prizes, even when $L > \frac{I}{2}$, a reverse-lottery contest could induce more effort. The following example illustrates this remark.

**Example 1.** Consider a contest with three contestants. Let the prize sequence be the ordered set (3, 1, 0). In this case, $L(= 2) > \frac{I}{2}(= \frac{3}{2})$. In a conventional lottery contest, each contestant exerts an effort $x_c^* = H^{-1}(\frac{1}{3}\sum_{k=1}^{3}[V_k(1-\sum_{g=0}^{k-1}\frac{1}{3-g})]) = H^{-1}(\frac{1}{3}[3\times(1-\frac{1}{3}) + 1\times(1-\frac{1}{3}-\frac{1}{2})]) = H^{-1}(\frac{13}{18})$. In a reverse-lottery contest, each contestant exerts an effort $x_r^* = H^{-1}(\frac{1}{3}\sum_{k=1}^{3}[V_k(-1+\sum_{g=0}^{3-k}\frac{1}{3-g})]) = H^{-1}(\frac{1}{3}[3\times(-1+\frac{1}{3}+\frac{1}{2}+1)+(-1+\frac{1}{3}+\frac{1}{2})]) = H^{-1}(\frac{14}{18})$. Hence, $x_c^* < x_r^*$.

## 5. Concluding remarks

We have proposed an alternative multi-prize contest model that can be viewed as the mirror image of the nested lottery contest of Clark and Riis (1996a), which we label a reverse nested lottery contest. The model is shown to be underpinned by a unique noisy-ranking system. Its statistical nature allows us to interpret it as a worst-shot contest, in which a contestant's performance is determined by the least favorable shock realized to his performance and he is compelled to expend effort to improve the reliability of his performance. This novel contest model depicts a set of competitive activities that are not addressed by a conventional framework (e.g., conventional lottery contests), and it provides a plausible and handy approach for modeling these competitions.

The equilibrium of a reverse-lottery contest is characterized in a symmetric setting. The results immediately shed light on the optimal prize allocation in a reverse-lottery contest. We further compare the equilibrium behavior to that in a conventional lottery contest. The comparison reveals the ramifications of the different competitive environments on contestants' incentives to exert effort.

The reverse-lottery model provides an alternative framework to model multi-prize contests, and leaves abundant room for future extension and applications. For instance, existing insight on the optimal design of multi-stage elimination contests obtained in conventional contest settings (e.g. Gradstein and Konrad, 1999, and Fu and Lu, 2012a)[29] can be reexamined in a setting of reverse-lottery contests, as well as understanding of the optimal division (or aggregation) of multi-prize contests (e.g. Fu and Lu, 2009). In addition, the axiomatic foundation for the ratio-form CEF proposed in this paper merits serious study. The methodology of Skaperdas (1996) and Clark and Riis (1997) can be adopted for this purpose.

## Appendix A

In this Appendix, we establish the results of Section 2 by linking the noisy-ranking (minimum) contest model (4) to that of Fu and Lu (2012b) as the mirror-image of their noisy-ranking (maximum) contest model.[30] In Fu and Lu (2012b), an identical analytical framework of (4) is adopted. However, they adopt an additive noise term which instead follows a Type-I extreme value (maximum) distribution.

Let us first restate the main results of Fu and Lu (2012b), which are utilized as a benchmark in subsequent analysis. To create a direct link between the two noisy-ranking models, we use the same set of notations when restating Fu and Lu's results. Specifically, let $\mathbf{x}$ denote the effort entries of all contestants, and $\mathbf{g}$ denote the set of deterministic output functions. Further, let the sequence $\{i_k\}_{k=1}^{I}$ denote a complete ranking of the $I$ contestants according to their performance, where $i_k$ is the contestant being ranked in the $k$th position (with $k$th highest perceived output under descending-order ranking rule), and is awarded prize $V_k$ accordingly.

The following obtains in Fu and Lu (2012b) when the noise term follows a Type-I extreme value (maximum) distribution.

**Lemma A0** (*Fu and Lu, 2012b*)**.**

(a) *For any given $\mathbf{x} \geq 0$ such that $\sum_{j\in\mathbf{I}} g_j(x_j) > 0$, the ex ante likelihood that a contestant $i$ will achieve the top rank is*

$$p(i|\mathbf{x}) = \frac{g_i(x_i)}{\sum\limits_{j\in\mathbf{I}} g_j(x_j)}, \quad \forall i \in \mathbf{I}. \tag{17}$$

(b) *For any given effort entries $\mathbf{x} \geq 0$ such that $g_i(x_i) > 0$, $\forall i \in I$, the ex ante likelihood of any complete ranking outcome $\{i_k\}_{k=1}^{I}$ can be expressed as*

$$p(\{i_k\}_{k=1}^{I}) = \Pi_{k=1}^{I}\frac{g_{i_k}(x_{i_k})}{\sum\limits_{k'=k}^{I} g_{i_{k'}}(x_{i_{k'}})}. \tag{18}$$

Lemma A0(a) is due to McFadden (1973, 1974). Furthermore, following Yellott (1977), Fu and Lu (2012b) establish that with $I \geq 3$, the stochastic ranking outcomes of (17) and (18) would result if and only if the noise term $\varepsilon_i$ in their noisy performance ranking model follows a Type-I extreme value (maximum) distribution.

The following fact must be noted:

$$\begin{aligned}(4) &\Rightarrow -\log(y_i) = -\log(g_i(x_i)) + (-\varepsilon_i)\\ &\Rightarrow \log(y_i^{-1}) = \log(g_i(x_i)^{-1}) + (-\varepsilon_i).\end{aligned} \tag{19}$$

Hence, ranking contestants *descendingly* by $\log(y_i)$ in the model (4) is equivalent to an alternative contest model

$$\log(\overline{y}_i) = \log(\overline{g}_i(x_i)) + \overline{\varepsilon}_i, \tag{20}$$

where contestants are ranked by $\log(\overline{y}_i)$ ascendingly, with $\overline{y}_i = y_i^{-1}$, $\overline{g}_i(x_i) = [g_i(x_i)]^{-1}$ and $\overline{\varepsilon}_i = -\varepsilon_i$. Three important statistical facts are established immediately.

**Lemma A1.** *Consider a given set of effort entries $\mathbf{x}$. The ex ante likelihood of a contestant $i$'s output $\log(y_i)$ being ranked descendingly in the top $j$th position in the noisy-ranking (minimum) contest model (4) is equal to the likelihood of his corresponding output $\log(\overline{y}_i)$ in (20) being ranked descendingly in the top $(I-j+1)$th position.*

Lemma A1 can be interpreted intuitively. Ranking $\log(y_i)$ in (4) descendingly is equivalent to ranking $\log(\overline{y}_i)$ in (20) ascendingly. Hence, for a given $\mathbf{x}$, when $\log(y_i)$ is ranked in the top $j$th position

[29] They show that under concave contest technology, a multi-stage contest outperforms a single-stage contest. However, our preliminary analysis demonstrates that a single-stage contest may dominate a multi-stage contest under plausible conditions. A more comprehensive analysis will be conducted and presented in a separate study.

[30] While all results in this section can be directly shown, deriving them indirectly through the linkage to Fu and Lu (2012b) allows us to utilize their existing results. Besides analytical efficiency, linking the two models helps to stress the statistical connection and difference between the two paradigms.

descendingly, the corresponding output $\log(\overline{y}_i)$ in (20) must obtain the same rank when $\log(\overline{y}_i)$s are ranked ascendingly. The top $j$th rank, however, would become the bottom $j$th rank, or the top $(I-j+1)$th rank, if the ranking rule is turned upside-down, i.e., when $\log(\overline{y}_i)$s are ranked descendingly.

Again, let the sequence $\{i_k\}_{k=1}^{I}$ denote a complete ranking of the $I$ contestants according to their performance $y_i$, where $i_k$ is the contestant in the $k$th position according the particular ranking rule of the corresponding model.

**Lemma A2.** *Consider a given set of effort entries* $\mathbf{x}$*. The ex ante likelihood of a complete ranking outcome* $\{i_k\}_{k=1}^{I}$ *in model (4), with* $\log(y_i)$ *being ranked descendingly, is equal to that of a ranking outcome* $\{i'_k\}_{k=1}^{I}$ *in model (20), with* $\log(\overline{y}_i)$ *being ranked descendingly,* $i'_k$ *being the contestant in the kth position, and* $i'_k = i_{I+1-k}$.

Lemma A2 iterates the statistical relation between model (4) and model (20). Let us illustrate it with a simple example with four contestants, indexed by $i = 1, 2, 3, 4$. Consider a given set of effort entries $\mathbf{x}$. A ranking outcome $\{1, 3, 4, 2\}$ in model (4), when $\log(y_i)$s are ranked descendingly, would occur with the same probability as a ranking outcome $\{2, 4, 3, 1\}$ in model (20), when $\log(\overline{y}_i)$ are ranked descendingly. Intuitively, under descending-order ranking rule, models (4) and (20) are mirror images to each other. The next fact links model (20) to Fu and Lu (2012b).

**Lemma A3.** *If* $\varepsilon_i$ *follows a Type-I extreme value (minimum) distribution, then* $\overline{\varepsilon}_i(= -\varepsilon_i)$ *must follow a Type-I extreme value (maximum) distribution.*

**Proof.** Note that when a variable $t$ follows a Type-I extreme value (maximum) distribution, the c.d.f. is

$$F(t) = e^{-e^{-t}}, \quad t \in (-\infty, +\infty). \tag{21}$$

Define $G(\cdot)$ as the distribution function of $\overline{\varepsilon}$, we prove Lemma A3 by showing $G(\overline{\varepsilon})$ is exactly the distribution function of a variable following a Type-I extreme value (maximum) distribution.

$$G(\overline{\varepsilon}) = \Pr(\overline{\varepsilon}_i \le \overline{\varepsilon}) = 1 - \Pr(\overline{\varepsilon}_i \ge \overline{\varepsilon}).$$

Due to $\overline{\varepsilon}_i = -\varepsilon_i$, the probability of $\overline{\varepsilon}_i \ge \varepsilon$ is the same with the probability of $\varepsilon_i \le -\varepsilon$, then

$$\begin{aligned} G(\overline{\varepsilon}) &= 1 - \Pr(\varepsilon_i \le -\overline{\varepsilon}) \\ &= 1 - F(-\overline{\varepsilon}). \end{aligned} \tag{22}$$

Because $\varepsilon_i$ follows a Type-I extreme value (minimum) distribution, using (5) we write

$$F(-\overline{\varepsilon}) = 1 - e^{-e^{-\overline{\varepsilon}}}. \tag{23}$$

Substituting (23) into (22), we derive

$$\begin{aligned} G(\overline{\varepsilon}) &= 1 - (1 - e^{-e^{-\overline{\varepsilon}}}) \\ &= e^{-e^{-\overline{\varepsilon}}}. \end{aligned} \tag{24}$$

(24) is equivalent to (21), which completes the proof. ■

As $\overline{\varepsilon}_i$ follows a Type-I extreme value (maximum) distribution, the noisy-ranking (minimum) model (4) is transformed into a noisy-ranking (maximum) contest model (20) with a set of output functions $\bar{\mathbf{g}} = (g_1(x_1)^{-1}, \ldots, g_I(x_I)^{-1})$. Model (20) with an ascending-order ranking rule, simply boils down to that of Fu and Lu (2012b) with output functions $\bar{\mathbf{g}}$ and a descending-order ranking rule.[31]

These facts allow us to borrow directly from Fu and Lu (2012b) the entire set of analytical results. The noisy-ranking (minimum) model (4) boils down to a mirror image of the noisy-ranking (maximum) model of Fu and Lu (2012b). For given effort entries $\mathbf{x}$ and output functions $\mathbf{g}$, one's output $\log(y_i)$ in the noisy-ranking (minimum) model (4) being ranked descendingly in the top $j$th position, must be the same as its counterpart of $\log(\overline{y}_i)$ in a noisy-ranking (maximum) model (Fu and Lu, 2012b) being ranked descendingly in the top $(I-j+1)$th position (or the bottom $j$th position).

Theorems 1 and 2 are apparent due to Lemmas A0–A3. Theorem 3 obtains due to the mirror image relation established above and the uniqueness of noise distribution in the conventional nested lottery contest.[32]

## Appendix B

**Proofs of Theorems 5 and 6.** The proof begins with a general case, and then proceeds to a winner-take-all case.

In a symmetric pure-strategy equilibrium (SPSE) of reverse-lottery model with $I$ players, given everyone else exerting equilibrium effort $y$, player $i$ considers exerting effort $e_i$, his expected utility can be expressed as

$$\begin{aligned} EU_i = {} & \frac{e_i^{-r}}{e_i^{-r} + (I-1)y^{-r}} V_I \\ & + \frac{(I-1)y^{-r}e_i^{-r}}{[e_i^{-r} + (I-1)y^{-r}][e_i^{-r} + (I-2)y^{-r}]} V_{I-1} + \cdots \\ & + \frac{(I-1)(I-2)\ldots(I-t)y^{-tr}e_i^{-r}}{[e_i^{-r} + (I-1)y^{-r}][e_i^{-r} + (I-2)y^{-r}]\ldots[e_i^{-r} + (I-t-1)y^{-r}]} V_{I-t} + \cdots \\ & + \frac{(I-1)(I-2)\ldots 1 \times y^{-(I-1)r}e_i^{-r}}{[e_i^{-r} + (I-1)y^{-r}][e_i^{-r} + (I-2)y^{-r}]\ldots[e_i^{-r} + y^{-r}]e_i^{-r}} V_1 - e_i. \end{aligned}$$

Defining $\lambda = \frac{e_i^{-r}}{y^{-r}}$, where $y$ is the equilibrium effort if a SPSE exists and

$$y = \frac{r}{I}\sum_{i=1}^{I}\left(-1 + \sum_{t=0}^{i-1}\frac{1}{I-t}\right)V_{I-i+1}, \tag{25}$$

which can be derived straightforwardly from Theorem 7. Then $EU_i$ can be expressed as:

$$\begin{aligned} EU_i = {} & \frac{\lambda}{\lambda + (I-1)} V_I + \frac{(I-1)\lambda}{[\lambda + (I-1)][\lambda + (I-2)]} V_{I-1} + \cdots \\ & + \frac{(I-1)(I-2)\ldots(I-t)\lambda}{[\lambda + (I-1)][\lambda + (I-2)]\ldots[\lambda + (I-t-1)]} V_{I-t} \\ & + \cdots + \frac{(I-1)(I-2)\ldots 1 \times \lambda}{[\lambda + (I-1)][\lambda + (I-2)]\ldots[\lambda + 1]} V_1 - e_i. \end{aligned}$$

As $\lambda = \frac{e_i^{-r}}{y^{-r}}$, we get $e_i = \lambda^{-\frac{1}{r}}y$.[33] Mathematically, $EU_i$ can be written in the following form:

$$\begin{aligned} EU_i = {} & \frac{1}{I}\sum_{i=1}^{I} V_{I-i+1}\left[\left(\lambda\prod_{t=0}^{i-1}\frac{I-t}{I-t+\lambda-1}\right)\right. \\ & \left. - r\left(-1 + \sum_{t=0}^{i-1}\frac{1}{I-t}\right)\lambda^{-\frac{1}{r}}\right], \end{aligned} \tag{26}$$

[31] To put it more precisely, (20) is different from a typical output function in a noisy-ranking (maximum) contest model defined by Fu and Lu (2012b). The output function $\overline{g}_i(x_i)$ is decreasing in $x_i$. However, it does not change Fu and Lu's (2012b) technical results on the likelihoods of stochastic ranking outcomes for given effort entries $\mathbf{x}$.

[32] Please refer to Theorem 2 in Fu and Lu (2012b) for details.

[33] Notice that $\lambda$ can also be expressed as $\lambda = (x/e_i)^r$, thus when $e_i$ increases from 0 to $+\infty$, $\lambda$ decreases from $+\infty$ to 0 accordingly.

which can also be expressed as

$$EU_i = \frac{1}{I}\sum_{i=1}^{I} b_{I-i+1} \times V_{I-i+1}, \quad \text{where} \tag{27}$$

$$b_{I-i+1} = \lambda \prod_{t=0}^{i-1} \frac{I-t}{I-t+\lambda-1} - r\left(-1+\sum_{t=0}^{i-1}\frac{1}{I-t}\right)\lambda^{-\frac{1}{r}}. \tag{28}$$

Hence, the expected utility $EU_i$ is a function of $\lambda$. Using

$$\frac{\partial}{\partial\lambda}\prod_{t=0}^{i-1}\frac{I-t}{I-t+\lambda-1} = -\left(\prod_{t=0}^{i-1}\frac{I-t}{I-t+\lambda-1}\right)\left(\sum_{t=0}^{i-1}\frac{1}{I-t+\lambda-1}\right), \tag{29}$$

$$\frac{\partial}{\partial\lambda}\sum_{t=0}^{i-1}\frac{\lambda}{I-t+\lambda-1} = \left(\sum_{t=0}^{i-1}\frac{1}{I-t+\lambda-1}\right) - \lambda\sum_{t=0}^{i-1}\left(\frac{1}{I-t+\lambda-1}\right)^2, \tag{30}$$

we derive the first and second order conditions for a local maximum:

$$\frac{\partial EU_i}{\partial\lambda} = \frac{1}{I}\sum_{i=1}^{I} V_{I-i+1} \times \left[-\left(\prod_{t=0}^{i-1}\frac{I-t}{I-t+\lambda-1}\right)\left(-1+\sum_{t=0}^{i-1}\frac{\lambda}{I-t+\lambda-1}\right) + \left(-1+\sum_{t=0}^{i-1}\frac{1}{I-t}\right)\lambda^{-\frac{1}{r}-1}\right] = 0; \tag{31}$$

$$\frac{\partial^2 EU_i}{\partial\lambda^2} = \frac{1}{I}\sum_{i=1}^{I} V_{I-i+1}\left\{\left(\prod_{t=0}^{i-1}\frac{I-t}{I-t+\lambda-1}\right) \times \left(\sum_{t=0}^{i-1}\frac{1}{I-t+\lambda-1}\right)\left(-1+\sum_{t=0}^{i-1}\frac{\lambda}{I-t+\lambda-1}\right) + \left(\prod_{t=0}^{i-1}\frac{I-t}{I-t+\lambda-1}\right) \times \left[-\left(\sum_{t=0}^{i-1}\frac{1}{I-t+\lambda-1}\right) + \lambda\sum_{t=0}^{i-1}\left(\frac{1}{I-t+\lambda-1}\right)^2\right] - \left(\frac{1}{r}+1\right)\left(-1+\sum_{t=0}^{i-1}\frac{1}{I-t}\right)\lambda^{-\frac{1}{r}-2}\right\}. \tag{32}$$

Substituting (31) into (32), we derive

$$\left.\frac{\partial^2 EU_i}{\partial\lambda^2}\right|_{\frac{\partial EU_i}{\partial\lambda}=0} = \frac{1}{I}\sum_{i=1}^{I} V_{I-i+1}\frac{1}{\lambda}\left(\prod_{t=0}^{i-1}\frac{I-t}{I-t+\lambda-1}\right) \times \left(-1+\sum_{t=0}^{i-1}\frac{\lambda}{I-t+\lambda-1}\right) \times \left[\left(-1+\sum_{t=0}^{i-1}\frac{\lambda}{I-t+\lambda-1}\right) - \frac{1}{r} - \frac{\sum_{t=0}^{i-1}\frac{\lambda}{I-t+\lambda-1}\left(1-\frac{\lambda}{I-t+\lambda-1}\right)}{-1+\sum_{t=0}^{i-1}\frac{\lambda}{I-t+\lambda-1}}\right]. \tag{33}$$

It can be verified that $\lambda = 1$ satisfies the first order condition. So the point $\lambda = 1$ is a local maximum if $\frac{\partial^2 EU_i}{\partial\lambda^2}|_{\lambda=1} < 0$, which holds if and only if

$$r < \frac{\sum_{i=1}^{I} V_{I-i+1}\left(-1+\sum_{t=0}^{i-1}\frac{1}{I-t}\right)}{\sum_{i=1}^{I} V_{I-i+1}\left[\left(-1+\sum_{t=0}^{i-1}\frac{1}{I-t}\right)^2 - \sum_{t=0}^{i-1}\frac{1}{I-t}\left(1-\frac{1}{I-t}\right)\right]} = r^{soc} \tag{34}$$

if the denominator of $r^{soc}$ is positive, and $r > 0$ when the denominator is negative.[34]

Participation constraint requires

$$EU_i = \frac{1}{I}\sum_{i=1}^{I} V_{I-i+1} - \frac{r}{I}\sum_{i=1}^{I} V_{I-i+1}\left(-1+\sum_{t=0}^{i-1}\frac{1}{I-t}\right) \geq 0,$$

which follows that

$$r \leq \frac{\sum_{i=1}^{I} V_{I-i+1}}{\sum_{i=1}^{I} V_{I-i+1}\left(-1+\sum_{t=0}^{i-1}\frac{1}{I-t}\right)} = r^{pc}. \tag{35}$$

Consider a single element $b_{I-i+1}$ which is given by (28). Let $\underline{k}$ be the minimal value of $k$ such that $\sum_{t=0}^{k}\frac{1}{I-t} > 1$, which follows that $\underline{k} \approx 0.632I$. When $I-i+1 \leq I-\underline{k}+1$, i.e., $i \geq \underline{k}$, it can be shown that every $b_{I-i+1}$, with $b_{I-i+1}(\lambda = 1) \geq 0$, is maximized at $\lambda = 1$ globally as long as $r$ satisfies condition (C1) in Theorem 6.[35] The proof is as follows.

First, it is straightforward to verify that $\lambda = 1$ is a local maximum for $b_{I-i+1}$. Next, we show that when $r \leq \frac{1}{I-1}$, there exists at most one local maximum, then $\lambda = 1$ must be a global maximum with participation constraint holding. Assume that there exists more than one local maximum, consider any two local maximums, denote the higher value of $\lambda$ that is a local maximum by $\lambda'$, and the lower one by $\lambda''$. From (26), we can see that $EU_i$ is a continuous function with respect to $\lambda$, then between the two maximums, there must exist at least one local minimum, say $\lambda = \lambda_{\min}$ where $\lambda'' < \lambda_{\min} < \lambda'$. At any critical point with $\frac{\partial b_{I-i+1}}{\partial\lambda} = 0$, using (31) and (33), we derive

$$\frac{\partial b_{I-i+1}}{\partial\lambda} = -\left(\prod_{t=0}^{i-1}\frac{I-t}{I-t+\lambda-1}\right)\left(-1+\sum_{t=0}^{i-1}\frac{\lambda}{I-t+\lambda-1}\right) + \left(-1+\sum_{t=0}^{i-1}\frac{1}{I-t}\right)\lambda^{-\frac{1}{r}-1}. \tag{36}$$

$$\left.\frac{\partial^2 b_{I-i+1}}{\partial\lambda^2}\right|_{\frac{\partial b_{I-i+1}}{\partial\lambda}=0} = \underbrace{\frac{1}{\lambda}\left(\prod_{t=0}^{i-1}\frac{I-t}{I-t+\lambda-1}\right)\left(-1+\sum_{t=0}^{i-1}\frac{\lambda}{I-t+\lambda-1}\right)}_{\text{first term}}$$

[34] Notice that the numerator of $r^{soc}$ is always bigger than zero as $\sum_{i=1}^{n} V_{n-i+1}(-1+\sum_{t=0}^{i-1}\frac{1}{n-t}) = y > 0$, so $r^{soc}$ is positive/negative when the denominator is positive/negative.

[35] Notice that $b_{I-i+1}$ with $i \geq \underline{k}$ can also be expressed as $b_l$ with $l \leq I - \underline{k} + 1$.

$$\times \underbrace{\left[ \underbrace{\left(-1+\sum_{t=0}^{i-1}\frac{\lambda}{I-t+\lambda-1}\right)-\frac{1}{r}}_{\text{part I}} - \underbrace{\frac{\sum_{t=0}^{i-1}\frac{\lambda(I-t-1)}{(I-t+\lambda-1)^2}}{\left(-1+\sum_{t=0}^{i-1}\frac{\lambda}{I-t+\lambda-1}\right)}}_{\text{part II}} \right]}_{\text{second term}}. \quad (37)$$

When $\frac{\partial EU_i}{\partial\lambda}=0$, from (36) we derive that $-1+\sum_{t=0}^{i-1}\frac{\lambda}{I-t+\lambda-1}>0$ due to the fact that $-1+\sum_{t=0}^{i-1}\frac{1}{I-t}>0$ for all $i\geq \underline{k}$, otherwise (36) cannot be zero. As $\lambda\in(0,+\infty)$ and using

$$\frac{\partial}{\partial\lambda}\sum_{t=0}^{i-1}\frac{\lambda}{I-t+\lambda-1}=\sum_{t=0}^{i-1}\frac{I-t-1}{(I-t+\lambda-1)^2}\geq 0, \quad (38)$$

we derive that part I in (37) approaches its maximum as $\lambda\to+\infty$, which is

$$\lim_{\lambda\to+\infty}\left(-1+\sum_{t=0}^{i-1}\frac{\lambda}{I-t+\lambda-1}\right)=i-1.$$

Therefore, for all $i\geq\underline{k}$, we have $0<-1+\sum_{t=0}^{i-1}\frac{\lambda}{I-t+\lambda-1}<I-1$. Then when $\frac{1}{r}\geq I-1$, part I in (37) is negative, then (37) must be negative as part II in (37) is negative. This implies that $\frac{\partial^2 b_{I-i+1}}{\partial\lambda^2}|_{\lambda=\lambda_{\min}}<0$, which contradicts our assumption that $\lambda=\lambda_{\min}$ is a local minimum.

Now we show that when the prizes are weakly decreasing, $r^{pc}>\frac{1}{I-1}$, where $r^{pc}$ is given by (35). As prizes are weakly decreasing, let $\Gamma=\sum_{i=1}^{I}V_{I-i+1}$, we can see that in the denominator of $r^{pc}$, the coefficient $(-1+\sum_{t=0}^{i-1}\frac{1}{I-t})$ is increasing with the order of the prize, thus the coefficient of the first prize $V_1$ is $(-1+\sum_{t=0}^{I-1}\frac{1}{I-t})$ which is the largest among all coefficients, so any $r^{pc}$ with decreasing prizes must be larger than or equal to the $r^{pc}$ with a single first prize $V_1=\Gamma$. Then we derive

$$r^{pc}\geq\frac{\Gamma}{\Gamma\left(-1+\sum_{t=0}^{I-1}\frac{1}{I-t}\right)}=\frac{1}{-1+\sum_{t=0}^{I-1}\frac{1}{I-t}}. \quad (39)$$

As $\sum_{t=0}^{I-1}\frac{1}{I-t}<I$, thus from (39), we have $r^{pc}>\frac{1}{I-1}$, which implies that participation constraint must be satisfied when $r\leq\frac{1}{I-1}$.

Next, we show that when $r\leq\frac{1}{I-1}$, we must have $r^{soc}>\frac{1}{I-1}$ when $r^{soc}$ is positive.[36] By (34), we derive that when the denominator of $r^{soc}$ is positive,

$$r^{soc}>\frac{\sum_{i=1}^{I}V_{I-i+1}\left(-1+\sum_{t=0}^{i-1}\frac{1}{I-t}\right)}{\sum_{i=1}^{I}V_{I-i+1}\left(-1+\sum_{t=0}^{i-1}\frac{1}{I-t}\right)^2}. \quad (40)$$

As $(-1+\sum_{t=0}^{i-1}\frac{1}{I-t})$ is increasing in $i$ and prizes are weakly decreasing, following similar arguments as those when showing $r^{pc}>\frac{1}{I-1}$, we further derive

$$r^{soc}>\frac{\Gamma\left(-1+\sum_{t=0}^{I-1}\frac{1}{I-t}\right)}{\Gamma\left(-1+\sum_{t=0}^{I-1}\frac{1}{I-t}\right)^2}=\frac{1}{-1+\sum_{t=0}^{I-1}\frac{1}{I-t}}>\frac{1}{I-1}.$$

Therefore, the second order condition for a local maximum at $\lambda=1$ must be satisfied when $r\leq\frac{1}{I-1}$.

Finally, we can conclude that under conditions (C1) and (C2) in Theorem 6, every $b_{I-i+1}$ is globally maximized at $\lambda=1$ as it is the only local maximum with participation constraint holding. As $EU_i$ is a weighted sum of the $b_{I-i+1}$ functions, therefore, $EU_i$ is maximized at $\lambda=1$ when every $b_{I-i+1}$ is maximized.

We now proceed to the winner-take-all case.

From (26), we write $EU_i=\frac{1}{I}b_1V_1$, where

$$b_1=\lambda\prod_{t=0}^{I-1}\frac{I-t}{I-t+\lambda-1}-r\left(-1+\sum_{t=0}^{I-1}\frac{1}{I-t}\right)\lambda^{-\frac{1}{r}}. \quad (41)$$

It is clear that maximizing $EU_i$ is equivalent to maximizing $b_1$. Hence, the expected utility is expressed as a function of $\lambda$. From (36) and (37), the first and second order conditions for a local maximum are:

$$\begin{aligned}\frac{\partial b_1}{\partial\lambda} &= -\left(\prod_{t=0}^{I-1}\frac{I-t}{I-t+\lambda-1}\right)\left(-1+\sum_{t=0}^{I-1}\frac{\lambda}{I-t+\lambda-1}\right)\\ &\quad+\left(-1+\sum_{t=0}^{I-1}\frac{1}{I-t}\right)\lambda^{-\frac{1}{r}-1}=0;\end{aligned} \quad (42)$$

$$\left.\frac{\partial^2 b_1}{\partial\lambda^2}\right|_{\frac{\partial b_1}{\partial\lambda}=0}=\underbrace{\frac{1}{\lambda}\prod_{t=0}^{I-1}\frac{I-t}{I-t+\lambda-1}\left(-1+\sum_{t=0}^{I-1}\frac{\lambda}{I-t+\lambda-1}\right)}_{\text{first term}}$$

$$\times\underbrace{\left[\left(-1+\sum_{t=0}^{I-1}\frac{\lambda}{I-t+\lambda-1}\right)-\frac{1}{r}-\frac{\sum_{t=0}^{I-1}\frac{\lambda(I-t-1)}{(I-t+\lambda-1)^2}}{\left(-1+\sum_{t=0}^{I-1}\frac{\lambda}{I-t+\lambda-1}\right)}\right]}_{\text{second term}}. \quad (43)$$

It can be verified that $\lambda=1$ satisfies the first order condition. So the point $\lambda=1$ is a local maximum if $\frac{\partial^2 b_1}{\partial\lambda^2}|_{\lambda=1}<0$, which holds if and only if

$$r<\frac{-1+\sum_{t=0}^{I-1}\frac{1}{I-t}}{\left(-1+\sum_{t=0}^{I-1}\frac{1}{I-t}\right)^2-\sum_{t=0}^{I-1}\frac{1}{I-t}\left(1-\frac{1}{I-t}\right)}=r^{soc} \quad (44)$$

if the denominator of $r^{soc}$ is positive, and $r>0$ when the denominator is negative. When the above condition holds, $\lambda=1$ is a local maximum.

Next, we show that as long as the participation constraint holds, i.e., $b_1(\lambda=1)\geq 0$, it is necessary and sufficient to show $\lambda=1$ is a global maximum. At any $\lambda>0$, participation constraint requires

$$\begin{aligned}b_1 &= \lambda\prod_{t=0}^{I-1}\frac{I-t}{I-t+\lambda-1}-r\left(-1+\sum_{t=0}^{I-1}\frac{1}{I-t}\right)\lambda^{-\frac{1}{r}}\geq 0\\ &\Rightarrow\prod_{t=0}^{I-1}\frac{I-t}{I-t+\lambda-1}\geq r\left(-1+\sum_{t=0}^{I-1}\frac{1}{I-t}\right)\lambda^{-\frac{1}{r}-1}.\end{aligned} \quad (45)$$

Assume that with participation constraint holding, there exists more than one local maximum, and denote the highest value of $\lambda$ that is a local maximum by $\lambda'$. In a critical point with $\frac{\partial b_1}{\partial\lambda}=0$, by (42) we write

$$\begin{aligned}&\prod_{t=0}^{I-1}\frac{I-t}{I-t+\lambda-1}\left(-1+\sum_{t=0}^{I-1}\frac{\lambda}{I-t+\lambda-1}\right)\\ &\quad=\left(-1+\sum_{t=0}^{I-1}\frac{1}{I-t}\right)\lambda^{-\frac{1}{r}-1}.\end{aligned} \quad (46)$$

[36] When $r^{soc}$ is negative, we only need $r>0$ to ensure $\lambda=1$ is a local maximum.

From (45) and (46), we derive that with participation constraint holding, at the local maximum $\lambda = \lambda'$,

$$\frac{1}{r} \geq \left(-1 + \sum_{t=0}^{I-1} \frac{\lambda'}{I - t + \lambda' - 1}\right). \tag{47}$$

Assume with participation constraint holding, there exists another local maximum at $\lambda'' < \lambda'$. From (45) we can see $b_1$ is a continuous function with respect to $\lambda$, then between the two maximums, there must exist at least one local minimum, say $\lambda = \lambda_{\min}$ where $\lambda'' < \lambda_{\min} < \lambda'$. Then we must have $\frac{\partial b_1}{\partial \lambda}|_{\lambda=\lambda_{\min}} = 0$ and $\frac{\partial^2 b_1}{\partial \lambda^2}|_{\lambda=\lambda_{\min}} > 0$. However, as $\lambda_{\min} < \lambda'$, using (43) and the fact that

$$\frac{\partial}{\partial \lambda} \sum_{t=0}^{i-1} \frac{\lambda}{I - t + \lambda - 1} = \sum_{t=0}^{i-1} \frac{I - t - 1}{(I - t + \lambda - 1)^2} > 0, \tag{48}$$

we derive

$$\frac{1}{r} \geq \left(-1 + \sum_{t=0}^{I-1} \frac{\lambda'}{I - t + \lambda' - 1}\right) \geq -1 + \sum_{t=0}^{I-1} \frac{\lambda_{\min}}{I - t + \lambda_{\min} - 1},$$

which implies that the second term in (43) is negative at $\lambda = \lambda_{\min}$. This contradicts the initial assumption that $\lambda = \lambda_{\min}$ is a local minimum, which indicates that there exists at most one local maximum with the participation constraint holding.

At $\lambda = 1$, using (45), participation constraint requires $r \leq r^{pc}$ where

$$r^{pc} = \frac{1}{\left(-1 + \sum_{t=0}^{I-1} \frac{1}{I-t}\right)}. \tag{49}$$

By comparing $r^{soc}$ and $r^{pc}$, which are given by (44) and (49) respectively, it is straightforward to show $r^{pc} < r^{soc}$ when $r^{soc}$ is positive. When $r^{soc}$ is negative, we only need $r > 0$ to ensure $\lambda = 1$ is a local maximum. Therefore, we are safe to conclude $r \leq r^{pc}$ is a sufficient condition to ensure the existence of a SPSE in a winner-take-all reverse-lottery contest.

**Proof of Theorem 7.** Again, we obtain the solution to symmetric equilibrium by utilizing existing results from its "mirror image", i.e., conventional lottery contest models (Clark and Riis, 1996a). We first state the following fact.

**Lemma A4.** *A reverse nested lottery contest with (weakly) decreasing prizes $V_k$, $k = 1, 2, \ldots, I$ and increasing impact functions $g_i(\cdot) = g(\cdot), \forall i \in \mathbf{I}$, is strategically equivalent to a nested lottery contest of Clark and Riis (1996a) with (weakly) increasing prizes $\tilde{V}_k$, $k = 1, 2, \ldots, I$ and decreasing impact functions $\tilde{g}_i(\cdot), i = 1, 2, \ldots, I$, with $\tilde{V}_k = V_{I+1-k}$, $k = 1, 2, \ldots, I$ and $\tilde{g}_i(\cdot) = \tilde{g}(\cdot) = [g(\cdot)]^{-1}$, $\forall i \in \mathbf{I}$.*

Lemma A4 simply follows from the fact that the two contest models are inverse to each other, which was stated more formally by Lemma A1. We apply the equilibrium solution of Fu and Lu (2009) for the nested lottery contest (which is restated in Lemma 1 in this paper), and replace each $V_k$ by $V_{I+1-k}$ in (15). We obtain that in a symmetric equilibrium of the strategically equivalent lottery contest the following must hold:

$$\frac{\widetilde{g}(x^*)}{\widetilde{g}'(x^*)} = \frac{1}{I} \sum_{k=1}^{I} \left[V_{I+1-k}\left(1 - \sum_{j=0}^{k-1} \frac{1}{I-j}\right)\right].$$

Further note $\frac{\tilde{g}(\cdot)}{\tilde{g}'(\cdot)} = -\frac{g(\cdot)}{g'(\cdot)}$. We then have

$$H(x^*) = \frac{g(x^*)}{g'(x^*)} = \frac{1}{I} \sum_{k=1}^{I} \left[-V_{I+1-k}\left(1 - \sum_{j=0}^{k-1} \frac{1}{I-j}\right)\right],$$

which leads to the claim of Theorem 7.

**Proof of Theorem 8.** The number of the contestants $I (\geq 3)$ can be even or odd. Hence, we consider the following two cases: $I = 2n$ and $I = 2n + 1$, where $n$ is a positive integer, i.e., $n$ can be any positive integer.

In both cases when $I = 2n$ and $I = 2n + 1$, when $L = n$, using (13), (14), (15), (16) and Remark 1, we can see that comparing $x_c^*$ and $x_r^*$ is equivalent to comparing $\sum_{k=1}^{n} c_k$ and $\sum_{k=1}^{n}(-c_{I+1-k})$ where

$$c_k = 1 - \sum_{g=0}^{k-1} \frac{1}{I-g}.$$

We can derive that $x_r^* > x_c^*$ if and only if

$$\sum_{k=1}^{n}(-c_{I+1-k}) - \sum_{k=1}^{n} c_k = \sum_{k=1}^{n} d_k > 0,$$

where

$$d_k = \left(-1 + \sum_{j=0}^{I-k} \frac{1}{I-j}\right) - \left(1 - \sum_{i=0}^{k-1} \frac{1}{I-i}\right). \tag{50}$$

Using (50), we can further derive:

$$d_1 = \frac{1}{I} + \left(\frac{1}{I} + \frac{1}{I-1} + \cdots + \frac{1}{2}\right) - 1 > 0 \quad \text{for all } I \geq 3;$$

$$d_k - d_{k-1} = \left[\frac{1}{I-(k-1)} - \frac{1}{k-1}\right] < 0$$

as $I > 2(n-1) \geq 2(k-1)$.

$$d_n = \left[\left(-1 + \frac{1}{I} + \cdots + \frac{1}{n}\right) - \left(1 - \frac{1}{I} - \cdots - \frac{1}{I-(n-1)}\right)\right]. \tag{51}$$

It is easy to verify that when $n = 1$ or $n = 2$, Theorem 8 holds.[37] Next, we are going to prove that Theorem 8 holds in general for all $n \geq 3$.

First, we seek to determine the sign of $d_n$. When $I = 2n$, (51) becomes

$$d_n = \left[\left(-1 + \frac{1}{2n} + \cdots + \frac{1}{n+1} + \frac{1}{n}\right) - \left(1 - \frac{1}{2n} - \cdots - \frac{1}{n+1}\right)\right] = 2\left[\left(\frac{1}{2n} + \cdots + \frac{1}{n}\right) - 1\right] - \frac{1}{n}; \tag{52}$$

when $I = 2n + 1$, (51) becomes

$$d_n = \left[\left(-1 + \frac{1}{2n+1} + \cdots + \frac{1}{n}\right) - \left(1 - \frac{1}{2n+1} - \cdots - \frac{1}{n+2}\right)\right] = 2\left[\left(\frac{1}{2n} + \cdots + \frac{1}{n}\right) - 1\right] - \frac{2n^2 + 2n + 1}{n(n+1)(2n+1)}. \tag{53}$$

[37] Notice that $I = 3$ when $n = 1$ (recall that here we do not consider the case $I = 2$), while when $n = 2$, $I = 4$ or $I = 5$.

In the following, we show that when $n \geq 3$,

$$\left(\frac{1}{2n} + \cdots + \frac{1}{n}\right) < 1. \tag{54}$$

When $n = 3$, (54) holds since $1/6 + 1/5 + 1/4 + 1/3 - 1 = -1/20 < 0$. When $n$ increases from $t$ to $t + 1$ where $t \geq 3$, LHS of (54) will increase by

$$\frac{1}{2t+2} + \frac{1}{2t+1} - \frac{1}{t} = -\frac{3t+2}{2t(t+1)(2t+1)} < 0.$$

Thus, for all $n \geq 3$, (54) holds, which implies (52) $< 0$ and (53) $< 0$, i.e., $d_n < 0$.

Up to now, we have shown that: for all $n \geq 3$, when $k$ gets larger, $d_k$ decreases from $d_1 > 0$ to $d_n < 0$. When $I = 2n$,

$$\begin{aligned}\sum_{k=1}^{n} d_k &= \sum_{k=1}^{n}\left[\left(-1 + \sum_{j=0}^{2n-k}\frac{1}{2n-j}\right) - \left(1 - \sum_{i=0}^{k-1}\frac{1}{2n-i}\right)\right] \\ &= -\left(\sum_{k=1}^{n} c_{2n-k+1} + \sum_{k=1}^{n} c_k\right) \\ &= -\left(\sum_{k=n+1}^{2n} c_k + \sum_{k=1}^{n} c_k\right) \\ &= -\sum_{k=1}^{2n} c_k = -\sum_{k=1}^{I} c_k.\end{aligned}$$

When $I = 2n + 1$,

$$\begin{aligned}\sum_{k=1}^{n} d_k &= \sum_{k=1}^{n}\left[\left(-1 + \sum_{j=0}^{2n+1-k}\frac{1}{2n+1-j}\right) - \left(1 - \sum_{i=0}^{k-1}\frac{1}{2n+1-i}\right)\right] \\ &= -\left(\sum_{k=1}^{n} c_{2n-k+2} + \sum_{k=1}^{n} c_k\right) \\ &= -\left(\sum_{k=n+2}^{2n+1} c_k + \sum_{k=1}^{n} c_k\right) \\ &= -\left(\sum_{k=1}^{2n+1} c_k - c_{n+1}\right) \\ &= -\sum_{k=1}^{I} c_k + c_{n+1};\end{aligned}$$

$$\begin{aligned}\sum_{k=1}^{n+1} d_k &= \sum_{k=1}^{n+1}\left[\left(-1 + \sum_{j=0}^{2n+1-k}\frac{1}{2n+1-j}\right) - \left(1 - \sum_{i=0}^{k-1}\frac{1}{2n+1-i}\right)\right] \\ &= -\left(\sum_{k=1}^{n+1} c_{2n-k+2} + \sum_{k=1}^{n+1} c_k\right) \\ &= -\left(\sum_{k=n+1}^{2n+1} c_k + \sum_{k=1}^{n+1} c_k\right) \\ &= -\left[\left(\sum_{k=1}^{2n+1} c_k\right) + c_{n+1}\right] \\ &= -\sum_{k=1}^{I} c_k - c_{n+1}.\end{aligned}$$

Notice that

$$\begin{aligned}\sum_{k=1}^{I} c_k &= \left(1 - \frac{1}{I}\right) + \left(1 - \frac{1}{I} - \frac{1}{I-1}\right) \\ &\quad + \cdots + \left(1 - \frac{1}{I} - \cdots - \frac{1}{2} - 1\right) \\ &= I - I\left(\frac{1}{I}\right) - (I-1)\left(\frac{1}{I-1}\right) - \cdots - 2\left(\frac{1}{2}\right) - 1 \\ &= 0;\end{aligned}$$

when $I = 2n + 1$,

$$\begin{aligned}c_{n+1} &= 1 - \sum_{g=0}^{n}\frac{1}{2n+1-g} \\ &= 1 - \left(\frac{1}{2n+1} + \frac{1}{2n} + \cdots + \frac{1}{n+1}\right) \\ &= \frac{n+1}{(2n+1)n} + \left[1 - \left(\frac{1}{2n} + \cdots + \frac{1}{n}\right)\right] > 0,\end{aligned}$$

since (54) holds for all $n \geq 3$.

Hence, when $I = 2n$, $\sum_{k=1}^{n} d_k = -\sum_{k=1}^{I} c_k = 0$. Because $\sum_{k=1}^{n} d_k = 0$, $d_k$ decreases as $k$ increases, and $d_n < 0$, we can further derive that when $L <, > n$, $\sum_{k=1}^{L} d_k >, < 0$. When $I = 2n + 1$,

$$\sum_{k=1}^{n} d_k = -\sum_{k=1}^{I} c_k + c_{n+1} = c_{n+1} > 0,$$

$$\sum_{k=1}^{n+1} d_k = -\sum_{k=1}^{I} c_k - c_{n+1} = -c_{n+1} < 0.$$

With $d_k$ decreasing when $k$ gets larger, we can further derive that when $L <, > n$, $\sum_{k=1}^{L} d_k >, < 0$. In summary, $\sum_{k=1}^{L} d_k = 0$ only when $I = 2n$ and $L = n$, i.e., (ii) holds; when $L <, > n$, $\sum_{k=1}^{L} d_k >, < 0$ in both cases where $I = 2n$ and $I = 2n + 1$, i.e., (i) holds no matter whether $I$ is even or odd.

**Proof of Theorem 9.** Again, we consider two cases which depend on $I$ being even or odd: When $I = 2n, L \leq I/2 = n$; when $I = 2n + 1, L \leq I/2 = n + \frac{1}{2}$, i.e., $L \leq n$. In either case, the maximal $L$ is $n$. From the proof of Theorem 8, we can see that whenever $I = 2n$ or $I = 2n + 1$, we always have $\sum_{k=1}^{n} d_k \geq 0$ for all $n \geq 3$, which further implies $\sum_{k=1}^{n}(V_k d_k) > 0$ since $V_k$ is decreasing with $k$, i.e., when $L = n$, $x_r^* > x_c^*$ for $n \geq 3$. It is simple to show that when $I = 3\,(n = 1), I = 4\,(n = 2)$ and $I = 5\,(n = 2)$, $x_r^* > x_c^*$. Hence, the following can be concluded: For all cases with $I \geq 3$, when $L = n$, we always have $x_r^* > x_c^*$. This result can be generalized to all cases where $L \leq n$ since $L < n$ are just special cases of $L = n$ with one or several bottom prizes being zero, which completes the proof.

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